EXERCISES 10.2

15-11-30-S13

1–6 ■ Find at least 10 partial sums of the series. Graph both the sequence of terms and the sequence of partial sums on the same screen. Does it appear that the series is convergent or divergent? If it is convergent, find the sum. If it is divergent, explain why.

1. $\sum_{n=1}^{\infty} \frac{10}{3^n}$
2. $\sum_{n=1}^{\infty} \sin n$
3. $\sum_{n=1}^{\infty} \frac{n}{n+1}$
4. $\sum_{n=4}^{\infty} \frac{3}{n(n-1)}$
5. $\sum_{n=1}^{\infty} \left(\frac{1}{n^{1.5}} - \frac{1}{(n+1)^{1.5}}\right)$
6. $\sum_{n=1}^{\infty} \left(-\frac{2}{7}\right)^{n-1}$

7–36 ■ Determine whether the series is convergent or divergent. If it is convergent, find its sum.

7. $4 + \frac{8}{5} + \frac{16}{25} + \frac{32}{125} + \cdots$
8. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$
9. $\frac{2}{3} - \frac{2}{9} + \frac{2}{27} - \frac{2}{81} + \cdots$
10. $-\frac{81}{100} + \frac{9}{10} - 1 + \frac{10}{9} - \cdots$
11. $\sum_{n=1}^{\infty} 2\left(\frac{3}{4}\right)^{n-1}$
12. $\sum_{n=1}^{\infty} \left(-\frac{3}{\pi}\right)^{n-1}$
13. $\sum_{n=1}^{\infty} 5\left(\frac{e}{3}\right)^n$
14. $\sum_{n=1}^{\infty} \frac{1}{e^{2n}}$
15. $\sum_{n=0}^{\infty} \frac{5^n}{8^n}$
16. $\sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n}$
17. $\sum_{n=1}^{\infty} 3^{-n} 8^{n+1}$
18. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^{2n}}{2^{3n+1}}$
19. $\sum_{n=1}^{\infty} \frac{1}{2n}$
20. $\sum_{n=1}^{\infty} \frac{n^2}{3(n+1)(n+2)}$
21. $\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)}$
22. $\sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} + \frac{2}{3^{n-1}}\right)$
23. $\sum_{n=1}^{\infty} [2(0.1)^n + (0.2)^n]$
24. $\sum_{n=1}^{\infty} \left(\frac{1}{n} + 2^n\right)$
25. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{1+n^2}}$
26. $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$
27. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$
28. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+5}\right)$
29. $\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n}$
30. $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$
31. $\sum_{n=1}^{\infty} \left[\sin\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n+1}\right)\right]$
32. $\sum_{n=1}^{\infty} \frac{1}{5 + 2^{-n}}$
33. $\sum_{n=1}^{\infty} \arctan n$
34. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$
35. $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$
36. $\sum_{n=2}^{\infty} \ln \frac{n^2 - 1}{n^2}$

37–42 ■ Express the number as a ratio of integers.

37. $0.\overline{5} = 0.5555\ldots$
38. $0.\overline{15} = 0.15151515\ldots$
39. $0.\overline{307} = 0.307307307307\ldots$
40. $1.\overline{123}$
41. $0.123\overline{456}$
42. $4.\overline{1570}$

43–48 ■ Find the values of x for which the series converges. Find the sum of the series for those values of x.

43. $\sum_{n=0}^{\infty} (x-3)^n$
44. $\sum_{n=0}^{\infty} 3^n x^n$
45. $\sum_{n=2}^{\infty} \frac{x^n}{5^n}$
46. $\sum_{n=0}^{\infty} \frac{1}{x^n}$
47. $\sum_{n=0}^{\infty} 2^n \sin^n x$
48. $\sum_{n=0}^{\infty} \tan^n x$

49–50 ■ Use the partial fraction command on your CAS to find a convenient expression for the partial sum, and then use this expression to find the sum of the series. Check your answer by using the CAS to sum the series directly.

49. $\sum_{n=1}^{\infty} \frac{1}{(4n+1)(4n-3)}$
50. $\sum_{n=1}^{\infty} \frac{n^2 + 3n + 1}{(n^2+n)^2}$

51. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = \frac{n-1}{n+1}$$

find a_n and $\sum_{n=1}^{\infty} a_n$.

52. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = 3 - n2^{-n}$$

find a_n and $\sum_{n=1}^{\infty} a_n$.

53. When money is spent on goods and services, those that receive the money also spend some of it. The people receiving some of the twice-spent money will spend some of that, and so on. Economists call this chain reaction the *multiplier effect.* In a hypothetical isolated community, the local government begins the process by spending D dollars. Suppose that each recipient of spent money spends $100c\%$ and saves $100s\%$ of the money that he or she receives. The values c and s are called the *marginal propensity to consume* and the *marginal propensity to save* and, of course, $c + s = 1$.
   (a) Let S_n be the total spending that has been generated after n transactions. Find an equation for S_n.
   (b) Show that $\lim_{n \to \infty} S_n = kD$, where $k = 1/s$. The number k is called the *multiplier.* What is the multiplier if the marginal propensity to consume is 80%?

From Example 6 we know that

$$\int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$$

so

$$s_{10} + \frac{1}{2(11)^2} \leqslant s \leqslant s_{10} + \frac{1}{2(10)^2}$$

Using $s_{10} \approx 1.197532$, we get

$$1.201664 \leqslant s \leqslant 1.202532$$

If we approximate s by the midpoint of this interval, then the error is at most half the length of the interval. So

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2021 \qquad \text{with error} < 0.0005$$

■

If we compare Example 7 with Example 6, we see that the improved estimate in (5) can be much better than the estimate $s \approx s_n$. To make the error smaller than 0.0005 we had to use 32 terms in Example 6 but only 10 terms in Example 7.

EXERCISES 10.3

1–18 ■ Test the series for convergence or divergence.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt[3]{n}}$

2. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{n\sqrt{n}} + \frac{3}{n^3}\right)$

3. $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n^{1.0001}}$

4. $\displaystyle\sum_{n=1}^{\infty} n^{-0.99}$

5. $\displaystyle\sum_{n=5}^{\infty} \frac{1}{(n-4)^2}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+3}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$

8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

9. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

12. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2n^2-n-1}$

13. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2+1}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{1+n^2}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+2n+2}$

18. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln n\ln(\ln n)}$

19–22 ■ Find the values of p for which the series is convergent.

19. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

20. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln n[\ln(\ln n)]^p}$

21. $\displaystyle\sum_{n=1}^{\infty} n(1+n^2)^p$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$

23. The Riemann zeta-function ζ is defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

and is used in number theory to study the distribution of prime numbers. What is the domain of ζ?

24. (a) Find the partial sum s_{10} of the series $\sum_{n=1}^{\infty} 1/n^4$. Estimate the error in using s_{10} as an approximation to the sum of the series.
(b) Use (5) with $n = 10$ to give an improved estimate of the sum.
(c) Find a value of n so that s_n is within 0.00001 of the sum.

25. (a) Use the sum of the first 10 terms to estimate the sum of the series $\sum_{n=1}^{\infty} 1/n^2$. How good is this estimate?
(b) Improve this estimate using (5) with $n = 10$.
(c) Find a value of n that will ensure that the error in the approximation $s \approx s_n$ is less than 0.001.

26. Find the sum of the series $\sum_{n=1}^{\infty} 1/n^5$ correct to three decimal places.

27. Estimate $\sum_{n=1}^{\infty} n^{-3/2}$ to within 0.01.

28. How many terms of the series $\sum_{n=2}^{\infty} 1/[n(\ln n)^2]$ would you need to add to find its sum to within 0.01?

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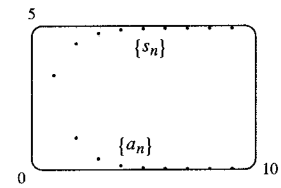
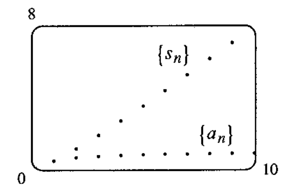
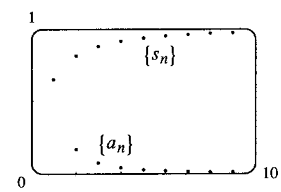
1. 3.33333, 4.44444, 4.81481, 4.93827, 4.97942, 4.99314, 4.99771, 4.99924, 4.99975, 4.99992
Convergent, sum = 5



3. 0.50000, 1.16667, 1.91667, 2.71667, 3.55000, 4.40714, 5.28214, 6.17103, 7.07103, 7.98012
Divergent (terms do not approach 0)



5. 0.64645, 0.80755, 0.87500, 0.91056, 0.93196, 0.94601, 0.95581, 0.96296, 0.96838, 0.97259
Convergent, sum = 1



7. $\frac{20}{3}$ **9.** $\frac{1}{2}$ **11.** 8 **13.** $5e/(3 - e)$ **15.** $\frac{8}{3}$
17. Divergent **19.** Divergent **21.** $\frac{1}{3}$ **23.** $\frac{17}{36}$
25. Divergent **27.** $\frac{3}{4}$ **29.** $\frac{3}{2}$ **31.** $\sin 1$ **33.** Divergent
35. Divergent **37.** $\frac{5}{9}$ **39.** $\frac{307}{999}$ **41.** 41,111/333,000
43. $2 < x < 4$, $1/(4 - x)$ **45.** $-5 < x < 5$, $x^2/[5(5 - x)]$
47. $|x - n\pi| < \pi/6$, n any integer, $1/(1 - 2\sin x)$ **49.** $\frac{1}{4}$
51. $a_1 = 0$, $a_n = 2/[n(n + 1)]$ for $n > 1$, sum = 1
53. (a) $S_n = D(1 - c^n)/(1 - c)$ (b) 5 **55.** $(\sqrt{3} - 1)/2$
57. $1/[n(n + 1)]$ **59.** The series is divergent
65. $\{s_n\}$ is bounded and increasing
67. (a) $0, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}, 1$
69. (a) $\frac{1}{2}, \frac{5}{6}, \frac{23}{24}, \frac{119}{120}$; $[(n + 1)! - 1]/(n + 1)!$ (c) 1

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Abbreviations: *C, convergent*; *D, divergent*

1. D **3.** C **5.** C **7.** D **9.** C **11.** D **13.** D
15. C **17.** C **19.** $p > 1$ **21.** $p < -1$ **23.** $(1, \infty)$
25. (a) 1.54977, error $\leqslant$ 0.1 (b) 1.64522, error $\leqslant$ 0.005
(c) $n > 1000$
27. 2.6124 **31.** $b < 1/e$