Chapter 1

Preliminary ideas

Some of the ideas you will encounter in mathematics will seem obvious to you. It is my responsibility to convince you that, sometimes, obvious ideas lead to spectacularly non-obvious conclusions.

The early bird gets the worm

Case 1. More birds than worms:

bird	*bird*	*bird*	*bird*	*bird*	*bird*	*bird*	*bird*
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
worm	*worm*	*worm*	*worm*	*worm*	*worm*	*worm*	

Case 2. More worms than birds:

bird	*bird*	*bird*	*bird*	*bird*	*bird*	*bird*	
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
worm	*worm*	*worm*	*worm*	*worm*	*worm*	*worm*	*worm*

Case 3. Exactly the same number of birds and worms:

bird	*bird*	*bird*	*bird*	*bird*	*bird*	*bird*	*bird*
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
worm	*worm*	*worm*	*worm*	*worm*	*worm*	*worm*	*worm*

In this last case, there is a 1-1 correspondence of birds and worms which was lacking in the first two cases. Seeing that there is a 1-1 correspondence is all that is needed to know there are exactly as many birds as worms. Let us commemorate this obvious idea by making it our first theorem.

Theorem 1.1
Two collections have exactly the same number of members if and only if there is a 1-1 correspondence between the members of the two collections. ■

Numbers and even numbers

Case 1. Stop at 10:

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ & 2 & & 4 & & 6 & & 8 & & 10 \end{array}$$

Half as many even numbers as numbers.

Case 2. Stop at 100:

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & 100 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots & \updownarrow \\ & 2 & & 4 & & 6 & & 8 & \cdots & 100 \end{array}$$

Half as many even numbers as numbers.

Case 3. Stop at 10000:

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & 10000 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots & \updownarrow \\ & 2 & & 4 & & 6 & & 8 & \cdots & 10000 \end{array}$$

Yup, half as many!

Case 4. Never stop:

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & \cdots \end{array}$$

There are *exactly the same number of even numbers as there are numbers!* We know this, because there is a 1-1 correspondence between the even numbers and the numbers.

Moral of story: even the most obvious of ideas can sometimes produce surprising consequences. So, it may be wise to show patience for obvious ideas.

When we write $1,2,3,\cdots,70$, we mean the "$\cdots$" to include the numbers from 4 to 69. When we write $1,2,3,\cdots$, we mean that this pattern continues without end..

Exercise 1.1

1. Using $\cdots$, write the numbers from 0 to 1000.
2. Using $\cdots$, write the numbers from 7 to 93.
3. Using $\cdots$, write the numbers from 5 on.
4. Are there the same number of odd numbers as even numbers? Show why your answer is correct.
5. Are there the same number of odd numbers as numbers? Show why your answer is correct.
6. Are there the same number of multiples of 5 as numbers? Show why your answer is correct.

1.1. Letters

There is nothing magical or mysterious about the use of letters in mathematics. The letter is merely a very brief name that we use instead of a long name or descriptive phrase.

Some school textbooks call letters "variables". This is unfortunate, because not every letter is a variable. We will call letters "letters".

Example 1.1
You know that

$$1+2=2+1,$$
$$1+3=3+1,$$
$$2+5=5+2.$$

You know much more than this, because you know this is true for every pair of numbers. How can we communicate this idea? Here is a convenient way.

Let a and b represent any numbers. Then $a+b=b+a$.

Example 1.2
The following statements are true.

$$(1+2)+3 = 1+(2+3).$$
$$(3+4)+5 = 3+(4+5).$$
$$(3+7)+8 = 3+(7+8).$$

Assert that this idea holds true for *every* triplet of numbers.

Solution

Let a, b, c represent any numbers. Then $(a+b)+c = a+(b+c)$. ■

In a previous grade, you learned that the product of two fractions is a fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators, provided that no denominator is zero. For instance, $\frac{2}{3} \times \frac{5}{7} = \frac{2\times 5}{3\times 7}$. That was a mouthful. Using letters, the idea is communicated simply and concisely.

Example 1.3
Write the rule for finding the product of fractions.

Solution

For any numbers a, b, c, d with b and d different from zero,

$$\frac{a}{b} \times \frac{c}{d} = \frac{a\times c}{b\times d}.$$

Example 1.4
Write the rule for canceling common factors in fractions.

Solution

For any numbers a, b, c with b and c different from zero, $\frac{a\times c}{b\times c} = \frac{a}{b}$. ■

The choice of letters is of no consequence. The statements "$a+b = b+a$" and "$x+y = y+x$" have exactly the same meaning. Each statement expresses the idea that the terms of a sum may be added in any order.

Exercise 1.2

1. Using the letter a, express the idea that a number multiplied by 1 is that same number.
2. Using the letter a, express the idea that a fraction in which the numerator and the denominator are identical is equal to 1.
3. Using the letter b, express the idea that zero added to a number results in the number.
4. Using the letters a, b, and c, show the addition of two fractions that have a common denominator.
5. Using the letters a and b, express the idea that the order in which two numbers are multiplied does not affect the product.
6. Using the letter y, write all the even numbers.

Appendix A

Answers to Exercises

Answers to Exercise 1.1

(**1**) $0, 1, 2, \cdots, 1000$. (**2**) $7, 8, 9, \cdots, 93$. (**3**) $5, 6, 7, \cdots$.
(**4**) Yes. Exhibit a 1-1 correspondence. (**5**) Yes. Exhibit a 1-1 correspondence. (**6**) Yes. Exhibit a 1-1 correspondence.

Answers to Exercise 1.2

(**1**) $a = 1 \times a$. (**2**) $\frac{a}{a} = 1$. (**3**) $0 + b = 0$. (**4**) $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$
(**5**) $a \times b = b \times a$ (**6**) $2 \times y$, where $y = 1, 2, 3, \cdots$.

Answers to Exercise 1.3

(**1**) $101 + a$. (**2**) $a + 3 = b + 5$. (**3**) Symmetric. (**4**) The Principle of Substitution
(**5**) (1) Adding the same number to each side of an equation preserves the equality. (2) Multiplying both sides of an equation by the same number preserves the equality.
(**6**) Proof
Let a, b and c be any numbers. Suppose that $a = b$.

$$a \times c = a \times c, \quad \text{equality is reflexive}$$
$$a \times c = b \times c, \quad \text{substitution, we supposed } a = b.$$

Therefore, if $a = b$ then $a \times c = b \times c$. ■

Answers to Exercise 1.4

(**1**) 11 (**2**) 36 (**3**) 15 (**4**) 17