

Course Companion Math-11
Advanced Algebra and Trigonometry
-DRAFT-

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ABSTRACT. This is intended to be used with the textbook “Basic Analysis; Japanese Grade 11”. Edited by Kunihiko Kodaira. American Mathematical Society, 1991.

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Chapter 1

Exponential and Logarithmic Functions

1.1. The idea of a function

Our intuitive idea of a function is that it is a rule that pairs each member of a set called the domain of the function with exactly one member of a set called the range of the function. Graphically, this means that a vertical line drawn through any value on the horizontal axis that lies within the domain of the function will cut the graph of the function exactly once. Figure (1.1) shows the vertical line test applied to the function $f : y = x^2$, $\mathcal{D} = \mathbb{R}$ and function $y^2 = x$, $x \geq 0$.

The symbol \mathcal{D} means the function's

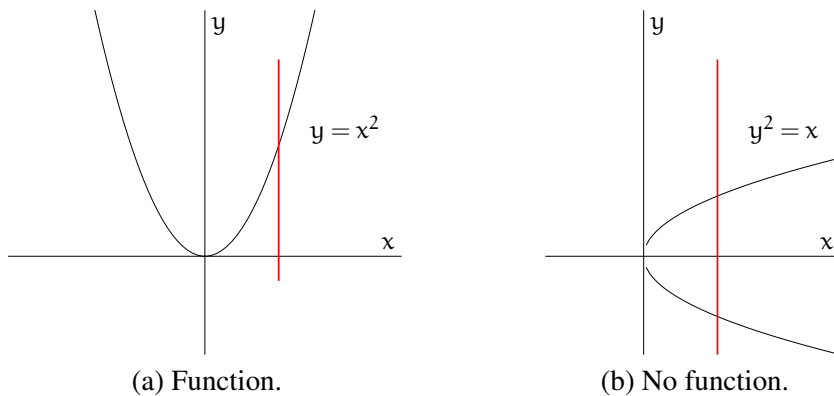


FIGURE 1.1. Vertical line test for function

Our intuitive idea is descriptive and easily understood. The vertical line test is simple and suggestive. Neither will establish beyond doubt that a given equation is a function. We need to make our intuition precise. The following definition answers our need.

Definition 1.1 (Function)

Let F be a relation rule that pairs all elements x of one set A with elements y of another set B . Let x_1 be paired with y_1 and x_2 with y_2 . If $x_1 = x_2$ implies $y_1 = y_2$, then y is a function of x .

Example 1.1

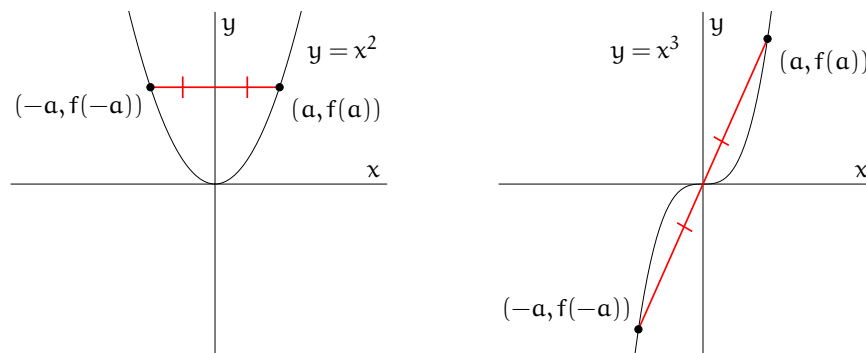
Show that for $y = x^3$, $x \in \mathbb{R}$, y is a function of x .

Solution

Let x_1 be paired with y_1 and x_2 with y_2 . That is, $y_1 = x_1^3$ and $y_2 = x_2^3$. Suppose $x_1 = x_2$. Then $y_2 = x_1^3$. But, $x_1^3 = y_1$. So that $y_1 = y_2$. Since $x_1 = x_2 \implies y_1 = y_2$, $y = x^3$ gives y as a function of x .

1.1.1. Symmetry: odd and even functions

The function $f : y = x^2$, $\mathcal{D} = \mathbb{R}$ is symmetric with respect to the y -axis. The function $g : y = x^3$, $\mathcal{D} = \mathbb{R}$ is symmetric with respect to the origin. This is illustrated in Figure (1.2).



(a) Symmetry with respect to y -axis. (b) Symmetry with respect to origin.

FIGURE 1.2. Vertical line test for function

Geometry should convince the reader that the labeled points in Figure (1.2) (a) and (b) have the symmetries and coordinates as shown. Notice that in Figure (1.2) (a), $f(-a) = f(a)$; in (b), $f(-a) = -f(a)$. The two symmetries issue from these facts. The function $f(x) = x^2$ is an example of

what is called an “even function”. The function $f(x) = x^3$ is an example of an “odd function”.

Definition 1.2 (Even function, Odd function)

A function $f(x)$ is an **even function** of x if $f(-x) = f(x)$, and an **odd function** of x if $f(-x) = -f(x)$, for every x in the function’s domain.

Theorem 1.1

All even functions are symmetric with respect to the vertical axis. All odd functions are symmetric with respect to the origin.

Example 1.2

The function $f(x) = x^5$, $\mathcal{D} = \mathbb{R}$, is symmetric to the origin. To see this, let a be any real number and note that $f(-a) = (-a)^5 = -a^5 = -(a^5) = -f(a)$. This means f is an odd function. Then Theorem (1.1) guarantees symmetry with respect to the origin.

Example 1.3

Show that the function $f(x) = x^5 + 2x^4$, $\mathcal{D} = \mathbb{R}$, is neither an even nor an odd function.

Solution

Consider $f(1)$ and $f(-1)$. $f(1) = 3$. $f(-1) = 1$. Since $3 \neq 1$, f is not an even function. Since $3 \neq -1$, f is not an odd function.

Exercise 1.1 ---

1. Prove that $y = ax + b$, $x \in \mathbb{R}$ and a, b constants gives y as a function of x .
2. Prove that $y = ax^2$, $x \in \mathbb{R}$ and a constant gives y as a function of x .
3. Show that $y = ax^2 + bx + c$, $x \in \mathbb{R}$ and a, b, c constants gives y as a function of x .
4. Try to prove that $y^2 = x$, $y \in \mathbb{R}$ gives y as a function of x and note at which step (and why) the proof cannot be continued.
5. Prove (by giving a counterexample) that the rule $y^2 = x$, $y \in \mathbb{R}$ does not give y as a function of x .

6. Show that $g(x) = x^4$ is symmetric with respect to the vertical axis.
 7. Show that $g(x) = x^7$ is symmetric with respect to the origin.
 8. Show that $f(x) = |x|$ is symmetric with respect to the vertical axis.
 9. Show that $f(x) = x^2 + 3x - 2$ is not an even function.
-

1.1.2. Increasing and decreasing functions

The function $f : y = x^2$ is defined for all real numbers x . But f behaves differently on the interval $(-\infty, 0]$ than it does on the interval $[0, \infty)$. The graph of f in Figure (1.3) shows that to the left of 0, the values of f decrease as the values of x increase. To the right of 0, $f(x)$ increases as x increases.

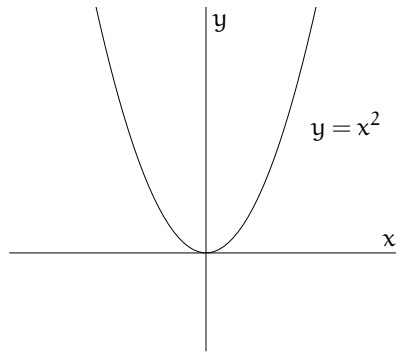


FIGURE 1.3. f is a decreasing function of x on $(-\infty, 0]$, but increasing on $[0, \infty)$.

Definition (1.3) makes precise our ideas of increasing and decreasing.

Definition 1.3 (Increasing (decreasing) on an interval)

Let $y = f(x)$ be defined on an interval I and let a and b be numbers in I such that $b > a$. If $f(b) > f(a)$, f is an **increasing** function of x on I . If $f(b) < f(a)$, f is a **decreasing** function of x on I .

Example 1.4

Show that $y = x^2$ is an increasing function of x on $[0, \infty)$.

Solution

Let a and b be any two numbers in $[0, \infty)$. Suppose that $b > a$. We will show that the difference $f(b) - f(a)$ is positive and that amounts to showing

that $f(b) > f(a)$.

$$\begin{aligned} f(b) - f(a) &= b^2 - a^2 \\ f(b) - f(a) &= (b - a)(b + a) \end{aligned}$$

$(b + a)$ is positive and so is $(b - a)$, so

$$> 0.$$

Since $b > a \implies f(b) > f(a)$ on $[0, \infty)$, f is an increasing function of x on $[0, \infty)$.

Example 1.5

Show that $y = x^2$ is a decreasing function of x on $(-\infty, 0]$.

Solution

Let a and b be any two numbers in $(-\infty, 0]$. Suppose that $b > a$. We will show that the difference $f(b) - f(a)$ is negative and that amounts to showing that $f(b) < f(a)$.

$$\begin{aligned} f(b) - f(a) &= b^2 - a^2 \\ &= (b - a)(b + a) \end{aligned}$$

$(b + a)$ is negative but $(b - a)$ is positive, so

$$f(b) - f(a) < 0.$$

Since $b > a \implies f(b) < f(a)$ on $(-\infty, 0]$, f is a decreasing function of x on $(-\infty, 0]$.

Example 1.6

Show that $y = x^3$ is an increasing function of x on $(-\infty, \infty)$.

Solution

Let a and b be any two numbers in $(-\infty, \infty)$. Suppose that $b > a$. We will show that the difference $f(b) - f(a)$ is positive and that amounts to showing that $f(b) > f(a)$.

$$\begin{aligned} f(b) - f(a) &= b^3 - a^3 \\ (1.1) \qquad &= (b - a)(a^2 + ab + b^2) \end{aligned}$$

$(b - a)$ is positive on all of $(-\infty, \infty)$, but the sign of $(a^2 + ab + b^2)$ is not so obvious. If a and b are in $(-\infty, 0]$ then ab is positive. If a and b are in $[0, \infty)$ then ab is again positive. So $(a^2 + ab + b^2)$ is positive on $(-\infty, 0] \cup [0, \infty) = (-\infty, \infty)$. Thus,

$$(1.2) \quad f(b) - f(a) > 0.$$

Since $b > a \implies f(b) > f(a)$ on $(-\infty, \infty)$, f is an increasing function of x on $(-\infty, \infty)$. ■

Although the proof in Example (1.6) is correct, it might be instructive to handle the case of $a < 0 < b$ explicitly. Our worry is the factor $(a^2 + ab + b^2)$ in Equation (1.1). If $a < 0 < b$, then ab is negative. The question is whether ab is negative enough to make the factor $(a^2 + ab + b^2)$ negative. If $|ab| > a^2 + b^2$, then $(a^2 + ab + b^2)$ is negative. Now suppose that $|ab| > a^2 + b^2$.

$$(1.3) \quad \begin{aligned} |ab| &> a^2 + b^2 \\ |ab|^2 &> (a^2 + b^2)^2 \\ \left(\sqrt{(ab)^2}\right)^2 &> (a^2 + b^2)^2 \\ (ab)^2 &> (a^2 + b^2)^2 \\ a^2b^2 &> a^4 + b^4 + 2a^2b^2 \\ 0 &> a^4 + b^4 + a^2b^2 \end{aligned}$$

But this is a contradiction, so $a^2 + b^2 \geq |ab|$ and since $a \neq b$, we conclude $a^2 + b^2 > |ab|$. And that is all we needed to show.

Remark 1.1

You should be able to justify each statement of the proof and in particular Equation (1.3).

Exercise 1.2

1. Let $f(x) = x^2 - 6x + 5$. Determine where f it is increasing and where it is decreasing. Provide proof.

2. Show that the function $y = mx + c$ is either increasing or decreasing depending on whether m is positive or negative.
 3. Show that the function $y = |x|$ is increasing on $[0, \infty)$ but decreasing on $(-\infty, 0]$.
 4. Show that adding a constant to an even function results in an even function.
-

1.2. Power function

The familiar functions $f(x) = x^2$ and $f(x) = x^3$ are examples of power functions $f(x) = x^n$ where $n \in \mathbb{Z}^+$ is a constant.

1.2.1. Power function for $x \in [-1, 1]$

When $x \in [-1, 1]$, the magnitude of the values of the power function, $|x^n|$, increase more slowly for greater values of n . Figure (1.4) shows the graphs of x^2 and x^4 .

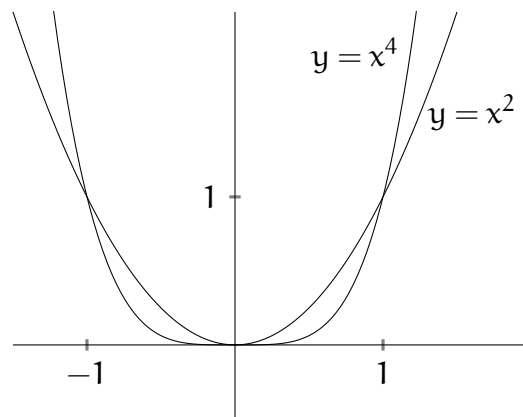


FIGURE 1.4. Comparison of x^2 and x^4 for $x \in [-1, 1]$.

1.3. Roots

You know that since $2 \cdot 2 = 4$, the number 2 is the second root (commonly called a “square root”) of 4. You also know that since $2 \cdot 2 \cdot 2 = 8$, the number 2 is the third root (commonly called a “cube root”) of 8. This idea may be generalized,

$\sqrt[n]{4}$ is usually written $\sqrt[n]{4}$.

$$\begin{aligned}
 2 \cdot 2 &= 4 \iff \sqrt{4} = 2 \\
 2 \cdot 2 \cdot 2 &= 8 \iff \sqrt[3]{8} = 2 \\
 2 \cdot 2 \cdot 2 \cdot 2 &= 16 \iff \sqrt[4]{16} = 2 \\
 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 &= 32 \iff \sqrt[5]{32} = 2 \\
 &\vdots \\
 \underbrace{2 \cdot 2 \cdot 2 \cdot \dots \cdot 2}_{n \text{ factors of } 2} &= b \iff \sqrt[n]{b} = 2.
 \end{aligned}$$

Using exponents,

$$\begin{aligned}
 2^2 &= 4 \iff \sqrt{4} = 2 \\
 2^3 &= 8 \iff \sqrt[3]{8} = 2 \\
 2^4 &= 16 \iff \sqrt[4]{16} = 2 \\
 2^5 &= 32 \iff \sqrt[5]{32} = 2 \\
 &\vdots \\
 2^n &= b \iff \sqrt[n]{b} = 2.
 \end{aligned}$$

Remark 1.2

The equivalence $2^n = b \iff \sqrt[n]{b} = 2$ means that the expressions $2^n = b$ and $\sqrt[n]{b} = 2$ express the same idea. Each relates the number of factors of 2 and the number that equals the product of those factors. ■

Remark 1.3

It is a fact that 4 has two roots, 2 and -2 . But the symbol $\sqrt{4}$ names the root 2 not -2 . If you wish to refer to the negative root of 4, write $-\sqrt{4}$. This is general. The symbol $\sqrt[n]{a}$ when n is an even number denotes the positive root of a . ■

The graph in Figure (1.5) is typical for $y = x^n$. When n is an even number the graph is symmetric with respect to the y -axis. This means that a given value of y will correspond to two values of x that are themselves symmetric to the y -axis.

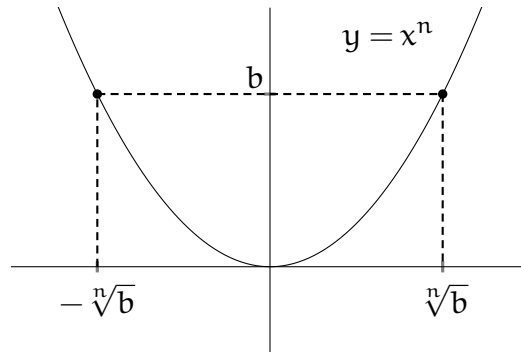


FIGURE 1.5. An even function has two symmetrically placed roots for a given value of y

Example 1.7

Find the 4th roots of 81.

Solution

Since 4 is an even number, the 4th roots of 81 are $\sqrt[4]{81} = 3$ and $-\sqrt[4]{81} = -3$. It is a mistake to write $\sqrt[4]{81} = \pm 3$.

Of course we wish to generalize to bases other than 2. We can do so, but with care. Note that $(-2)(-2) = 4$ but $\sqrt{4} = 2$ not -2 . On the other hand, $(-2)(-2)(-2) = -8$ and $\sqrt[3]{-8} = -2$. The generalization to bases other than 2 requires cases.

Theorem 1.2

Let a and b represent a real numbers.

$$a^n = b \iff \begin{cases} \sqrt[n]{b} = a & \text{if } a \geq 0, b \geq 0 \text{ and } n \text{ is an even integer,} \\ \sqrt[n]{b} = a & \text{if } n \text{ is an odd integer.} \end{cases}$$

Example 1.8

Find $\sqrt[3]{125}$.

Solution

Since 3 is an odd integer, $\sqrt[3]{125} = b \iff b^3 = 125$. So, $b = 5$, because $5^3 = 125$.

Example 1.9

Find $\sqrt[3]{-27}$.

Solution

Since 3 is an odd integer, $\sqrt[3]{-27} = b \iff b^3 = -27$. So, $b = -3$, because $(-3)^3 = -27$.

Example 1.10

Find $\sqrt[4]{81}$.

Solution

Since 4 is an even integer and $81 \geq 0$, $\sqrt[4]{81} = b \iff b^4 = 81$. So, $b = 3$, because $3^4 = 81$.

Example 1.11

Find $\sqrt[3]{16632}$.

Solution

The prime factorization of 16632 is $2^3 \cdot 3^3 \cdot 7 \cdot 11$. Thus, $\sqrt[3]{16632} = \sqrt[3]{2^3 \cdot 3^3 \cdot 7 \cdot 11}$. Since $\sqrt[3]{2^3} = 2$ and $\sqrt[3]{3^3} = 3$, the result is $6\sqrt[3]{77}$.

Example 1.12

Find $\sqrt[4]{-81}$.

Solution

Theorem (1.2) has nothing to say about this, because the conditions of the theorem are not met: $b = -81 \not\geq 0$.

If you think $\sqrt[4]{-81}$ is not a real number, you are correct. We will be able to prove this later.

1.3.1. Approximate values of irrational roots

We derived the equation

$$(1.4) \quad \sqrt{N} \approx \frac{1}{2} \left(a + \frac{N}{a} \right)$$

Example 1.13

Find an approximation of $\sqrt{5}$.

Solution

Choose a positive integer a whose square is close to 5. The obvious candidates for a are 2 and 3. $2^2 = 4$ and $3^2 = 9$. Since 4 is closer to 5 than is 9, select $a = 2$. Then,

$$\begin{aligned} \sqrt{N} &\approx \frac{1}{2} \left(2 + \frac{5}{2} \right) \\ &= \frac{1}{2} \left(\frac{9}{2} \right) \\ &= \left(\frac{9}{4} \right) \\ &= 2.25. \quad \blacksquare \end{aligned}$$

How good is our approximation? A computer produced value for $\sqrt{5}$ is 2.236. Our approximation differs from this by only 0.014. The relative error is 0.6%. Not bad for such little effort.

No law says we must stop with the approximation to $\sqrt{5}$ obtained in Example (1.13). We could continue the approximation of $\sqrt{5}$ by using the value 2.25 that we *just obtained* as a in Equation (1.4).

$$\begin{aligned} \sqrt{N} &\approx \frac{1}{2} \left(2.25 + \frac{5}{2.25} \right) \\ &= 2.23611. \end{aligned}$$

The approximation 2.23611 differs from the computer produced value of 2.23607 by only 0.00004. If an even better approximation were desired, one would simply use the value just obtained in a in Equation (1.4) to compute a better approximation. Of course, the arithmetic becomes messy.

When we substitute into Equation (1.4) the obtained value to produce a better approximation and then substitute *that* value to obtain the next approximation etcetera, we use Equation (1.4) recursively and each recursion is called an iteration.

The approximation equation Equation (1.4) requires few iterations to produce a result that is very close to the actual value of, in our example, 5. An approximation formula that yields results close to the target value with very few iterations is said to *converge rapidly*.

Example 1.14

Find an approximation of $\sqrt{7}$.

Solution

Choose a positive integer a whose square is close to 7. The obvious candidates for a are 2 and 3. $2^2 = 4$ and $3^2 = 9$. Since 9 is closer to 7 than is 4, select $a = 3$. Then,

$$\begin{aligned}\sqrt{N} &\approx \frac{1}{2} \left(3 + \frac{7}{3} \right) \\ &= \frac{1}{2} \left(\frac{16}{3} \right) \\ &= \left(\frac{8}{3} \right) \\ &= 2.667.\end{aligned}$$

The result, 2.2667, differs from the computer produced value of 2.64575 by about 0.02.

1.3.2. Rules of n th roots

Theorem 1.3 (Product rule)

If a and b are real numbers, not both negative, and n is a natural number, then

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.$$

Example 1.15

Find the product.

$$(1) \sqrt[3]{3} \cdot \sqrt[3]{12} = \sqrt[3]{36}$$

$$(2) \sqrt[4]{3} \cdot \sqrt[4]{5} = \sqrt[4]{15}$$

$$(3) \sqrt[3]{3} \cdot \sqrt[4]{5} = \sqrt[6]{15} \text{ Product rule does not apply, because roots differ. One is a 4th root, the other is a 6th root.}$$

It is often advantageous to simplify radicals before computing the product. For example, $\sqrt[4]{81} \cdot \sqrt[4]{16} = \sqrt[4]{1296}$. Simplifying before multiplying makes

the work easy

$$\sqrt[4]{81} \cdot \sqrt[4]{16} = 3 \cdot 2 = 6.$$

Chapter 2

Trigonometry

2.1. Radian measure

The degree is one of several arbitrary measures of angle. It is convenient for applied work. On the other hand, there is a natural measure of angle called the radian that is more convenient than the degree for mathematics and theoretical work.

Definition 2.1 (Radian)

Given a circle of radius r , the central angle of **one radian** is subtended by an arc of length r .

Several useful facts follow almost immediately.

Let s be an arc of a circle of radius r , and let θ the central angle subtended by arc s .

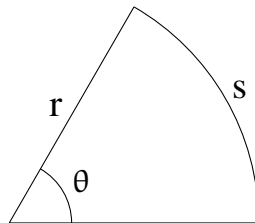


FIGURE 2.1. Sector of circle radius r , arc length s , central angle θ .

Then

$$(2.1) \quad \theta = \frac{s}{r} \quad \text{or equivalently} \quad s = r\theta.$$

2.1.1. Area of sector of circle

Let A_s be the area of the sector of figure (2.1). Since the area of a sector of a circle should compare to the circle's area as the length of the arc compares to the circumference of the circle,

$$\frac{A_s}{\pi r^2} = \frac{s}{2\pi r}$$

which implies that

$$(2.2) \quad A_s = \frac{1}{2}rs$$

and by substitution

$$(2.3) \quad A_s = \frac{1}{2}r^2\theta.$$

Equations (2.2) and (2.3) are often useful.

2.1.2. Unit Circle

Definition 2.2 (Unit circle)

The circle of radius 1 unit is called the **unit circle**.

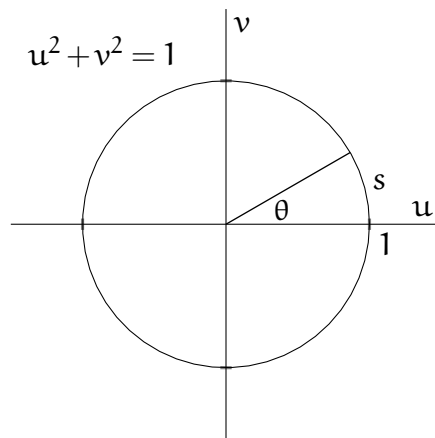


FIGURE 2.2. The unit circle.

In Figure (2.2), $\theta = s$, because $r = 1$.

2.2. Real numbers around a circle

Why anyone would wish to associate each real number with a point on a circle will become clear a few pages from now. Until then, perhaps you take my word that something good will come of it.

Everyone knows that one complete trip around a circle radius r is a distance of $2\pi r$. If the radius of the circle is one, then the complete circuit is distance 2π . The circle of radius one is called the “unit circle.”

2.3. Pre-image

Example 2.1

Find $x \ni \sin x = \frac{1}{2}$.

Solution.

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\sin\left(\pi - \frac{\pi}{6}\right) = \frac{1}{2}$$

$$\sin\left(\frac{\pi}{6} + 2n\pi\right) = \frac{1}{2}$$

$$\sin\left(\frac{5\pi}{6} + 2n\pi\right) = \frac{1}{2}$$

$$\therefore x = \frac{\pi}{6} + 2n\pi, x = \frac{5\pi}{6} + 2n\pi, \quad n \in \mathbb{Z}$$

Example 2.2

Find $x \ni \sin x = \frac{-1}{2}$.

Solution.

Remember that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

$$\sin\left(\pi + \frac{\pi}{6}\right) = \frac{-1}{2}$$

$$\sin\left(2\pi - \frac{\pi}{6}\right) = \frac{-1}{2}$$

$$\sin\left(\frac{7\pi}{6} + 2n\pi\right) = \frac{-1}{2}$$

$$\sin\left(\frac{11\pi}{6} + 2n\pi\right) = \frac{-1}{2}$$

$$\therefore x = \frac{7\pi}{6} + 2n\pi, x = \frac{11\pi}{6} + 2n\pi, \quad n \in \mathbb{Z}$$

Example 2.3

Find $x \ni \cos x = \frac{-\sqrt{2}}{2}$.

Solution.

Remember that $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

$$\cos\left(\pi - \frac{\pi}{4}\right) = \frac{-\sqrt{2}}{2}$$

$$\cos\left(\pi + \frac{\pi}{4}\right) = \frac{-\sqrt{2}}{2}$$

$$\cos\left(\frac{3\pi}{4} + 2n\pi\right) = \frac{-\sqrt{2}}{2}$$

$$\cos\left(\frac{5\pi}{4} + 2n\pi\right) = \frac{-\sqrt{2}}{2}$$

$$\therefore x = \frac{3\pi}{4} + 2n\pi, x = \frac{5\pi}{4} + 2n\pi, \quad n \in \mathbb{Z}$$

2.4. Trigonometric Equations

2.4.1. Preliminary observations

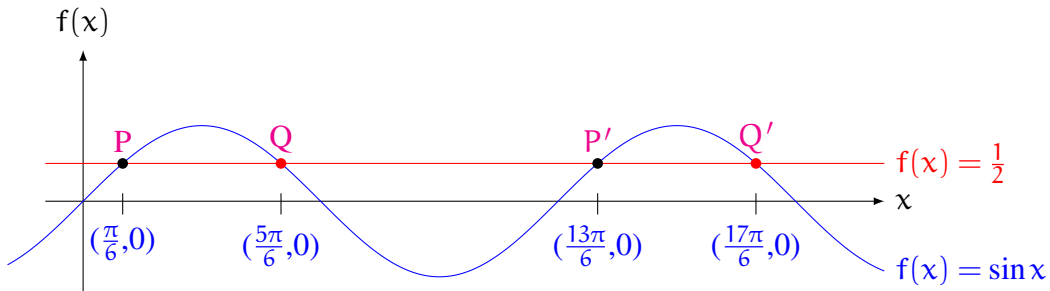


FIGURE 2.3. The sine function is not a 1-1 function on \mathbb{R} .

No trigonometric function is 1-1 on \mathbb{R} . Even the simple equation $\sin(x) = \frac{1}{2}$ has multiple solutions. In fact, there are two sequences of solutions. One sequence is $\{\frac{\pi}{6} + 2n\pi, n \in \mathbb{Z}\}$. The other is $\{\frac{5\pi}{6} + 2n\pi, n \in \mathbb{Z}\}$. It is no surprise that adjacent points in each sequence are 2π apart, since the period of $f(x) = \sin(x)$ is 2π .

Example 2.4

Solve $\sin(x) = \frac{1}{2}$, $x \in \mathbb{R}$.

Solution.

One sequence of solutions is based on $\sin(x + 2n\pi) = \frac{1}{2}$.

$$\begin{aligned} \sin(x + 2n\pi) = \frac{1}{2} &\implies x + 2n\pi = \frac{\pi}{6} \\ (2.4) \qquad \qquad \qquad &\implies x = \frac{\pi}{6} + 2n\pi. \end{aligned}$$

And the other sequence of solutions is based on $\sin(\pi - x + 2n\pi) = \frac{1}{2}$.

$$\begin{aligned} \sin(\pi - x + 2n\pi) = \frac{1}{2} &\implies \pi - x + 2n\pi = \frac{\pi}{6} \\ (2.5) \qquad \qquad \qquad &\implies -x = \frac{\pi}{6} - \pi + 2n\pi \\ &\implies x = \frac{5\pi}{6} + 2n\pi. \end{aligned}$$

Therefore,

$$x = \frac{\pi}{6} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{6} + 2n\pi, \quad n \in \mathbb{Z}. \quad \blacksquare$$

Are equations (2.4) and (2.5) correct? Should they have included “ $-2n\pi$ ” instead of “ $+2n\pi$ ”?

Let's agree to call the solutions that lie within one period of the function to the right of 0 the **principal solutions** of the equation. The principal solutions of $\sin(x) = \frac{1}{2}$ are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. Do note that the period is not always 2π . For example, the period of $f(x) = \sin(3x)$ is $\frac{2\pi}{3}$.

2.4.2. Solving equations

Several strategies for solving trigonometric equations are shown in the following examples. With practice, you will gain skill at solving equations that involve trigonometric functions. There are, however, equations that require mathematics beyond what you are learning here; for example, the equation $x \sin x = -1$.

Example 2.5

Solve $2 \sin(x) + \sqrt{3} = 0$, $x \in \mathbb{R}$.

Solution.

First, rewrite $2 \sin(x) + \sqrt{3} = 0$,

$$2 \sin(x) + \sqrt{3} = 0 \iff \sin(x) = \frac{-\sqrt{3}}{2}$$

$$\iff -\sin(x) = \frac{\sqrt{3}}{2}.$$

Second, remember that $-\sin(\theta) = \sin(-\theta) = \sin(\pi + \theta)$.

One sequence of solutions is based on $\sin(-x + 2n\pi) = \frac{\sqrt{3}}{2}$.

$$\sin(-x + 2n\pi) = \frac{\sqrt{3}}{2} \implies -x + 2n\pi = \frac{\pi}{3}$$

$$\iff x = -\frac{\pi}{3} + 2n\pi$$

$$\iff x = \frac{5\pi}{3} + 2n\pi.$$

And the other sequence of solutions is based on $\sin(\pi + x + 2n\pi) = \frac{\sqrt{3}}{2}$.

$$\begin{aligned}\sin(\pi + x + 2n\pi) = \frac{\sqrt{3}}{2} &\implies \pi + x + 2n\pi = \frac{\pi}{3} \\ &\iff x = \frac{\pi}{3} - \pi + 2n\pi \\ &\iff x = \frac{-2\pi}{3} + 2n\pi \\ &\iff x = \frac{4\pi}{3} + 2n\pi.\end{aligned}$$

Therefore,

$$x = \frac{4\pi}{3} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{3} + 2n\pi, \quad n \in \mathbb{Z}. \quad \blacksquare$$

It may be illuminating to view graphs of the functions considered in this handout. Since the purpose would be to see the graphs, rather than learn how to produce them (you already learned that), you might as well use a machine to produce the graphs.

Example 2.6

Solve $\sin^2(x) + 7\sin(x) + 6 = 0$, $x \in \mathbb{R}$.

Solution.

$$\begin{aligned}\sin^2(x) + 7\sin(x) + 6 = 0 &\iff (\sin x + 1)(\sin x + 6) = 0 \\ &\iff \sin x + 1 = 0 \quad \text{or} \quad \sin x + 6 = 0 \\ &\iff \sin x = -1 \quad \text{or} \quad \sin x = -6 \\ &\implies x + 2n\pi = \frac{3\pi}{2} \quad \text{or} \quad x = \emptyset.\end{aligned}$$

Therefore,

$$x = \frac{3\pi}{2} + 2n\pi, \quad n \in \mathbb{Z}.$$

Example 2.7

Solve $2\sin 4x = 1$, $x \in \mathbb{R}$.

Solution.

Rewrite $2\sin 4x = 1$ as $\sin 4x = \frac{1}{2}$.

One sequence of solutions is based on $\sin(4x + 2n\pi) = \frac{1}{2}$.

$$\sin(4x + 2n\pi) = \frac{1}{2} \implies 4x + 2n\pi = \frac{\pi}{6}$$

$$\iff 4x = \frac{\pi}{6} + 2n\pi$$

$$\iff x = \frac{\pi}{24} + \frac{n\pi}{2}.$$

And the other sequence of solutions is based on $\sin(\pi - 4x + 2n\pi) = \frac{1}{2}$.

$$\sin(\pi - 4x + 2n\pi) = \frac{1}{2} \implies \pi - 4x + 2n\pi = \frac{\pi}{6}$$

$$\iff -4x = \frac{\pi}{6} - \pi + 2n\pi$$

$$\iff 4x = \pi - \frac{\pi}{6} + 2n\pi$$

$$\iff 4x = \frac{5\pi}{6} + 2n\pi$$

$$\iff x = \frac{5\pi}{24} + \frac{n\pi}{2}.$$

Therefore,

$$x = \frac{\pi}{24} + \frac{n\pi}{2} \quad \text{or} \quad x = \frac{5\pi}{24} + \frac{n\pi}{2}, \quad n \in \mathbb{Z}.$$

Example 2.8

Solve $\sec^2 x + \tan^2 x = 3$, $x \in \mathbb{R}$.

Solution.

Rewrite $\sec^2 x + \tan^2 x = 3$ as $\sec^2 x + (\sec^2 x - 1) = 3$.

$$\begin{aligned}
 \sec^2 x + (\sec^2 x - 1) = 3 &\iff 2\sec^2 x - 1 = 3 \\
 &\iff 2\sec^2 x - 4 = 0 \\
 &\iff \sec^2 x - 2 = 0 \\
 &\iff (\sec x + \sqrt{2})(\sec x - \sqrt{2}) = 0 \\
 (2.6) \quad &\iff \sec x = -\sqrt{2} \quad \text{or} \quad \sec x = \sqrt{2}.
 \end{aligned}$$

Each of equations (2.6) produces two sequences of solutions.

Two sequences of solutions from $\sec x = \sqrt{2}$ are

$$\sec x = \sqrt{2} \implies x + 2n\pi = \frac{\pi}{4} \iff x = \frac{\pi}{4} + 2n\pi$$

and

$$\sec x = \sqrt{2} \implies \sec(-x) = \sqrt{2} \implies -x + 2n\pi = \frac{\pi}{4} \iff x = \frac{-\pi}{4} + 2n\pi = \frac{7\pi}{4} + 2n\pi.$$

Two sequences of solutions from $\sec x = -\sqrt{2}$ are

$$\sec x = -\sqrt{2} \implies \sec(\pi - x) = \sqrt{2} \implies \pi - x + 2n\pi = \frac{\pi}{4} \iff x = \frac{3\pi}{4} + 2n\pi$$

and

$$\sec x = -\sqrt{2} \implies \sec(\pi + x) = \sqrt{2} \implies \pi + x + 2n\pi = \frac{\pi}{4} \iff x = \frac{-3\pi}{4} + 2n\pi = \frac{5\pi}{4} + 2n\pi.$$

So,

$$x = \frac{\pi}{4} + 2n\pi \vee x = \frac{3\pi}{4} + 2n\pi \vee x = \frac{4\pi}{4} + 2n\pi \vee x = \frac{7\pi}{4} + 2n\pi, \quad n \in \mathbb{Z}.$$

You should convince yourself that all four of these sequences may be produced from

$$x = \frac{\pi}{4} + \frac{n\pi}{2}, \quad n \in \mathbb{Z}.$$

Example 2.9

Solve $\sin x - \cos x = 1$, $x \in \mathbb{R}$.

Solution.

The trouble is two functions. If we rewrite $\sin x - \cos x = 1$ as $\sin x = 1 + \cos x$, then square both sides, we can replace $\sin^2 x$ with $1 - \cos^2 x$.

$$\begin{aligned} \sin x - \cos x = 1 &\iff \sin x = 1 + \cos x \\ &\implies \sin^2 x = 1 + 2\cos x + \cos^2 x \\ &\iff 1 - \cos^2 x = 1 + 2\cos x + \cos^2 x \\ &\iff 2\cos x + 2\cos^2 x = 0 \\ &\iff \cos x + \cos^2 x = 0 \\ &\iff \cos x(1 + \cos x) = 0 \\ &\implies \cos x = 0 \quad \text{or} \quad \cos x = -1. \end{aligned}$$

So *possible* solutions are

$$x = \frac{\pi}{2} + 2n\pi, \quad x = \frac{3\pi}{2} + 2n\pi, \quad x = \pi + 2n\pi.$$

But some of these may be extraneous having been introduced when we squared both sides of $\sin x = 1 + \cos x$. Checking,

$$x = \frac{\pi}{2} \implies \sin x = 1, \cos x = 0, \sin x - \cos x = 1 - 0 = 1, \quad (\text{OK})$$

$$x = \frac{3\pi}{2} \implies \sin x = -1, \cos x = 0, \sin x - \cos x = -1 - 0 = -1, \quad (\text{NO})$$

$$x = \pi \implies \sin x = 0, \cos x = -1, \sin x - \cos x = 0 - (-1) = 1. \quad (\text{OK})$$

Therefore,

$$x = \frac{\pi}{2} + 2n\pi \quad \text{or} \quad x = \pi + 2n\pi, \quad n \in \mathbb{Z}.$$

Example 2.10

Solve $\sin 3x = \sin 7x$, $x \in \mathbb{R}$.

Solution.

Remembering that $\sin \theta = \sin(\pi - \theta)$, we realize $\sin 3x = \sin 7x$ whenever $7x = 3x + 2n\pi$ or $7x = \pi - 3x + 2n\pi$. Then

$$7x = 3x + 2n\pi \quad \text{or} \quad 7x = \pi - 3x + 2n\pi$$

$$4x = 2n\pi \quad \text{or} \quad 10x = \pi + 2n\pi.$$

Therefore,

$$x = \frac{n\pi}{2} \quad \text{or} \quad x = \frac{\pi}{10} + \frac{n\pi}{5}.$$

Example 2.11

Solve $\cos 5x = -\cos 2x$, $x \in \mathbb{R}$.

Solution.

We use $-\cos \theta = \cos(\pi - \theta)$ and $-\cos \theta = \cos(\pi + \theta)$. So,

$$2x = \pi - 5x + 2n\pi \quad \text{or} \quad 2x = \pi + 5x + 2n\pi$$

$$7x = \pi + 2n\pi \quad \text{or} \quad -3x = \pi + 2n\pi.$$

Therefore,

$$x = \frac{\pi}{7} + \frac{2n\pi}{7} \quad \text{or} \quad x = \frac{-\pi}{3} + \frac{2n\pi}{3}.$$

Since $\cos(-\theta) = \cos \theta$, we could write

$$x = \frac{\pi}{7} + \frac{2n\pi}{7} \quad \text{or} \quad x = \frac{\pi}{3} + \frac{2n\pi}{3}.$$

Example 2.12

Solve $\sin 2x = \cos 3x$, $x \in \mathbb{R}$.

Solution.

Use $\cos \theta$ as $\sin(\frac{\pi}{2} - \theta)$. Then,

One sequence of solutions is based on $\sin 2x = \sin(\frac{\pi}{2} - 3x + 2n\pi)$.

$$\sin 2x = \sin\left(\frac{\pi}{2} - 3x + 2n\pi\right) \implies 2x = \frac{\pi}{2} - 3x + 2n\pi$$

$$\iff 5x = \frac{\pi}{2} + 2n\pi$$

$$\iff x = \frac{\pi}{10} + \frac{2n\pi}{5}.$$

And the other sequence of solutions is based on $\sin 2x = \sin(\pi - (\frac{\pi}{2} - 3x) + 2n\pi)$.

$$\sin 2x = \sin\left(\pi - \left(\frac{\pi}{2} - 3x\right) + 2n\pi\right) \implies 2x = \pi - \left(\frac{\pi}{2} - 3x\right) + 2n\pi$$

$$\iff 2x = \pi - \frac{\pi}{2} + 3x + 2n\pi$$

$$\iff -x = \frac{\pi}{2} + 2n\pi$$

$$\iff x = \frac{-\pi}{2} + 2n\pi.$$

Therefore,

$$x = \frac{\pi}{10} + \frac{2n\pi}{5} \quad \text{or} \quad x = \frac{-\pi}{2} + 2n\pi, \quad n \in \mathbb{Z}. \quad \blacksquare$$

Example 2.13

Solve $\cos 2x = \sin x$, $x \in \mathbb{R}$.

Solution.

Use $\cos \theta = \sin(\frac{\pi}{2} - \theta)$. Then,

One sequence of solutions is based on $\cos 2x = \sin(\frac{\pi}{2} - 2x + 2n\pi)$.

$$\begin{aligned} \sin x &= \sin\left(\frac{\pi}{2} - 2x + 2n\pi\right) \implies x = \frac{\pi}{2} - 2x + 2n\pi \\ &\iff 3x = \frac{\pi}{2} + 2n\pi \\ &\iff x = \frac{\pi}{6} + \frac{2n\pi}{3}. \end{aligned}$$

And the other sequence of solutions is based on $\sin x = \sin(\pi - x)$.

$$\begin{aligned} \sin(\pi - x) &= \sin\left(\frac{\pi}{2} - 2x + 2n\pi\right) \implies \pi - x = \frac{\pi}{2} - 2x + 2n\pi \\ &\iff x = \frac{\pi}{2} - \pi + 2n\pi \\ &\iff x = \frac{-\pi}{2} + 2n\pi. \end{aligned}$$

Therefore,

$$x = \frac{\pi}{6} + \frac{2n\pi}{3} \quad \text{or} \quad x = \frac{3\pi}{2} + 2n\pi, \quad n \in \mathbb{Z}. \quad \blacksquare$$

2.4.3. Simplified solutions

The conclusion of Example (2.13) was

$$(2.7) \quad x = \frac{\pi}{6} + \frac{2n\pi}{3} \quad \text{or} \quad x = \frac{3\pi}{2} + 2n\pi, \quad n \in \mathbb{Z}.$$

It may not be obvious, but the expression $\frac{\pi}{6} + \frac{2n\pi}{3}$ produces all the numbers that $\frac{3\pi}{2} + 2n\pi$ produces. It will be easier to discuss this, if we use letters m and n to represent the integers in the expressions. Then one sequence of solutions is

$$(2.8) \quad x = \frac{\pi}{6} + \frac{2n\pi}{3}, \quad n \in \mathbb{Z}.$$

and the other sequence is

$$(2.9) \quad x = \frac{3\pi}{2} + 2m\pi, \quad m \in \mathbb{Z}.$$

By Equation (2.9),

$$\text{when } m = 0, \quad x = \frac{3\pi}{2},$$

$$\text{when } m = 1, \quad x = \frac{7\pi}{2}.$$

But, each of these values of x can be produced by Equation (2.8),

$$\text{when } n = 2, \quad x = \frac{3\pi}{2},$$

$$\text{when } n = 5, \quad x = \frac{7\pi}{2}.$$

Naturally, we wonder if all the solutions from Equation (2.9) are produced by Equation (2.8). If for every integer m in Equation (2.9), we can find an *integer* n such that Equation (2.8) produces the same solution as m in Equation (2.9), then all we need is Equation (2.8). We now do just that.

Let m be given. Solve for n .

$$\frac{\pi}{6} + \frac{2n\pi}{3} = \frac{3\pi}{2} + 2m\pi$$

$$\pi + 4n\pi = 9\pi + 12m\pi$$

$$4n = 8 + 12m$$

$$n = 8 + 6m.$$

Since the integers are closed under addition and multiplication, n is an integer and we are done.

Therefore, we can write the solution of Example (2.13) as simply

$$x = \frac{\pi}{6} + \frac{2n\pi}{3}, \quad n \in \mathbb{Z}.$$

Could the solution of Example (2.12) have been simplified? The solution to Example (2.12) is

$$x = \frac{\pi}{10} + \frac{2n\pi}{5}, n \in \mathbb{Z} \quad \text{or} \quad x = \frac{-\pi}{2} + 2m\pi, m \in \mathbb{Z}.$$

Let m be given. Solve for n .

$$\frac{\pi}{10} + \frac{2n\pi}{5} = \frac{3\pi}{2} + 2m\pi$$

$$\pi + 4n\pi = 15\pi + 20m\pi$$

$$n = \frac{14}{4} + 5m.$$

Since the integers are not closed under division, n need not be an integer. Since when $m = 1$, $n = \frac{34}{4} \notin \mathbb{Z}$. So it is false that for every m there is an n .

We conclude that

$$x = \frac{\pi}{10} + \frac{2n\pi}{5}, n \in \mathbb{Z} \quad \text{or} \quad x = \frac{-\pi}{2} + 2m\pi, m \in \mathbb{Z}.$$

cannot be simplified.

2.4.4. Approximate solutions

So far, all of the solutions have been exact. When an exact solution does not exist, we will write an approximate solution accurate to 5 significant figure.

Example 2.14

Solve $3 \cos \frac{x}{3} + 2 = 0$, $x \in \mathbb{R}$.

Solution.

Note that

$$\cos \frac{x}{3} = \frac{-2}{3} \iff -\cos \frac{x}{3} = \frac{2}{3} \implies \cos \left(\pi - \frac{x}{3} \right) = \frac{2}{3} \quad \text{and} \quad \cos \left(\pi + \frac{x}{3} \right) = \frac{2}{3}.$$

One sequence of solutions is based on $\cos \left(\pi - \frac{x}{3} \right) = \frac{2}{3}$.

$$\begin{aligned} (2.10) \quad \cos \left(\pi - \frac{x}{3} \right) &= \frac{2}{3} \\ \implies \pi - \frac{x}{3} &= 0.84107 + 2n\pi \\ -\frac{x}{3} &= 0.84107 - \pi + 2n\pi \\ x &= 3\pi - 2.5232 + 6n\pi. \end{aligned}$$

$$\begin{aligned} \cos \left(\pi + \frac{x}{3} \right) &= \frac{2}{3} \\ \pi + \frac{x}{3} &= 0.84107 + 2n\pi \\ \frac{x}{3} &= 0.84107 - \pi + 2n\pi \\ x &= 2.5232 - 3\pi + 6n\pi. \end{aligned}$$

Therefore,

$$x = 3\pi - 2.5232 + 6n\pi \quad \text{or} \quad x = 2.5232 - 3\pi + 6n\pi, \quad n \in \mathbb{Z}. \quad \blacksquare$$

In equation (2.10), the value 0.84107 is obtained from a calculator as the inverse cosine of $\frac{2}{3}$.

Example 2.15

Suppose $\sin 2x = \frac{\sqrt{3}}{2}$. Find (a) the principal solutions, and (b) all the solutions in the interval $[0, 2\pi]$.

(a) Solution.

The period of $\sin 2x$ is π . The two sequences of solutions are

$$\frac{\pi}{6} + n\pi, n \in \mathbb{Z}$$

and

$$\frac{\pi}{3} + n\pi, n \in \mathbb{Z}.$$

The first few terms of the sequence $\left\{ \frac{\pi}{6} + n\pi \right\}$ are $\left\{ \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6} \right\}$ when $n = 0, 1, 2$. Only $\frac{\pi}{6}$ is in $[0, \pi]$.

The first few terms of the sequence $\left\{ \frac{\pi}{3} + n\pi \right\}$ are $\left\{ \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3} \right\}$ when $n = 0, 1, 2$. Only $\frac{\pi}{3}$ lies in $[0, \pi]$.

So, the only principal solutions are $\frac{\pi}{6}$ and $\frac{\pi}{3}$.

(b) Solution.

Since the frequency of $\sin 2x$ is 2, we expect that each sequence of solutions will produce two solutions in $[0, 2\pi]$. A glance at the solution to (a) above convinces us that this is indeed the case. The four solutions in $[0, 2\pi]$ are

$$\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}. \quad \blacksquare$$

Call the sequence of solutions based on integer m the m -sequence and call the sequence of solutions based on integer n the n -sequence.

To show that all solutions of one sequence depending on integer m are included in another sequence depending on the integer n , you must show that for every integer m , there exists an integer n such that n used in the n -sequence produces the same value as does m used in the m -sequence.

To show that not all solutions of one sequence depending on integer m are included in another sequence depending on the integer n , you must show that there exists an integer m that produces a solution in the m -sequence, but there is no integer n that produces the same solution in the n -sequence.

Exercise 2.1 ---

1. Show that the sequence of solutions $x = \frac{5\pi}{6} + \frac{2m\pi}{3}, m \in \mathbb{Z}$ is included in the sequence $x = \frac{\pi}{6} + \frac{2n\pi}{3}, n \in \mathbb{Z}$.
 2. Show that the sequence of solutions $x = \frac{3\pi}{4} + 2m\pi, m \in \mathbb{Z}$ is included in the sequence $x = \frac{\pi}{4} + \frac{n\pi}{2}, n \in \mathbb{Z}$.
 3. Show that the sequence of solutions $x = \frac{2m\pi}{3}, m \in \mathbb{Z}$ is included in the sequence $x = \frac{2n\pi}{9}, n \in \mathbb{Z}$.
 4. Show neither sequence of solutions is included in the other sequence for the sequences $x = \frac{\pi}{7} + \frac{2m\pi}{7}, m \in \mathbb{Z}$ and $x = \frac{\pi}{3} + \frac{2n\pi}{3}, n \in \mathbb{Z}$.
-

2.5. Inverse functions

2.6. The sum sine and cosine

In general, the sum of the sine and cosine functions is difficult to compute. But when the frequency of the functions match, the sum is not too hard to find. Though a little ingenuity is required. Let's see if we can find the following sum.

$$(2.11) \quad y = a \sin kx + b \cos kx.$$

We multiply by 1 in a form that will initially seem arbitrary, but really isn't.

$$(2.12) \quad \begin{aligned} a \sin kx + b \cos kx &= \frac{\sqrt{a^2 + b^2}}{\sqrt{a^2 + b^2}} (a \sin kx + b \cos kx) \\ &= \sqrt{a^2 + b^2} \left(\frac{a}{\sqrt{a^2 + b^2}} \sin kx + \frac{b}{\sqrt{a^2 + b^2}} \cos kx \right). \end{aligned}$$

Now, notice that

$$\left(\frac{a}{\sqrt{a^2 + b^2}} \right)^2 + \left(\frac{b}{\sqrt{a^2 + b^2}} \right)^2 = 1$$

This means that for some number (angle) β ,

$$\cos \beta = \frac{a}{\sqrt{a^2 + b^2}}, \quad \sin \beta = \frac{b}{\sqrt{a^2 + b^2}}.$$

So, equation (2.12) may be rewritten

$$a \sin kx + b \cos kx = \sqrt{a^2 + b^2} (\cos \beta \sin kx + \sin \beta \cos kx).$$

So that,

$$a \sin kx + b \cos kx = \sqrt{a^2 + b^2} [\sin(\beta + kx)].$$

Or, equivalently,

$$a \sin kx + b \cos kx = \sqrt{a^2 + b^2} \left(\sin k \left(x + \frac{\beta}{k} \right) \right).$$

Evidently, the sum of the sine and cosine functions with matching frequencies and amplitudes a and b respectively is a sine function amplitude $\sqrt{a^2 + b^2}$ shifted $\frac{\beta}{k}$ left.

2.7. The complex numbers

It turns out that it is often more convenient to perform certain operations on complex numbers when such are written in what is called their “trigonometric form”. This is especially so when we wish to raise a complex number to an exponent or find the roots of a complex number. We begin by showing how a complex number may be rewritten in trigonometric form.

2.7.1. Non-unit circle

As you know, the equation of a circle radius r centered at the origin in the uv -plane is

$$(2.13) \quad u^2 + v^2 = r^2.$$

It is a fact that

$$(2.14) \quad \cos^2 x + \sin^2 x = 1.$$

Multiplying Equation (2.14) by r^2 produces

$$(2.15) \quad r^2 \cos^2 x + r^2 \sin^2 x = r^2.$$

Equations (2.13) and (2.15) together imply that on the circle radius r ,

$$u = r \cos \theta \quad \text{and} \quad v = r \sin \theta.$$

Figure (2.4) illustrates this.

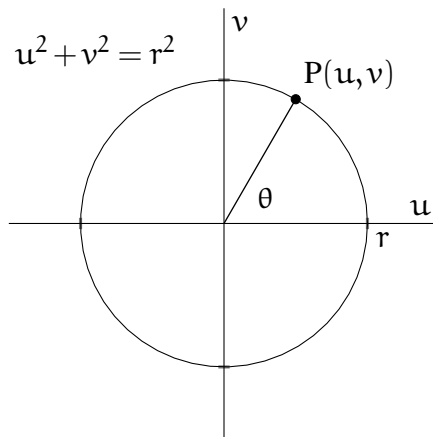


FIGURE 2.4. Circle radius r center origin.

2.7.2. The complex plane

Complex numbers can be represented as points in the complex plane. The horizontal axis is called the real axis and the vertical axis is called the

imaginary axis. The complex number $z = a + bi$ consists of a real part, a , and an imaginary part, b . If $b = 0$, then z is the real number a . If $a = 0, b \neq 0$, then z is the pure imaginary number bi . When we represent a complex number $z = a + bi$ in the complex plane as shown in Figure (2.5).

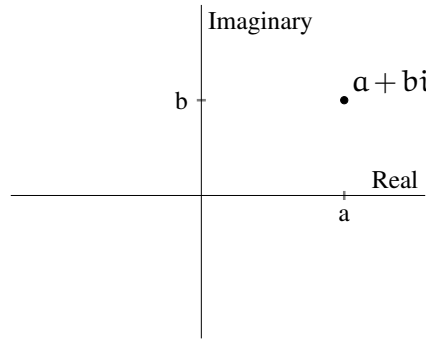


FIGURE 2.5. $z = a + bi$ in complex plane

2.7.3. The polar form (“trigonometric form”) of a complex number

The rectangular coordinates of Figure (2.5) are not the only way we can specify the location of a point in the plane. We could, in the alternative, state the location of a point P by imagining it at the end of a ray of length r that makes an angle θ , positive is CCW, with respect to the positive horizontal axis. This is shown in Figure (2.6).

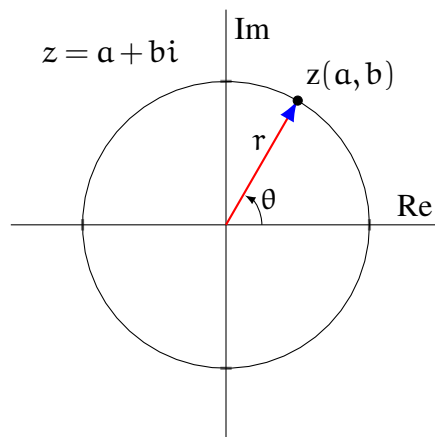


FIGURE 2.6. Complex number $z = a + bi$ has coordinates a and b .

As we noted in Section (2.7.1) on page 36, the coordinates at z are

$$a = r \cos \theta \quad \text{and} \quad b = r \sin \theta.$$

So, a complex number may be written in either rectangular form, LHS of Equation (2.16), or trigonometric form, RHS of Equation (2.16)

$$(2.16) \quad a + bi = r \cos \theta + (r \sin \theta)i.$$

When z is written in trigonometric form, the number r is called the “modulus” of z and the number θ is called the “argument” or “amplitude” of z .

We often prefer Equation (2.16) rearranged in the form of Equation (2.17)

$$(2.17) \quad a + bi = r(\cos \theta + i \sin \theta).$$

Complex numbers $z = r_1(\cos \theta + i \sin \theta)$ and $w = r_2(\cos \phi + i \sin \phi)$ are equal when $r_1 = r_2$ and $\theta = \phi + 2n\pi$, $n \in \mathbb{Z}$.

It is also helpful to know that since $\tan \theta = \frac{\sin \theta}{\cos \theta}$,

$$(2.18) \quad \tan \theta = \frac{b}{a}.$$

Example 2.16

Write the complex number $z = 1 + i\sqrt{3}$ in trigonometric form.

Solution

$$r = \sqrt{1^2 + (\sqrt{3})^2} = 2.$$

Then

$$(2.19) \quad a = r \cos \theta \implies 1 = 2 \cos \theta \implies \cos \theta = \frac{1}{2}$$

$$(2.20) \quad b = r \sin \theta \implies \sqrt{3} = 2 \sin \theta \implies \sin \theta = \frac{\sqrt{3}}{2}.$$

Two comments should now be made. First, only one of Equations (2.19) or (2.20) is needed to determine the value of θ . Second, because of the symmetry of trigonometric functions, there are two values of θ for which $\sin \theta = \frac{\sqrt{3}}{2}$. The values are $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. Selecting the correct alternative requires knowing the quadrant in which $1 + i\sqrt{3}$ resides.

To finish this example, note that $1 + i\sqrt{3}$ is in Quadrant I, so $\theta = \frac{\pi}{3}$.

Therefore, $1 + i\sqrt{3} = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$. ■

Someone is sure to mention that since $\tan \theta = \frac{b}{a}$, one might simply find the value of $\tan^{-1} \frac{\sqrt{3}}{2}$ in order to work Example 2.16. This is fine. In general, one should proceed in which ever way is easiest, given the numbers used in the particular case.

Example 2.17

Write the complex number $z = -3\sqrt{2} - 3i\sqrt{2}$ in trigonometric form.

Solution

$$r = \sqrt{(-3\sqrt{2})^2 + (-3\sqrt{2})^2} = 6.$$

Then,

$$\cos \theta = \frac{-3\sqrt{2}}{6} = -\frac{\sqrt{2}}{2} \implies \theta = \frac{3\pi}{4} \quad \text{or} \quad \theta = \frac{5\pi}{4}.$$

Since, z is in Quadrant IV,

$$\theta = \frac{5\pi}{4}.$$

Therefore, $-3\sqrt{2} - 3i\sqrt{2} = 6 \left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right)$. ■

Appendices

Appendix A

Answers to Exercises

Answers to Exercise 1.1

(1) Let x_1 be paired with y_1 and x_2 with y_2 . Suppose $x_1 = x_2$. Then $y_2 = ax_2 + b = ax_1 + b = y_1$. Since, $x_1 = x_2 \implies y_1 = y_2$, y is a function of x .

(2) Let x_1 be paired with y_1 and x_2 with y_2 . Suppose $x_1 = x_2$. Then $y_2 = ax_2^2 = ax_1^2 = y_1$. Since, $x_1 = x_2 \implies y_1 = y_2$, y is a function of x .

(3) Let x_1 be paired with y_1 and x_2 with y_2 . Suppose $x_1 = x_2$. Then $y_2 = ax_2^2 + bx_2 + c = ax_1^2 + bx_1 + c = y_1$. Since, $x_1 = x_2 \implies y_1 = y_2$, y is a function of x .

(4) Let x_1 be paired with y_1 and x_2 with y_2 . Suppose $x_1 = x_2$. Then $y_2^2 = x_2 = x_1 = y_1^2$. The proof cannot continue as did the previous proofs, because $y_1^2 = y_2^2$ does not necessarily imply that $y_1 = y_2$.

(5) $2^2 = 4$ and $(-2)^2 = 4$. But $-2 \neq 2$. In other words, one value of x , 4, is paired with two values of y , namely 2 and (-2) . Formally, $x_1 = x_2$ and $y_1 \neq y_2$, so it is false that $x_1 = x_2 \implies y_2 = y_2$.

(6) $g(x) = x^4$ and $g(-x) = (-x)^4 = x^4$. Since g is an even function, g is symmetric with respect to the vertical axis.

(7) $g(x) = x^7$ and $g(-x) = (-x)^7 = -(x^7)$. Since $g(x) = -g(-x)$, is an odd function, and as such is symmetric with respect to the origin.

(8) Suppose $x \geq 0$. $f(x) = |x| = x$ and $f(-x) = |-x| = -(-x) = x$. Since $f(x) = f(-x)$, f is an even function. In the alternative, suppose $x < 0$. $f(x) = |x| = -x$ and $f(-x) = |-x| = -x$. Since $f(x) = f(-x)$, f is an even

function. In any case, f is an even function so symmetric with respect to the vertical axis. [Note: If these computations with absolute value do not make *perfect* sense to you, then by all means do review absolute value.]

(9) Note that 1 and -1 are in the domain of f . Now, $f(1) = 2$, but $f(-1) = -4$. So f is not an even function.

Answers to Exercise 1.2

(1) The graph of $f(x) = x^2 - 6x + 5$ suggests that f is decreasing to the left of $x = 3$ and increasing to the right of $x = 3$. Suppose $a < b \leq 3$. Then,

$$\begin{aligned} f(b) - f(a) &= (b^2 - 6b + 5) - (a^2 - 6a + 5) \\ &= b^2 - a^2 - 6(b - a) \\ &= (b - a)(b + a) - 6(b - a) \\ &= (b - a)(a + b - 6) \end{aligned}$$

Since $(b - a)$ is positive on the interval, the last expression is negative only if $(a + b) < 6$. And $(a + b) < 6$, because b is no greater than 3 and a is less than 3. So,

$$f(b) - f(a) < 0.$$

Therefore, f is a decreasing function of x on $(-\infty, 3]$.

To see that f is increasing in the interval $3 \leq a < b$, consider $f(b) - f(a)$.

$$\begin{aligned} f(b) - f(a) &= (b^2 - 6b + 5) - (a^2 - 6a + 5) \\ &= b^2 - a^2 - 6(b - a) \\ &= (b - a)(b + a) - 6(b - a) \\ &= (b - a)(a + b - 6) \end{aligned}$$

Since $(b - a)$ is positive on the interval, the last expression is positive only if $(a + b) > 6$. And $(a + b) > 6$, because a is no less than 3 and b is greater than 3. So,

$$f(b) - f(a) > 0.$$

Therefore, f is an increasing function of x on $[3, -\infty)$.

(2) The function $f(x) = mx + c$ with m, c constants is defined for all real numbers x . Suppose a and b are real numbers with $b > a$.

$$\begin{aligned} f(b) - f(a) &= mb + c - ma - c \\ &= m(b - a) \end{aligned}$$

Since $(b - a)$ is positive, $f(b) - f(a)$ is positive or negative (hence increasing or decreasing) depending on the sign of m .

(3) The function $f(x) = |x|$ is defined for all real numbers x . Suppose a and b are real numbers with $a < b \leq 0$.

$$\begin{aligned} f(b) - f(a) &= |b| - |a| \\ &= -b - (-a) \\ &= a - b \\ &< 0. \end{aligned}$$

Since $f(b) < f(a)$, f is decreasing on $a < b \leq 0$.

Suppose $0 \leq a < b$.

$$\begin{aligned} f(b) - f(a) &= |b| - |a| \\ &= b - a \\ &> 0. \end{aligned}$$

Since $f(b) > f(a)$ when $b > a$, f is increasing on $0 \leq a < b$.

(4) Suppose f is an even function of x . This means $f(-x) = f(x)$ for all x in the domain of f . If we create a new function $h(x) = f(x) + k$, k a constant, then $h(-x) = f(-x) + k$. Since $f(-x) = f(x)$, $h(-x) = f(x) + k$. But this is $h(x)$. Since $h(-x) = h(x)$, $h(x)$ is an even function. Therefore, $f(x) + k$ is an even function whenever $f(x)$ is.

Answers to Exercise 2.1

(1) The graph of $f(x) = x^2 - 6x + 5$ suggests that f is decreasing to the left of $x = 3$ and increasing to the right of $x = 3$. Suppose $a < b \leq 3$. Then,

(2) The function $f(x) = mx + c$ with m, c constants is defined for all real numbers x . Suppose a and b are real numbers with $b > a$.