Theorem 1. *The limit of a sequence, when it exists, is unique.*

Proof. (Indirect) Let a_n be a sequence. Suppose that $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} a_n = B$ and that $A \neq B$. Choose $\varepsilon = |A - B|/3$. Then, there exist M_A and M_B for which

$$n \geq M_A \implies |a_n - A| < \varepsilon$$

and

$$n \geq M_B \implies |a_n - B| < \varepsilon.$$

Let $M = \max\{M_A, M_B\}$. Then $n \geq M$ implies both

$$|a_n - A| < \varepsilon$$
$$|a_n - B| < \varepsilon$$

or, what is equivalent,

$$-\varepsilon < a_n - A < \varepsilon$$
$$-\varepsilon < a_n - B < \varepsilon.$$

Then

$$A - \varepsilon < a_n < A + \varepsilon \quad (1)$$
$$B - \varepsilon < a_n < B + \varepsilon. \quad (2)$$

No generality is lost if we suppose that $B > A$. Since $\varepsilon = |A - B|/3$,

$$A + 3\varepsilon = B.$$

This means

$$A + 2\varepsilon = B - \varepsilon$$
$$A + \varepsilon < B - \varepsilon. \quad (3)$$

Using inequalities (1), (2), and (3),

$$a_n < A + \varepsilon < B - \varepsilon < a_n.$$

This contradiction proves the theorem. ∎