

1.3.1. Approximate values of irrational roots

We derived the equation

$$(1.4) \quad \sqrt{N} \approx \frac{1}{2} \left(a + \frac{N}{a} \right)$$

Example 1.12

Find an approximation of $\sqrt{5}$.

Solution

Choose a positive integer a whose square is close to 5. The obvious candidates for a are 2 and 3. $2^2 = 4$ and $3^2 = 9$. Since 4 is closer to 5 than is 9, select $a = 2$. Then,

$$\begin{aligned} \sqrt{N} &\approx \frac{1}{2} \left(2 + \frac{5}{2} \right) \\ &= \frac{1}{2} \left(\frac{9}{2} \right) \\ &= \left(\frac{9}{4} \right) \\ &= 2.25. \quad \blacksquare \end{aligned}$$

How good is our approximation? A computer produced value for $\sqrt{5}$ is 2.236. Our approximation differs from this by only 0.014. The relative error is 0.6%. Not bad for such little effort.

No law says we must stop with the approximation to $\sqrt{5}$ obtained in Example (1.12). We could continue the approximation of $\sqrt{5}$ by using the value 2.25 that we *just obtained* as a in Equation (1.4).

$$\begin{aligned} \sqrt{N} &\approx \frac{1}{2} \left(2 + \frac{5}{2.25} \right) \\ &= 2.23611. \end{aligned}$$

The approximation 2.23611 differs from the computer produced value of 2.23607 by only 0.00004. If an even better approximation were desired, one would simply use the value just obtained in a in Equation (1.4) to compute a better approximation. Of course, the arithmetic becomes messy.

When we substitute into Equation (1.4) the obtained value to produce a better approximation and then substitute *that* value to obtain the next approximation etcetera, we use Equation (1.4) recursively and each recursion is called an iteration.

The approximation equation Equation (1.4) requires few iterations to produce a result that is very close to the actual value of, in our example, 5.

An approximation formula that yields results close to the target value with very few iterations is said to *converge rapidly*.

Example 1.13

Find an approximation of $\sqrt{7}$.

Solution

Choose a positive integer a whose square is close to 7. The obvious candidates for a are 2 and 3. $2^2 = 4$ and $3^2 = 9$. Since 9 is closer to 7 than is 4, select $a = 3$. Then,

$$\begin{aligned}\sqrt{N} &\approx \frac{1}{2} \left(3 + \frac{7}{3} \right) \\ &= \frac{1}{2} \left(\frac{16}{3} \right) \\ &= \left(\frac{8}{3} \right) \\ &= 2.667. \quad \blacksquare\end{aligned}$$

The result, 2.2667, differs from the computer produced value of 2.64575 by about 0.02.