Computation in Triangles

In the remaining part of this chapter we shall develop and use some formulas which facilitate computation in triangles which are not right triangles.

In this section of our work we shall be concerned only with angles which contain less than 180°, that is, with acute and obtuse angles. It is convenient to summarize the relationship between the sines and between the cosines of these angles by the following statements:

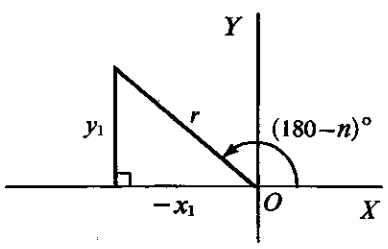
▶ **The sine of an obtuse angle is equal to the sine of its supplement; that is,**

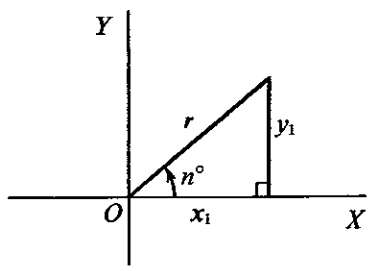
$$\sin(180 - n)^\circ = \sin n^\circ. \quad (0 < n < 90)$$

▶ **The cosine of an obtuse angle is the negative of the cosine of its supplement; that is,**

$$\cos(180 - n)^\circ = -\cos n^\circ. \quad (0 < n < 90)$$

The statements merely emphasize the relationships between the sines and cosines of angles in the first and second quadrants which follow from the general definitions.





Area of Any Triangle

Consider $\triangle ABC$ of any shape in a Cartesian coordinate system with A as origin and having the line AC as x-axis. Let a, b, c represent the sides of

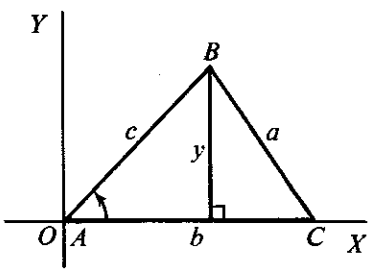


Fig. 1

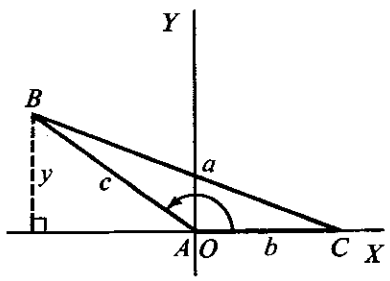


Fig. 2

the triangle opposite $\angle A$, $\angle B$, $\angle C$ as usual. Figures 1 and 2 illustrate the situation when $\angle A$ is acute and when $\angle A$ is obtuse.

Let the ordinate of B be y. Then if we regard b as the base of $\triangle ABC$, the altitude on the base is y. Hence, if the area of $\triangle ABC$ is represented by K square units, we have

$$K = \tfrac{1}{2}\, b \cdot y.$$

But $\frac{y}{c} = \sin A$ or $y = c \cdot \sin A$ (in both figures).

$$\therefore K = \tfrac{1}{2}\, bc \sin A.$$

A different choice of origin and x-axis would enable us to show that $K = \frac{1}{2}\, ab \sin C$ or $K = \frac{1}{2}\, ca \sin B$.

The area of $\triangle ABC$ may therefore be expressed as one-half the product of two sides and the sine of the angle formed by these sides.

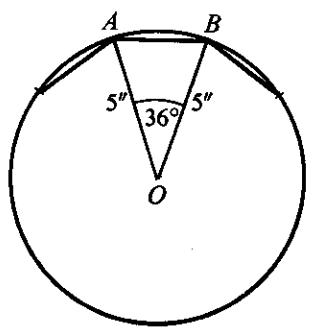
Example 1. Find the area of $\triangle ABC$ if $a = 20$ in., $b = 25$ in., $\angle C = 120°$.

Solution: Using $K = \frac{1}{2}\, ab \sin C$, we have

$$\begin{aligned} K &= \tfrac{1}{2} \cdot 20 \cdot 25 \cdot \sin 120° \\ &= 10 \cdot 25 \cdot \frac{\sqrt{3}}{2} \\ &= 125\sqrt{3} \text{ square inches.} \end{aligned}$$

Example 2. Find the area of a regular 10-sided polygon inscribed in a circle of radius 5 inches

Solution: Let AB be a side of the polygon and O the center of the circle. Then $\angle AOB = 36°$.



$$\begin{aligned} \text{Area of } \triangle AOB &= \tfrac{1}{2}\, AO \cdot BO \cdot \sin 36° \\ &= \tfrac{1}{2} \cdot 5 \cdot 5 \cdot \sin 36° \\ &= 12.5 \cdot \sin 36°. \end{aligned}$$

The polygon is made up of 10 such triangles.

$$\begin{aligned} \therefore \text{area of polygon} &= 10[12.5 \sin 36°] \\ &= 125 \sin 36° \\ &= 125(0.5878) \\ &= 73.5 \text{ square inches.} \end{aligned}$$

Exercises [A]

In exercises 1–6 find the area of $\triangle ABC$:

1. (a) $b = 6''$, $c = 10''$, $\angle A = 30°$;
   (b) $b = 6''$, $c = 10''$, $\angle A = 150°$.
2. (a) $b = 12''$, $c = 15''$, $\angle A = 45°$;
   (b) $b = 12''$, $c = 15''$, $\angle A = 135°$.
3. (a) $a = 10''$, $b = 8''$, $\angle C = 60°$;
   (b) $a = 10''$, $b = 8''$ $\angle C = 120°$.
4. $a = 12.5''$, $b = 20.0''$, $\angle C = 115°\ 30'$.

5. $c = 25.0''$, $a = 40.0''$, $\angle B = 32° 40'$.
6. $b = 17.5''$, $c = 8.0''$, $\angle A = 128° 20'$.
7. A regular pentagon is inscribed in a circle of radius 6 inches. Find the area of the pentagon.
8. A regular 9-sided polygon is inscribed in a circle of radius 4 inches. Find the area of the polygon.
9. A regular 8-sided polygon is inscribed in a circle of radius r inches. Obtain a formula for the area of the polygon.
10. A regular 12-sided polygon is inscribed in a circle of radius r inches. Obtain a formula for the area of the polygon.
11. $\triangle ABC$ has $a = 6$ in., $b = 10$ in. If the area of $\triangle ABC$ is 24 sq. in., find the two possible values for the size of $\angle C$.
12. $\triangle ABC$ has $b = 12$ in., $c = 15$ in. If the area of the triangle is 45 sq. in., find the size of $\angle A$.
13. $\triangle ABC$ has $a = 4$ in., $b = 6$ in. Write an expression for the area K of the triangle when $\angle C$ contains $x°$. State the domain and range of the area function so defined. Sketch the graph of the function.
14. Use two expressions for the area of $\triangle ABC$ to show that

$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

The Law of Sines

A useful relationship between the sides and the angles of a triangle can be derived from the expressions for the area.

Since $\quad bc \sin A = ca \sin B = ab \sin C \quad (= 2K)$,

we have $\quad \dfrac{abc}{bc \sin A} = \dfrac{abc}{ca \sin B} = \dfrac{abc}{ab \sin C}$, from which we deduce that:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The last statement is called the *Law of Sines*. It expresses the fact that for *any given triangle* the ratio of the length of a side to the sine of the opposite angle is constant.

In using the Law of Sines we should note that the proportion formed by two of the ratios can be expressed in other forms. If we start with

$$\frac{a}{\sin A} = \frac{b}{\sin b},$$

we can deduce $\quad a \sin B = b \sin A \quad$ or $\quad \dfrac{a}{b} = \dfrac{\sin A}{\sin B}.$

If three of the numbers in the proportion $\frac{a}{\sin A} = \frac{b}{\sin B}$ are known, the fourth number can be computed. Thus, solving the proportion for b we have

$$b = \frac{a \sin B}{\sin A},$$

and when a, $\sin A$, $\sin B$ are known, the value of b can be computed. Solving the proportion for $\sin B$, we have

$$\sin B = \frac{b \sin A}{a},$$

and if a, b, $\sin A$ are known, the value of $\sin B$ can be computed.

Example 1. $\triangle ABC$ has $a = 12''$, $b = 6''$, $\angle A = 45°$. Find the size of $\angle B$.

Solution: From the Law of Sines,

$$\sin B = \frac{b \sin A}{a}.$$

Hence, in this particular triangle we have

$$\begin{aligned} \sin B &= \frac{6 \sin 45°}{12} \\ \sin B &= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{1.4142}{4} \\ &= 0.3535 \\ \angle B &= 20° \, 42'. \end{aligned}$$

NOTE. In obtaining the size of an angle of a triangle from the value of its sine we must not overlook the fact that a number between 0 and 1 is the sine of an obtuse angle as well as the sine of an acute angle. Thus $\angle B = 159° \, 18'$ is also a solution of the statement $\sin B = 0.3535$. Why is this second solution not acceptable for the given triangle?

Example 2. $\triangle ABC$ has $\angle A = 45°$, $\angle B = 60°$. What is the ratio of a to b?

Solution: From the Law of Sines,

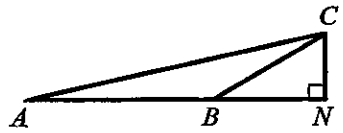
$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

Hence, in this particular triangle we have:

$$\begin{aligned} \frac{a}{b} &= \frac{\sin 45°}{\sin 60°} \\ &= \frac{\frac{1}{2}\sqrt{2}}{\frac{1}{2}\sqrt{3}} \\ &= \frac{\sqrt{2}}{\sqrt{3}} \\ &= \frac{\sqrt{6}}{3}. \end{aligned}$$

Exercises [A]

1. $\triangle ABC$ has $\angle A = 45°$, $\angle B = 60°$, and $a = 10.0$ inches. Find the lengths of b, c.
2. $\triangle ABC$ has $\angle A = 120°$, $\angle B = 18° 26'$, and $c = 25.0$ inches. Find the lengths of a, b.
3. In $\triangle ABC$, $\angle A = 120°$ and $b = \frac{2}{3} a$. Find the size of $\angle B$ and $\angle C$.
4. In $\triangle ABC$, $a = 15.75''$, $b = 20.25''$, $\angle B = 38° 30'$. Find the size of $\angle A$ and $\angle C$.
5. In $\triangle ABC$, $b = 12.0''$, $a = 10.0''$, $\angle A = 42° 30'$. Find the two possible values of $\angle B$ and the corresponding values of $\angle C$.
6. Find the size of $\angle C$ of $\triangle ABC$ if $\angle A = 38° 20'$ and $a = \frac{3}{4} b$.
7. Find $\sin C$ if $\triangle ABC$ has $a = 2''$, $c = 5''$, $\angle A = 30°$. What can be deduced concerning the given values?
8. Is it possible for $\triangle ABC$ to have $b = 25''$, $c = 12''$, $\angle C = 31° 30'$?
9. A light Z is observed from the ends X, Y of a base line 1500 yd. long. If $\angle ZXY = 32° 15'$ and $\angle ZYX = 43° 30'$, find the lengths ZX, ZY.
10. In the diagram, CN is perpendicular to the extension of AB. $AB = 175$ ft., $\angle CAB = 12° 30'$, $\angle CBN = 30°$. Find the length CN.



The Law of Cosines

The Law of Sines is useful in computations only if three terms of one of the proportions are given so that the fourth term can be found. This condition is not met when (**a**) the three sides of a triangle are given, (**b**) two sides and their included angle are given. The formula which we now develop enables us to deal directly with each of these two cases.

Consider $\triangle ABC$ of any shape. Choose A as the origin for a Cartesian coordinate system and the line AB as x-axis.

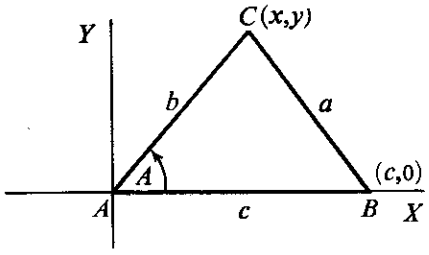


Fig. 1

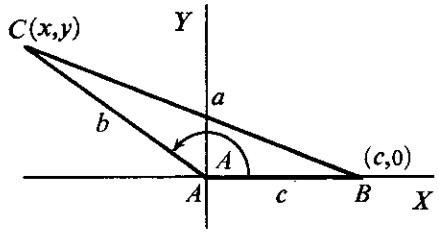


Fig. 2

Let a, b, c represent the lengths of the sides in the usual way. Let the coordinates of C be (x, y). The coordinates of B are $(c, 0)$. Figure 1 illustrates the situation when $\angle A$ is acute. Figure 2 illustrates the situation when $\angle A$ is obtuse.

In both cases $$\begin{cases} x^2 + y^2 = b^2 \\ x = b \cos A. \end{cases}$$

Using the distance formula for CB, we have

$$a^2 = (x - c)^2 + (y - 0)^2$$
$$a^2 = x^2 - 2\,cx + c^2 + y^2$$
$$a^2 = b^2 + c^2 - 2\,cx \quad (\text{since } x^2 + y^2 = b^2)$$

▶ $$\mathbf{a^2 = b^2 + c^2 - 2\,bc \cos A} \quad (\text{since } x = b \cos A)$$

This result may also be written

▶ $$\mathbf{\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}}.$$

In a similar manner it can be shown that

$$b^2 = c^2 + a^2 - 2\,ca \cos B \quad \text{or} \quad \cos B = \frac{c^2 + a^2 - b^2}{2\,ca},$$

$$c^2 = a^2 + b^2 - 2\,ab \cos C \quad \text{or} \quad \cos C = \frac{a^2 + b^2 - c^2}{2\,ab}.$$

Any one of these formulas is called the Law of Cosines.

Example 1. $\triangle ABC$ has $BC = 12.0''$, $AC = 10.0''$, $\angle C = 120°$. Find the length of the side AB.

Solution: Use the Law of Cosines in the form $c^2 = a^2 + b^2 - 2\,ab \cos C$.

Then,
$$c^2 = 144 + 100 - 2 \cdot 12 \cdot 10 \cdot \cos 120°$$
$$c^2 = 244 - 2 \cdot 12 \cdot 10 \cdot (-\tfrac{1}{2})$$
$$c^2 = 244 + 120 = 364$$
$$c = 19.1$$

The length of the side AB is 19.1 inches.

Example 2. Find the largest angle of a triangle with sides $10''$, $15''$, $21''$ long.

Solution: Let $a = 10$, $b = 15$, $c = 21$, so that we wish to find the size of $\angle C$.

From the Law of Cosines, $$\cos C = \frac{a^2 + b^2 - c^2}{2\,ab}$$
$$\cos C = \frac{100 + 225 - 441}{2 \cdot 10 \cdot 15}$$
$$\cos C = \frac{-116}{300} = -0.3867$$
$$\angle C = 180° - 67°\,15' = 112°\,45'$$

Exercises [A]

1. Find the largest angle of a triangle having sides 3 in., 5 in., 7 in. long.
2. Find the smallest angle of the triangle specified in exercise 1.
3. A triangle has sides 6 inches and 10 inches long forming an angle of 120°. Find the length of the third side.
4. A triangle has sides 15.0 inches and 20.0 inches long, forming an angle of 60°. Find the length of the third side to the nearest tenth of an inch.
5. $\triangle ABC$ has $a = 6$, $b = 7$, $c = 10$. (a) Find $\angle C$. (b) Find $\angle A$.
6. $\triangle ABC$ has $a = 6$, $b = 8$, $c = 12$. (a) Find $\angle A$. (b) Find $\angle C$.
7. $\triangle ABC$ has $\angle A = 53° 8'$, $AC = 10.5''$, $AB = 20.0''$. Find BC.
8. $\triangle ABC$ has $\angle B = 143° 8'$, $BC = 24.0''$, $AB = 35.0''$. Find AC.
9. $ABCD$ is a parallelogram with $AB = 15.0''$, $BC = 12.0''$, and $\angle B = 70°$. Find the length of each diagonal to the nearest tenth of an inch.
10. $ABCD$ is a parallelogram with $AB = 12.0''$, $BC = 10.0''$, and the diagonal $AC = 18.0''$. Find (a) the size of $\angle B$, (b) the length of the diagonal BD.
11. Find the lengths of the sides of a parallelogram if the diagonals intersect at an angle of 52° 15′ and have lengths 10.0 inches and 16.0 inches.
12. $\triangle ABC$ has $a = 10$ in., $b = 21$ in., $c = 17$ in. Without using tables, find (a) $\cos C$, (b) $\sin C$, (c) $\sin B$, (d) the area of the triangle.

Miscellaneous Exercises [A]

1. Find the value of:

(a) $\sin 120°$	(d) $\sin 405°$	(g) $\sin 300°$
(b) $\cos 240°$	(e) $\cos 220°$	(h) $\sin 450°$
(c) $\cos 700°$	(f) $\cos 360°$	(i) $\cos 720°$

2. If $0° \leqq \theta < 360°$, find all values of θ such that

(a) $\sin \theta = 0$	(e) $4 \sin \theta - 3 = 0$
(b) $\cos \theta = -1$	(f) $2 \cos^2\theta = \frac{1}{8}$
(c) $\sin \theta = -\frac{1}{2}\sqrt{3}$	(g) $\cos \theta - 2 = 7 \cos \theta$
(d) $\cos \theta = -\frac{1}{2}\sqrt{2}$	(h) $2 \cos^2\theta = 1$

3. If $\cos \theta = \frac{1}{2}\sqrt{2}$ and θ is a fourth-quadrant angle, find $\sin \theta$.
4. If $\sin \theta = -\frac{2}{3}$ and $180° < \theta < 270°$, find

(a) $\cos \theta$, (b) $\cos(180 - \theta)$.

11. $\frac{3}{5}$; $-\frac{3}{5}$

12. $-\frac{\sqrt{5}}{3}$; $\frac{\sqrt{5}}{3}$

13. $-\frac{2\sqrt{2}}{3}$

14. $-\frac{\sqrt{3}}{2}$

15. a. 90°
    b. No

16. a. 270°
    b. No

17. a. 0°
    b. No

18. a. 180°
    b. No

Page 411 [B]

1. First or fourth; $\sin X = \frac{\sqrt{b^2 - a^2}}{b}$ or $-\frac{\sqrt{b^2 - a^2}}{b}$.

2. $\{k \mid 0 < k \leqq 1\}$; $\angle A$ acute, $\cos A = \sqrt{1 - k^2}$; $\angle A$ obtuse $\cos A = -\sqrt{1 - k^2}$.

3. Third or fourth; $\cos Y = -\frac{\sqrt{n^2 - m^2}}{n}$ or $\frac{\sqrt{n^2 - m^2}}{n}$.

Page 415

1. $-\frac{\sqrt{3}}{2}$; $-\frac{1}{2}$

2. $-\frac{\sqrt{2}}{2}$; $\frac{\sqrt{2}}{2}$

3. -1

4. -1

5. a. 1
   b. $-\frac{\sqrt{3}}{2}$
   c. $\frac{\sqrt{3}}{2}$
   d. 0

6. a. 0
   b. $-\frac{\sqrt{3}}{2}$
   c. 0
   d. $-\frac{\sqrt{2}}{2}$

7. {150°, 210°}

8. $\theta = 270°$

9. {90°, 270°}

10. {210°, 330°}

11. {48°11', 311°49'}

12. {233°8', 306°52'}

13. {45°, 135°, 225°, 315°}

14. {60°, 120°, 240°, 300°}

15. $\sqrt{2}$

17. a. $\frac{1}{2}$
    b. $-\frac{\sqrt{3}}{4}$

18. a. $\sqrt{3}$
    b. 1

19. a. $\{y \mid -1 \leqq y \leqq 1\}$
    b. $\{y \mid -1 \leqq y \leqq 1\}$

20. $\theta = 45°, 225°$

Pages 417-418

1. a. 15 sq.in.
   b. 15 sq.in.

2. a. $45\sqrt{2}$ sq.in.
   b. $45\sqrt{2}$ sq.in.

3. a. $20\sqrt{3}$ sq.in.
   b. $20\sqrt{3}$ sq.in.
4. 112.8, or 113, sq.in.
5. 269.9, or 270, sq.in.
6. 54.91, or 54.9, sq.in.
7. 85.60 sq.in.
8. 46.3 sq.in.
9. $A = 2r^2\sqrt{2}$
10. $A = 3r^2$
11. $\angle C = 53°8'$ or $126°52'$
12. $\angle A = 30°$ or $150°$
13. $K = 12 \sin x°$;
    $\{x \mid 0 < x < 180\}$;
    $\{K \mid 0 < K \leq 12\}$.

Page 420

1. b = 12.25 in., c = 13.66 in.
2. a = 32.62 in., b = 11.91 in.
3. $\angle B = 35°16'$, $\angle C = 24°44'$
4. $\angle A = 28°57'$, $\angle C = 112°33'$
5. $\angle B = 54°10'$ or $125°50'$;
   $\angle C = 83°20'$ or $11°40'$
6. $\angle C = 85°52'$ or $17°28'$
7. $\sin C = 1.25$;
   $\triangle ABC$ does not exist.
8. $\sin B > 1$; no.
9. ZX = 1065 yd.; ZY = 825.7 yd.
10. 62.98, or 63.0, ft.

Page 422 (Top)

1. 120°
2. 21°47'
3. 14 in.
4. 18.0 in.
5. a. 100°17'
   b. 36°11'
6. a. 26°33'
   b. 117°17'
7. 16.1 in.
8. 56.1 in.
9. 15.7 in., 22.2 in.
10. a. 109°28'
    b. 12.8 in.
11. 6.33 in., 11.75 in.
12. a. $\frac{3}{5}$
    b. $\frac{4}{5}$
    c. $\frac{84}{85}$
    d. 84 sq.in.

Pages 422-423

1. a. $\frac{\sqrt{3}}{2}$ c. 0.9397 e. −0.7660 g. $-\frac{\sqrt{3}}{2}$ i. 1
   b. $-\frac{1}{2}$ d. $\frac{\sqrt{2}}{2}$ f. 1 h. 1

2. a. 0°, 180° b. 180° c. 240°, 300° d. 135°, 225°