[MT-6]

6

Complex numbers

6.1 The rectangular form of complex numbers

In Chapter 0 we indicated that the set C of complex numbers could be expressed as

$$C = \{x + yi \mid x \in R, y \in R, i = \sqrt{-1}\}.$$

We did not consider the complex numbers further at that point, but here we shall consider them in more detail.

Recall that in the set of real numbers we can always find the sum, the difference, the product, or the quotient of two real numbers, provided that we do not divide by zero. There is some difficulty with root extraction, since we cannot extract even roots of negative numbers. You will discover that in the set of complex numbers we can do all of the operations possible in the set of real numbers, and that root extraction is always possible in the set of complex numbers.

In the complex number $a + bi$, a is considered to be the **real part** of $a + bi$ and b is considered to be the **imaginary part** or, sometimes, the **coefficient of the imaginary part.**

The complex number $a + 0i = a$ is considered to be a **real number** and the complex number $0 + bi = bi$ is called a **pure imaginary number**.

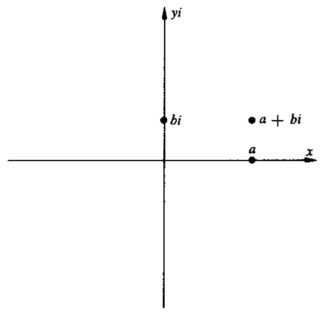

Figure 6.1

When we wish to represent complex numbers graphically, we often use what is called the **Argand plane**, in which we have a horizontal axis, called the **real axis**, and a vertical axis, called the **imaginary axis**. The plotting of the complex number $a + bi$, illustrated in Figure 6.1, is done in the same fashion as the plotting of the ordered pair (a, b) in R^2. Corresponding to the complex number $a + bi$ is a complex number called the **complex conjugate** of $a + bi$. The complex conjugate of $a + bi$ is $a - bi$. We can readily see that this complex conjugate is symmetric to the complex number $a + bi$ with respect to the real axis.

The complex conjugate of $3 + 4i$ is $3 - 4i$; the complex conjugate of $5 - 2i$ is $5 - (-2i) = 5 + 2i$. Figure 6.2 illustrates the symmetry of $3 + 4i$ and $3 - 4i$ with respect to the x-axis.

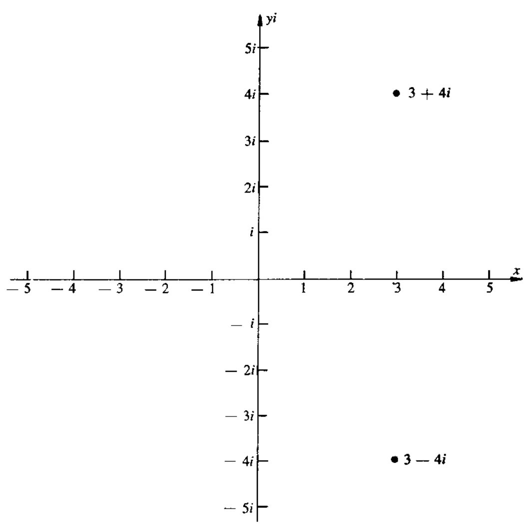

Figure 6.2

If $z = a + bi$, we often use $\bar{z}$ to refer to the complex conjugate of z. If $z = 4 + i$,

$$\bar{z} = 4 - i,$$

and if $w = 2 - 2i$,

$$\bar{w} = 2 + 2i.$$

Two complex numbers $a + bi$ and $c + di$ are considered to be equal if and only if $a = c$ and $b = d$. Thus, we can see that two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

The four standard operations, addition, subtraction, multiplication, and division, will be defined in this section. The operations of addition and subtraction are defined quite easily, as is shown below.

The ***sum*** *of $a + bi$ and $c + di$ is given by*

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

The ***difference*** *of $a + bi$ and $c + di$ is given by*

$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$

We can readily see that the sum (difference) of two complex numbers can be obtained by taking the sum (difference) of their real parts and of their imaginary parts, respectively. Examples of these two operations are shown below. Let

$$w = 3 - 4i, \quad z = 5 + 2i, \quad \text{and} \quad u = 4 + i.$$

Then

$$\begin{aligned} w + z &= 8 - 2i, & z + u &= 9 + 3i, \\ \bar{w} &= 3 + 4i, & \bar{z} &= 5 - 2i, \\ \bar{w} + u &= 7 + 5i, & w - z &= -2 - 6i, \\ z - u &= 1 + i, \quad \text{and} & \bar{z} - w &= 2 + 2i. \end{aligned}$$

The ***product*** *of $a + bi$ and $c + di$ is given by*

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

This definition might be developed if we assume that multiplication of complex numbers has the same properties as multiplication of real numbers—namely, that it is commutative and associative, and that it distributes over addition. This development is shown below.

$$\begin{aligned} (a + bi)(c + di) &= a(c + di) + bi(c + di) \\ &= [ac + (ad)i] + [(bc)i + (bd)i^2] \\ &= [ac + (bd)i^2] + [ad + bc]i \\ &= (ac - bd) + (ad + bc)i \quad (\text{since } i^2 = -1). \end{aligned}$$

Thus,

$$\begin{aligned}(2 + 3i)(5 - 7i) &= [2\cdot 5 - 3\cdot(-7)] + [2\cdot(-7) + 3\cdot 5]i \\ &= (10 + 21) + (-14 + 15)i \\ &= 31 + i,\end{aligned}$$

and

$$\begin{aligned}(3 + 4i)(3 - 4i) &= [3\cdot 3 - 4\cdot(-4)] + [3\cdot(-4) + 4\cdot 3]i \\ &= [9 + 16] + [-12 + 12]i \\ &= 25 + 0i = 25.\end{aligned}$$

The last example illustrated that the product of a complex number and its conjugate is a real number. This is shown in general below.

$$\begin{aligned}(a + bi)\cdot(a - bi) &= a^2 - (bi)^2 \\ &= a^2 - b^2i^2 \\ &= a^2 + b^2.\end{aligned}$$

The last fact is helpful in developing the definition of **division** of one complex number by another. We would like $(a + bi)/(c + di)$ to be a complex number. This means that this quotient should have a real part and an imaginary part. If we multiply both the numerator and the denominator of $(a + bi)/(c + di)$ by the complex conjugate of $c + di$, we have the following development.

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i.\end{aligned}$$

You may find it easier to remember the technique used in developing this definition than to remember the definition. Using the technique above, we find that

$$\begin{aligned}(7 - 2i) \div (2 + 5i) &= \frac{(7 - 2i)(2 - 5i)}{(2 + 5i)(2 - 5i)} \\ &= \frac{(14 - 10) + (-35 - 4)i}{4 + 25} \\ &= \frac{4 - 39i}{29} \\ &= \frac{4}{29} - \frac{39}{29}i.\end{aligned}$$

Either of these last two forms would be considered appropriate as an answer.

Exercises 6.1

1. Identify the real part, the imaginary part, and the complex conjugate of each of the following. Plot each of the following complex numbers.

a. $7 - i$	b. $3 + 2i$	c. $2 + 5i$
d. $1 - 3i$	e. $-4i$	f. $4 + i$
g. $9 - 2i$	h. -5	i. $\sqrt{3} + \sqrt{2i}$
j. $e + \pi i$	k. 3	l. $3i$

Perform the indicated operations.

2. $(3 - 2i) + (17 + i)$	3. $(\pi + 3i) + (-3\pi - 4i)$
4. $3i + (4 - 3i)$	5. $(-2 + 7i) + (2 - i)$
6. $(3 - 2i) - (17 + i)$	7. $(\pi + 3i) - (-3\pi - 4i)$
8. $(6 - i) - (4 - 5i)$	9. $(4 + 4i) - (3 + 2i)$
10. $(6 - 2i)(3 + i)$	11. $(e + 13i)(3e - 2i)$
12. $i(4 - 3i)$	13. $(2 - 3i)(3 - 4i)$
14. $(3 - 2i) \div (1 + i)$	15. $(1 + i) \div (3 - 2i)$
16. $(4 - 3i) \div (3 - 9i)$	17. $1 \div (c + di)$
18. $[(3 + i) + (2 - 3i)](4 + 5i)$	19. $[(5 - 2i)(3 + 4i)] \div (4 + 6i)$
20. $[(3 - i) \div (2 + 2i)](5 - 6i)$	21. $[(3 + 2i) - (-3 + i)](6 - 3i)$
22. $(1 + i)^4$	23. $(3 + 2i)^2 \div (1 + 2i)^3$
24. $(-5 + 2i)^3 - (4 - i)^2$	25. $(1 + i)^4 \div (4 - i)^2$

6.2 The polar form of complex numbers

In Section 6.1 we explored the method known as the rectangular form of representing complex numbers. In this section we shall examine the **polar form** of representing complex numbers.

First, we shall consider the *distance* that the point representing a complex number is from the origin. Through the use of the Pythagorean theorem we can easily see in Figure 6.3 that the distance of the point $x + yi$ from the origin is given by $\sqrt{x^2 + y^2}$. This distance is usually referred to as the **modulus** of the complex number, and it is denoted by r.

Second, we shall consider the *angle* that the segment connecting the point $x + yi$ to the origin makes with the positive half of the x-axis. This angle is denoted by θ in Figure 6.3, and it is called the **argument** or the **amplitude** of the complex number $x + yi$. *The argument of a complex number will be given in radians, not degrees, in this book.*

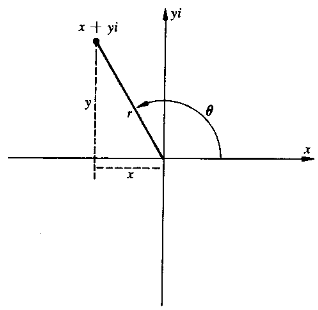

Figure 6.3

We can note that $\tan\theta = y/x$, but we cannot go further to state that $\theta = \text{Tan}^{-1}\, y/x$, the reason being that y/x has the same value for two points which are symmetric with respect to the origin even though they are on opposite half-lines emanating from the origin. It is important to notice the quadrant in which a complex number lies when determining the argument of the complex number. If we consider the complex numbers $-2 + 2i$ and $2 - 2i$, we note that $-2 + 2i$ lies in the second quadrant and $3\pi/4$ is an appropriate selection for the argument of $-2 + 2i$. The complex number $2 - 2i$ lies in the fourth quadrant and $-\pi/4$ is an appropriate selection for the argument of $2 - 2i$. These two complex numbers are graphed in Figure 6.4.

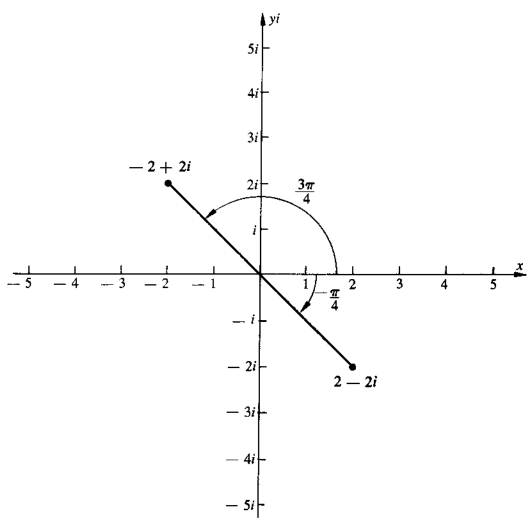

Figure 6.4

For the complex number $-2 + 2i$,

$$\frac{y}{x} = \frac{2}{-2} = -1\,,$$

and for the complex number $2 - 2i$,

$$\frac{y}{x} = \frac{-2}{2} = -1.$$

Since $\tan \theta = -1$ for each of these complex numbers, it is necessary to see which quadrant the complex number is in before we choose the argument of the complex number.

You may ask why pick $-\pi/4$ instead of $7\pi/4$ for the argument of $2 - 2i$. Remember that if the measures of two angles differ by an integral multiple of 2π radians, the two angles have the same terminal side. This means that if θ is an argument for the complex number $a + bi$, then any number of the form $\theta + 2n\pi$, where $n \in J$, is also an argument for the complex number $a + bi$. To achieve uniformity in the determination of the argument of a complex number, we shall agree that the **principal argument** of a complex number will be in the interval $(-\pi, \pi]$.

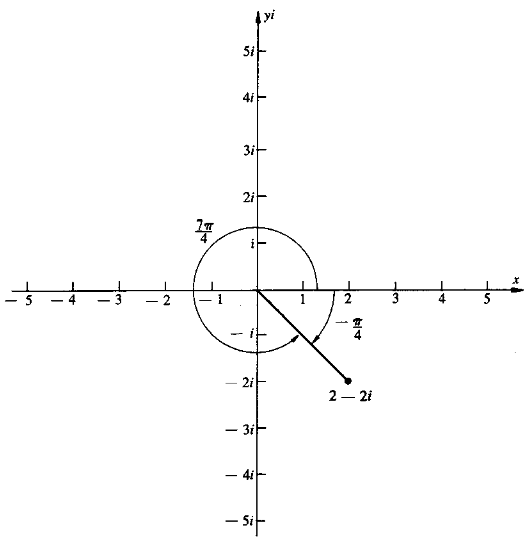

Figure 6.5

When writing a complex number in polar form, we usually consider the origin to be the **pole**, and the ray which corresponds to the positive half of the x-axis is called the **polar axis**. The complex number is written as $r(\cos \theta + i \sin \theta)$ where r and θ are as described above.

Two complex numbers

$$r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad r_2(\cos \theta_2 + i \sin \theta_2)$$

are the same complex number if $\boldsymbol{r_1 = r_2}$ and $\boldsymbol{\theta_1 = \theta_2 + 2n\pi}$ for some $n \in J$. Although we do not customarily write complex numbers with negative moduli,

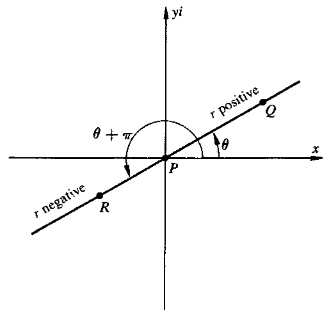

Figure 6.6

this is occasionally done. If r is negative, we first determine the ray emanating from the origin which makes an angle of θ radians with the polar axis. The other half of the line determined by this ray makes an angle of $\theta + \pi$ radians with the polar axis. We might consider that if the modulus r in the complex number $r(\cos\theta + i\sin\theta)$ is *positive*, we moved r units along the ray $(\overrightarrow{PQ})$, making an angle of θ radians with the polar axis. If r is *negative*, we might consider the ray $(\overrightarrow{PR})$, making an angle of $\theta + \pi$ radians with the polar axis, to be the "negative half" of the ray $(\overrightarrow{PQ})$ for the purposes of graphing.

We have included a sufficient amount of material to see that if we wish to convert a complex number $x + yi$ from rectangular form to polar form, we can do it by letting $r = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$.

Example 1

Convert $-3 - 3\sqrt{3}i$ to polar form.

Solution

$$r = \sqrt{(-3)^2 + (-3\sqrt{3})^2} = \sqrt{9 + 27} = \sqrt{36} = 6.$$

$-3 - 3\sqrt{3}i$ is in the third quadrant.

$\tan\theta = \dfrac{-3\sqrt{3}}{-3} = \sqrt{3}$, so $\theta = \dfrac{-2\pi}{3}$ and

$$-3 - 3\sqrt{3}i = 6\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right].$$

To convert the complex number $r(\cos\theta + i\sin\theta)$ to rectangular form, we let

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

This should be easy to see, since

$$r(\cos\theta + i\sin\theta) = (r\cos\theta) + (r\sin\theta)i.$$

Example 2

Convert $2\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$ to rectangular form.

Solution

$x = 2\cos\left(-\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}.$

$y = 2\sin\left(-\frac{\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}.$

$$2\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right] = \sqrt{2} - \sqrt{2}i.$$

Exercises 6.2

Convert the following complex numbers from rectangular form to polar form.

1. $-1 + i$
2. $1 - \sqrt{3}i$
3. $-4i$
4. $3 + \sqrt{3}i$
5. $6 - 2\sqrt{3}i$
6. $4 - 4i$
7. $-1 + \sqrt{3}i$
8. -9

Graph each of the following complex numbers. Then convert the complex number from polar form to rectangular form.

9. $5\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$
10. $3\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$
11. $4\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$
12. $10\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$
13. $8\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$
14. $6\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$
15. $-6\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$
16. $4[\cos(-\pi) + i\sin(-\pi)]$

6.3 De Moivre's theorem

The representation of complex numbers in polar form has some distinct advantages. One is that it is much easier to multiply or divide complex numbers in polar form than in rectangular form. If we let two complex numbers z_1 and z_2 be represented by

$$r_1(\cos\theta + i\sin\theta) \quad \text{and} \quad r_2(\cos\phi + i\sin\phi),$$

respectively, we have the following development for their product.

$$\begin{aligned}
z_1 \cdot z_2 &= [r_1(\cos\theta + i\sin\theta)]\cdot[r_2(\cos\phi + i\sin\phi)] \\
&= [r_1\cos\theta + (r_1\sin\theta)i]\cdot[r_2\cos\phi + (r_2\sin\phi)i] \\
&= (r_1 r_2\cos\theta\cos\phi - r_1 r_2\sin\theta\sin\phi) \\
&\quad + (r_1 r_2\cos\theta\sin\phi + r_1 r_2\sin\theta\cos\phi)i \\
&= r_1 r_2(\cos\theta\cos\phi - \sin\theta\sin\phi) \\
&\quad + r_1 r_2(\cos\theta\sin\phi + \sin\theta\cos\phi)i \\
&= r_1 r_2\cos(\theta+\phi) + r_1 r_2\sin(\theta+\phi)i \\
&= r_1 r_2[\cos(\theta+\phi) + i\sin(\theta+\phi)].
\end{aligned}$$

$$\therefore \quad \boldsymbol{z_1 \cdot z_2 = r_1 r_2[\cos(\theta+\phi) + i\sin(\theta+\phi)].}$$

Thus, we see that when we take the product of two complex numbers, the modulus of the product is the product of the moduli of the two factors, and the argument of the product is the sum of the arguments of the factors. This result can be generalized in the following way.

If $z = r(\cos\theta + i\sin\theta)$, and $n \in N$, then

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

This result, known as **De Moivre's theorem,** can be proved by mathematical induction, as follows:

(i) $z^1 = z = r(\cos\theta + i\sin\theta) = r^1(\cos 1\cdot\theta + i\sin 1\cdot\theta)$.

(ii) Assume that $z^k = r^k(\cos k\theta + i\sin k\theta)$ for some $k \in J$. Using this as the induction hypothesis, we wish to prove that

$$z^{k+1} = r^{k+1}[\cos(k+1)\theta + i\sin(k+1)\theta].$$

$z^{k+1} = z^k \cdot z$

$= [r^k(\cos k\theta + i\sin k\theta)]\cdot[r(\cos\theta + i\sin\theta)]$ (by the induction hypothesis)

$= r^k \cdot r[\cos(k\theta+\theta) + i\sin(k\theta+\theta)]$ (by the previous result about the product)

$= r^{k+1}[\cos(k+1)\theta + i\sin(k+1)\theta]$ (by properties of algebra).

$\therefore \quad z^{k+1} = r^{k+1}[\cos(k+1)\theta + i\sin(k+1)\theta]$.

$\therefore$ **If $z = r(\cos\theta + i\sin\theta)$ and $n \in N$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$.**

Examples using these results are given below.

Example 1

$$\left[3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]\cdot\left[\frac{2}{3}\left(\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\right)\right]$$

Solution

$$\left(3\cdot\frac{2}{3}\right)\left[\cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)\right] = 2\left[\cos\frac{7\pi}{12} = i\sin\frac{7\pi}{12}\right].$$

Example 2

Evaluate $(1 + i)^6$

Solution

$$\begin{aligned}(1+i)^6 &= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^6\\ &= (\sqrt{2})^6\left[\cos\left(6\cdot\frac{\pi}{4}\right) + i\sin\left(6\cdot\frac{\pi}{4}\right)\right]\\ &= 8\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)\\ &= 8(0 - i)\\ &= -8i.\end{aligned}$$

We have seen that finding products and powers is quite easy using polar coordinates. Finding quotients is also quite easy. We first express $1/z$ in polar coordinates. Using this result, we then find a way of expressing z_1/z_2. If

$$z = r(\cos\theta + i\sin\theta),$$

then

$$\begin{aligned}\frac{1}{z} &= \frac{1}{r(\cos\theta + i\sin\theta)}\\ &= \frac{1}{r(\cos\theta + i\sin\theta)}\cdot\frac{r(\cos\theta - i\sin\theta)}{r(\cos\theta - i\sin\theta)}\\ &= \frac{r(\cos\theta - i\sin\theta)}{r^2(\cos^2\theta - i^2\sin^2\theta)}\\ &= \frac{\cos\theta - i\sin\theta}{r(\cos^2\theta + \sin^2\theta)}\\ &= \frac{1}{r}(\cos\theta - i\sin\theta)\\ &= \frac{1}{r}[\cos(-\theta) + i\sin(-\theta)]\end{aligned}$$

[since $\cos(-\theta) \equiv \cos\theta$ and $\sin(-\theta) \equiv -\sin\theta$]. Using this result, if

$$z_1 = r_1(\cos\theta + i\sin\theta) \quad\text{and}\quad z_2 = r_2(\cos\phi + i\sin\phi),$$

then

$$\begin{aligned}\frac{z_1}{z_2} &= \left[r_1(\cos\theta + i\sin\theta)\right]\cdot\left[\frac{1}{r_2}(\cos(-\phi) + i\sin(-\phi))\right]\\ &= \frac{r_1}{r_2}[\cos(\theta - \phi) + i\sin(\theta - \phi)],\end{aligned}$$

using the rule for the product of two complex numbers in polar form. Thus, the modulus of the quotient is the quotient of the moduli, and the argument of the quotient is the difference of the arguments. Examples of this rule are shown below.

Example 3

$$\left[3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right] \div \left[6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]$$

Solution

$$\frac{3}{6}\left[\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right)\right] = \frac{1}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

Example 4

$$\frac{12\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}{\left[2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)\right]^2}$$

Solution

$$\frac{12\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}{\left[2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{3}\right)\right]^2} = \frac{12\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}{4(\cos 3\pi + i\sin 3\pi)}$$

$$= \frac{12}{4}\left[\cos\left(\frac{\pi}{4} - 3\pi\right) + i\sin\left(\frac{\pi}{4} - 3\pi\right)\right]$$

$$= 3\left[\cos\left(\frac{-11\pi}{4}\right) + i\sin\left(\frac{-11\pi}{4}\right)\right]$$

Exercises 6.3

Find the indicated products, powers, and quotients.

1. $\left[5\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right] \cdot \left[2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)\right]$
2. $\left[\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)\right] \cdot \left[13\left(\cos(-\pi) + i\sin(-\pi)\right)\right]$
3. $\left[\sqrt{6}\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)\right] \cdot \left[\sqrt{18}\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)\right]$
4. $\left[3\left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right)\right] \cdot \left[7\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]$
5. $\left[3\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)\right]^3$
6. $\left[\sqrt{5}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]^6$
7. $\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^8$
8. $\left[2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)\right]^{10}$

9. $\left[8\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right] \div \left[4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right]$
10. $\left[35\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)\right] \div \left[7\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]$
11. $\left[4\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)\right] \div \left[16\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]$
12. $\left[3\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)\right] \div \left[\sqrt{3}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)\right]$
13. $(1 - i)^8$
14. $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^6$
15. $(3 - \sqrt{3i})^4$
16. $(-2 - 2i)^8$
17. $(1 - i)^8 \div (3 - \sqrt{3}i)^4$
18. $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^6 \div (-2 - 2i)^8$

6.4 Roots of complex numbers

We indicated earlier that the extraction of roots is always possible within the system of complex numbers. This process is accomplished fairly easily by using an application of De Moivre's theorem.

Let $z = r(\cos\theta + i\sin\theta)$. We wish to find the complex number(s) w such that $w^n = z$. Assume that $w = \rho\,(\sin\phi + i\sin\phi)$. Then, by De Moivre's theorem,

$$w^n = \rho^n(\cos n\phi + i\sin n\phi).$$

In order to have $w^n = z$, we must have

$$\rho^n(\sin n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta).$$

This means that $\rho^n = r$ and $n\phi = \theta$ or $n\phi = \theta + 2k\pi$, where $k \in J$. If $\rho^n = r$ then $\rho = \sqrt[n]{r}$, and if $n\phi = \theta + 2k\pi$, $k \in J$, then

$$\phi = \frac{\theta + 2k\pi}{n}, k \in J.$$

If k takes the different values $0, 1, 2, \ldots, n - 1$, then

$$\phi_1 = \frac{\theta}{n}, \quad \phi_2 = \frac{\theta + 2\pi}{n} \quad \phi_3 = \frac{\theta + 4\pi}{n}, \ldots, \quad \phi_n = \frac{\theta + (n-1)2\pi}{n}$$

will be n *different* angles which are not coterminal. Since these angles are not coterminal,

$$w_1 = \sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right),$$

$$w_2 = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi}{n}\right) + i\sin\left(\frac{\theta + 2\pi}{n}\right)\right],$$

$$\cdots$$

$$w_n = \sqrt[n]{r}\left[\cos\frac{\theta + (n-1)2\pi}{n} + i\sin\frac{\theta + (n-1)2\pi}{n}\right]$$

are all different complex numbers, and z has n distinct nth roots.

Let us illustrate by finding the four fourth roots of -4.

$$-4 = 4(\cos\pi + i\sin\pi).$$

Let

$$w = \rho(\cos\phi + i\sin\phi).$$

If $w^4 = -4$, then

$$[\rho(\cos\phi + i\sin\phi)]^4 = 4(\cos\pi + i\sin\pi),$$
$$\rho^4(\cos 4\phi + i\sin 4\phi) = 4(\cos\pi + i\sin\pi).$$
$$\rho^4 = 4, \qquad 4\phi = \pi + 2k\pi, \quad k = 0, 1, 2, 3 \quad (\text{since } n = 4),$$
$$\rho^2 = 2, \qquad 4\phi = \pi \quad \text{or} \quad \pi + 2\pi \quad \text{or} \quad \pi + 4\pi \quad \text{or} \quad \pi + 6\pi,$$
$$\rho = \sqrt{2}, \qquad 4\phi = \pi \quad \text{or} \quad 3\pi \quad \text{or} \quad 5\pi \quad \text{or} \quad 7\pi,$$
$$\rho = \sqrt{2}, \qquad \phi_1 = \frac{\pi}{4}, \qquad \phi_2 = \frac{3\pi}{4}, \qquad \phi_3 = \frac{5\pi}{4}, \qquad \phi_4 = \frac{7\pi}{4}.$$

The four fourth roots of -4 are given by

$$w_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = 1 + i,$$
$$w_2 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \sqrt{2}\left(\frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = -1 + i,$$
$$w_3 = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \sqrt{2}\left(\frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -1 - i,$$
$$w_4 = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = 1 - i.$$

Anyone who is not convinced that $1 + i$, $-1 + i$, $-1 - i$, and $1 - i$ are all fourth roots of -4, can check this by using multiplication of complex numbers in rectangular form.

The n nth roots of 1 can be found in a similar fashion. These are often called the **n nth roots of unity**. Let us find the five fifth roots of unity. We can express 1 in the form $1(\cos 0 + i\sin 0)$. Thus, if

$$w = \rho(\cos\phi + i\sin\phi),$$

$\rho^5 = 1$ and $5\phi = 0,\ 2\pi,\ 4\pi,\ 6\pi$, or 8π. Thus, $\rho = 1$, $\phi_1 = 0$, $\phi_2 = 2\pi/5$, $\phi_3 = 4\pi/5$, $\phi_4 = 6\pi/5$, and $\phi_5 = 8\pi/5$. The five fifth roots of unity are given by

$$w_1 = 1(\cos 0 + i\sin 0) = 1,$$

$$w_2 = 1\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right),$$
$$w_3 = 1\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right),$$
$$w_4 = 1\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right),$$
$$w_5 = 1\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right).$$

These could be expressed without writing the modulus of 1. If no modulus appears, then it is understood that the modulus is 1.

Exercises 6.4

1. Find the 3 third roots of unity. Express them in rectangular form.
2. Find the 8 eighth roots of unity. Express them in rectangular form.
3. Find the 7 seventh roots of unity.
4. Find the 10 tenth roots of unity.
5. Find the 3 cube roots of $2 - 2i$.
6. Find the 5 fifth roots of $-4 - 4i$.
7. Find the 7 seventh roots of $5\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$.
8. Find the 6 sixth roots of $13\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.
9. Find the 6 sixth roots of $-64i$.
10. Find the 3 cube roots of $27i$. Express them in rectangular form.
11. Find the 3 cube roots of $-i$. Express them in rectangular form.
12. Find the 2 square roots of $-25i$. Express them in rectangular form.

Review Exercises

1. Convert $6\left(\cos\frac{13\pi}{6} + i\sin\frac{13\pi}{6}\right)$ to rectangular form.
2. Convert $12 - 12i$ to polar form.
3. Give the complex conjugate of

   a. $3 - 14i$, b. $6i$, c. -3, d. $6 + 2i$.
4. Find the 2 square roots of $4i$.
5. Find the 3 cube roots of -8.

6. Perform the indicated operations.

a. $(3 - 2i)(-5 + 7i)$
b. $(3 - 2i) - (-5 + 7i)$
c. $(25 + 50i) \div (3 - 4i)$
d. $(-17 + 16i) + (3 - 8i)$
e. $\left[3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^3$
f. $\left[2\left(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}\right)\right]^4$
g. $\left[3\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)\right] \cdot \left[\frac{1}{3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]$
h. $\left[3\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)\right] \div \left[\frac{1}{3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]$

7. Convert $-6 + 6\sqrt{3i}$ to polar form.

8. Convert $-4\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ to rectangular form.

9. Perform the indicated operations.

a. $\left[5\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right] \div \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]$
b. $\left[5\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right] \cdot \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]$
c. $(3 + 6i) + (-2 + 3i)$
d. $(3 + 6i) - (-2 + 3i)$
e. $(3 + 6i) \cdot (-2 + 3i)$
f. $(3 + 6i) \div (-2 + 3i)$
g. $(1 - i)^5$

9. $\alpha = 43°18'$, $\beta = 104°14'$, $b = 32.51$ yds or $\alpha = 136°42'$, $\beta = 10°40'$, $b = 6.207$ yds.
11. No solution
13. $\alpha = 130°22'$, $\gamma = 11°48'$, $a = 41.0$ in.
15. a. 210 ft b. 703 ft 17. 110 ft 19. 180 ft

Exercises 5.4

1. $\alpha = 65°40'$, $\beta = 40°20'$, $c = 40.09$ ft.
3. $\alpha = 31°45'$, $\beta = 51°15'$, $c = 105.6$ yds.
5. $\alpha = 62°42'$, $\gamma = 58°48'$, $b = 28.31$ in.
7. $\beta = 25°41'$, $\gamma = 22°29'$, $a = 134.8$ ft.
9. $\beta = 42°10'$, $\gamma = 42°10'$, $a = 63.30$ yds.
11. $\alpha = 50°51'$, $\gamma = 70°43'$, $\beta = 58°26'$.
13. $\alpha = 36°52'$, $\beta = 53°8'$, $\gamma = 90°$.
15. 240 ft 17. 37 ft 19. $32\sqrt{3}$ ft

Exercises 5.5

1. 193.9 sq ft 3. 4036 sq yds 5. 450.8 sq in.
7. 778.6 sq ft 9. 606.7 sq ft 11. 280.2 sq in.
13. 191,500 sq yds 15. 170.9 sq ft or 253.6 sq ft
17. No solution 19. 105.8 sq ft 21. $128\sqrt{3}$ sq ft

Review Exercises

1. $\beta = 70°$, $a = 218.7$ in., $c = 266.4$ in.
3. $\alpha = 40°32'$, $\beta = 49°28'$, $a = 2\sqrt{66}$ ft $= 16.25$ ft.
5. $\alpha = 64°59'$, $\gamma = 40°38'$, $a = 26.92$ ft.
7. a. $10°$ c. $-828°$ 9. 414.1 ft
11. a. $\frac{2\pi}{9}$ radians c. $-\frac{9\pi}{10}$ radians
13. $\alpha = 35°26'$, $\beta = 48°11'$, $\gamma = 96°23'$ 15. No solution
17. $b = 46.84$ yds, $\alpha = 26°58'$, $\gamma = 103°42'$.

Chapter 6

Exercises 6.1

1. a. $7, -1, 7 + i$ c. $2, 5, 2 - 5i$ e. $0, -4, 4i$
   g. $9, -2, 9 + 2i$ i. $\sqrt{3}, \sqrt{2}, \sqrt{3} - \sqrt{2}\,i$ k. $3, 0, 3$
3. $-2\pi - i$ 5. $6i$ 7. $4\pi + 7i$
9. $1 + 2i$ 11. $(3e^2 + 2b) + 37$ 13. $-6 - 17i$
15. $\frac{1}{13} + \frac{5}{13}i$ 17. $\frac{c}{c^2 + d^2} - \frac{d}{c^2 + d^2}$ 19. $\frac{88}{26} - \frac{41}{26}i$
21. $39 - 21i$ 23. $-\frac{79}{125} - \frac{122}{125}i$ 25. $-\frac{60}{289} - \frac{32}{289}i$

Exercises 6.2

1. $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$
3. $4\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$
5. $4\sqrt{3}\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$
7. $2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$
9. $-\frac{5}{\sqrt{2}} + \frac{5}{\sqrt{2}}i$
11. $2 - 2\sqrt{3}i$
13. $-4\sqrt{2} - 4\sqrt{2}i$
15. $-3\sqrt{3} + 3i$

Exercises 6.3

1. $10\left(\cos\frac{13\pi}{6} + i\sin\frac{13\pi}{6}\right)$
3. $6\sqrt{3}\left(\cos\frac{14\pi}{45} + i\sin\frac{14\pi}{45}\right)$
5. $27\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right)$
7. 1
9. $2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$
11. $\frac{1}{4}\left[\cos\left(-\frac{2\pi}{15}\right) + i\sin\left(-\frac{2\pi}{15}\right)\right]$
13. 16
15. $144\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$
17. $\frac{1}{9}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

Exercises 6.4

1. $1,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i$
3. $1,\ \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7},\ \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7},\ \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7},$
$\cos\frac{8\pi}{7} + i\sin\frac{8\pi}{7},\ \cos\frac{10\pi}{7} + i\sin\frac{10\pi}{7},\ \cos\frac{12\pi}{7} + i\sin\frac{12\pi}{7}$
5. $\sqrt{2}\left[\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right],\ \sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right),$
$\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$
7. $\sqrt[7]{5}\left(\cos\frac{\pi}{35} + i\sin\frac{\pi}{35}\right),\ \sqrt[7]{5}\left(\cos\frac{11\pi}{35} + i\sin\frac{11\pi}{35}\right),$
$\sqrt[7]{5}\left(\cos\frac{21\pi}{35} + i\sin\frac{21\pi}{35}\right),\ \sqrt[7]{5}\left(\cos\frac{31\pi}{35} + i\sin\frac{31\pi}{35}\right),$
$\sqrt[7]{5}\left(\cos\frac{41\pi}{35} + i\sin\frac{41\pi}{35}\right),\ \sqrt[7]{5}\left(\cos\frac{51\pi}{35} + i\sin\frac{51\pi}{35}\right),$
$\sqrt[7]{5}\left(\cos\frac{61\pi}{35} + i\sin\frac{61\pi}{35}\right)$
9. $\sqrt{2}\left[\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right],\ \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),$
$\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right),\ \sqrt{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right),$
$\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right),\ \sqrt{2}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$
11. $\frac{\sqrt{3}}{2} - \frac{i}{2},\ i,\ -\frac{\sqrt{3}}{2} - \frac{i}{2}$

Review Exercises

1. $3\sqrt{3} + 3i$
3. a. $3 + 14i$ c. -3
5. $1 + \sqrt{3}\,i,\ -2,\ 1 - \sqrt{3}\,i$
6. a. $-1 + 31i$ c. $-5 + 10i$ e. $27\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$
   g. $\cos \frac{8\pi}{15} + i \sin \frac{8\pi}{15}$
7. $12\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$
9. a. $\frac{5}{2}\left(\cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12}\right)$ c. $1 + 9i$
   e. $-24-3i$ g. $-4 + 4i$

Appendix

Exercises A.1

1. 3
3. $\frac{1}{2}$
5. 0
7. $\log 291 + \log 243$
9. $\log 291 - \log 243$
11. $\frac{1}{2} \log 31 + 3 \log 29$
13. $\frac{1}{3} \log 729 + 4 \log 281 - \log 927$
15. $4[2 \log 314 + \log 475 - \log 627]$
17. $1.967 \cdot 10^3$
19. $8.42 \cdot 10^{-2}$
21. $9.189 \cdot 10^{-5}$

Exercises A.2

1. 2.4639
3. 9.7143 − 10
5. 1.4053
7. 8.6894 − 10
9. 6.0208
11. 5.7912 − 10
13. 4.96
15. 820.8
17. 0.6512
19. 1953
21. 0.001328

Exercises A.3

1. $1.466 \cdot 10^7$
3. 4291
5. 29.94
7. $3.760 \cdot 10^{10}$
9. 1.239

Exercises A.4

1. 9.8821 − 10
3. 9.9004 − 10
5. 10.2307 − 10
7. 10.0547 − 10
9. 9.9838 − 10
11. 9.5813 − 10
13. 26°50′
15. 48°20′
17. 4°43′
19. 12°52′
21. 23°56′
23. 37°4′