Example 2.12

Solve $\sin 2x = \cos 3x,\ x \in \mathbb{R}$.

Solution.

Use $\cos\theta$ as $\sin(\frac{\pi}{2} - \theta)$. Then,

One sequence of solutions is based on $\sin 2x = \sin(\frac{\pi}{2} - 3x + 2n\pi)$.

$$\begin{aligned}\sin 2x = \sin\left(\frac{\pi}{2} - 3x + 2n\pi\right) &\implies 2x = \frac{\pi}{2} - 3x + 2n\pi \\ &\iff 5x = \frac{\pi}{2} + 2n\pi \\ &\iff x = \frac{\pi}{10} + \frac{2n\pi}{5}.\end{aligned}$$

And the other sequence of solutions is based on $\sin 2x = \sin(\pi - (\frac{\pi}{2} - 3x) + 2n\pi)$.

$$\begin{aligned}\sin 2x = \sin\left(\pi - \left(\frac{\pi}{2} - 3x\right) + 2n\pi\right) &\implies 2x = \pi - \left(\frac{\pi}{2} - 3x\right) + 2n\pi \\ &\iff 2x = \pi - \frac{\pi}{2} + 3x + 2n\pi \\ &\iff -x = \frac{\pi}{2} + 2n\pi \\ &\iff x = \frac{-\pi}{2} + 2n\pi.\end{aligned}$$

Therefore,

$x = \dfrac{\pi}{10} + \dfrac{2n\pi}{5}$ or $x = \dfrac{-\pi}{2} + 2n\pi,\ n \in \mathbb{Z}$. ■

Example 2.13

Solve $\cos 2x = \sin x,\ x \in \mathbb{R}$.

Solution.

Use $\cos\theta = \sin(\frac{\pi}{2} - \theta)$. Then,

One sequence of solutions is based on $\cos 2x = \sin(\frac{\pi}{2} - 2x + 2n\pi)$.

$$\begin{aligned} \sin x = \sin\left(\frac{\pi}{2} - 2x + 2n\pi\right) &\implies x = \frac{\pi}{2} - 2x + 2n\pi \\ &\iff 3x = \frac{\pi}{2} + 2n\pi \\ &\iff x = \frac{\pi}{6} + \frac{2n\pi}{3}. \end{aligned}$$

And the other sequence of solutions is based on $\sin x = \sin(\pi - x)$.

$$\begin{aligned} \sin(\pi - x) = \sin\left(\frac{\pi}{2} - 2x + 2n\pi\right) &\implies \pi - x = \frac{\pi}{2} - 2x + 2n\pi \\ &\iff x = \frac{\pi}{2} - \pi + 2n\pi \\ &\iff x = \frac{-\pi}{2} + 2n\pi. \end{aligned}$$

Therefore,

$$x = \frac{\pi}{6} + \frac{2n\pi}{3} \quad \text{or} \quad x = \frac{3\pi}{2} + 2n\pi,\ n \in \mathbb{Z}. \qquad \blacksquare$$

2.4.3. Simplified solutions

The conclusion of Example (2.13) was

$$x = \frac{\pi}{6} + \frac{2n\pi}{3} \quad \text{or} \quad x = \frac{3\pi}{2} + 2n\pi,\ n \in \mathbb{Z}. \tag{2.7}$$

It may not be obvious, but the expression $\frac{\pi}{6} + \frac{2n\pi}{3}$ produces all the numbers that $\frac{3\pi}{2} + 2n\pi$ produces. It will be easier to discuss this, if we use letters m and n to represent the integers in the expressions. Then one sequence of solutions is

$$x = \frac{\pi}{6} + \frac{2n\pi}{3},\ n \in \mathbb{Z}. \tag{2.8}$$

and the other sequence is

$$x = \frac{3\pi}{2} + 2m\pi,\ m \in \mathbb{Z}. \tag{2.9}$$

By Equation (2.9),

when $m = 0, \quad x = \frac{3\pi}{2},$

when $m = 1, \quad x = \frac{7\pi}{2}.$

But, each of these values of x can be produced by Equation (2.8),

when $n = 2, \quad x = \frac{3\pi}{2},$

when $n = 5, \quad x = \frac{7\pi}{2}.$

Naturally, we wonder if all the solutions from Equation (2.9) are produced by Equation (2.8). If for every integer m in Equation (2.9), we can find an *integer n* such that Equation (2.8) produces the same solution as m in Equation (2.9), then all we need is Equation (2.8). We now do just that.

Let m be given. Solve for n.

$$\frac{\pi}{6} + \frac{2n\pi}{3} = \frac{3\pi}{2} + 2m\pi$$

$$\pi + 4n\pi = 9\pi + 12m\pi$$

$$4n = 8 + 12m$$

$$n = 8 + 6m.$$

Since the integers are closed under addition and multiplication, n is an integer and we are done.

Therefore, we can write the solution of Example (2.13) as simply

$$x = \frac{\pi}{6} + \frac{2n\pi}{3},\ n \in \mathbb{Z}.$$

Could the solution of Example (2.12) have bee simplified? The solution to Example (2.12) is

$$x = \frac{\pi}{10} + \frac{2n\pi}{5},\ n \in \mathbb{Z} \quad \text{or} \quad x = \frac{-\pi}{2} + 2m\pi,\ m \in \mathbb{Z}.$$

Let m be given. Solve for n.

$$\frac{\pi}{10} + \frac{2n\pi}{5} = \frac{3\pi}{2} + 2m\pi$$

$$\pi + 4n\pi = 15\pi + 20m\pi$$

$$n = \frac{14}{4} + 5m.$$

Since the integers are not closed under division, n need not be an integer. Since when $m = 1$, $n = \frac{34}{4} \notin \mathbb{Z}$. So it is false that for every m there is an n. We conclude that

$$x = \frac{\pi}{10} + \frac{2n\pi}{5},\ n \in \mathbb{Z} \quad \text{or} \quad x = \frac{-\pi}{2} + 2m\pi,\ m \in \mathbb{Z}.$$

cannot be simplified.

2.4.4. Approximate solutions

So far, all of the solutions have been exact. When an exact solution does not exist, we will write an approximate solution accurate to 5 significant figure.

Example 2.14

Solve $3\cos\frac{x}{3}+2=0,\ x\in\mathbb{R}$.

Solution.

Note that

$$\cos\frac{x}{3}=\frac{-2}{3} \iff -\cos\frac{x}{3}=\frac{2}{3} \implies \cos\left(\pi-\frac{x}{3}\right)=\frac{2}{3} \quad \text{and} \quad \cos\left(\pi+\frac{x}{3}\right)=\frac{2}{3}.$$

One sequence of solutions is based on $\cos\left(\pi-\frac{x}{3}\right)=\frac{2}{3}$.

$$\begin{aligned} \cos\left(\pi-\frac{x}{3}\right)&=\frac{2}{3} \\ \implies \pi-\frac{x}{3}&=0.84107+2n\pi \\ -\frac{x}{3}&=0.84107-\pi+2n\pi \\ x&=3\pi-2.5232+6n\pi. \end{aligned} \tag{2.10}$$

$$\begin{aligned} \cos\left(\pi+\frac{x}{3}\right)&=\frac{2}{3} \\ \pi+\frac{x}{3}&=0.84107+2n\pi \\ \frac{x}{3}&=0.84107-\pi+2n\pi \\ x&=2.5232-3\pi+6n\pi. \end{aligned}$$

Therefore,

$$x=3\pi-2.5232+6n\pi \quad \text{or} \quad x=2.5232-3\pi+6n\pi,\ n\in\mathbb{Z}. \qquad \blacksquare$$

In equation (2.10), the value 0.84107 is obtained from a calculator as the inverse cosine of $\frac{2}{3}$.

Example 2.15

Suppose $\sin 2x = \frac{\sqrt{3}}{2}$. Find (a) the principal solutions, and (b) all the solutions in the interval $[0, 2\pi]$.

(a) Solution.

The period of $\sin 2x$ is π. The two sequences of solutions are

$$\frac{\pi}{6} + n\pi,\ n \in \mathbb{Z}$$

and

$$\frac{\pi}{3} + n\pi,\ n \in \mathbb{Z}.$$

The first few terms of the sequence $\left\{\frac{\pi}{6} + n\pi\right\}$ are $\left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}\right\}$ when $n = 0, 1, 2$. Only $\frac{\pi}{6}$ is in $[0, \pi]$.

The first few terms of the sequence $\left\{\frac{\pi}{3} + n\pi\right\}$ are $\left\{\frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}\right\}$ when $n = 0, 1, 2$. Only $\frac{\pi}{3}$ lies in $[0, \pi]$.

So, the only principal solutions are $\frac{\pi}{6}$ and $\frac{\pi}{3}$.

(b) Solution.

Since the frequency of $\sin 2x$ is 2, we expect that each sequence of solutions will produce two solutions in $[0, 2\pi]$. A glance at the solution to (a) above convinces us that this is indeed the case. The four solutions in $[0, 2\pi]$ are

$$\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}. \qquad \blacksquare$$

Call the sequence of solutions based on integer m the m-sequence and call the sequence of solutions based on integer n the n-sequence.

To show that all solutions of one sequence depending on integer m are included in another sequence depending on the integer n, you must show that for every integer m, there exists an integer n such that n used in the n-sequence produces the same value as does m used in the m-sequence.

To show that not all solutions of one sequence depending on integer m are included in another sequence depending on the integer n, you must show that there exists an integer m that produces a solution in the m-sequence, but there is no integer n that produces the same solution in the n-sequence.

Exercise 2.1

1. Show that the sequence of solutions $x = \frac{5\pi}{6} + \frac{2m\pi}{3}, m \in \mathbb{Z}$ is included in the sequence $x = \frac{\pi}{6} + \frac{2n\pi}{3}, n \in \mathbb{Z}$.
2. Show that the sequence of solutions $x = \frac{3\pi}{4} + 2m\pi, m \in \mathbb{Z}$ is included in the sequence $x = \frac{\pi}{4} + \frac{n\pi}{2}, n \in \mathbb{Z}$.
3. Show that the sequence of solutions $x = \frac{2m\pi}{3}, m \in \mathbb{Z}$ is included in the sequence $x = \frac{2n\pi}{9}, n \in \mathbb{Z}$.
4. Show neither sequence of solutions is included in the other sequence for the sequences $x = \frac{\pi}{7} + \frac{2m\pi}{7}, m \in \mathbb{Z}$ and $x = \frac{\pi}{3} + \frac{2n\pi}{3}, n \in \mathbb{Z}$.

2.5. Inverse functions

2.6. The sum sine and cosine

In general, the sum of the sine and cosine functions is difficult to compute. But when the frequency of the functions match, the sum is not too hard to find. Though a little ingenuity is required. Let's see if we can find the following sum.

$$y = a\sin kx + b\cos kx. \tag{2.11}$$

We multiply by 1 in a form that will initially seem arbitrary, but really isn't.

$$a\sin kx + b\cos kx = \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}}(a\sin kx + b\cos kx)$$

$$= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin kx + \frac{b}{\sqrt{a^2+b^2}}\cos kx\right). \tag{2.12}$$

Now, notice that

$$\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 = 1$$

This means that for some number (angle) β,

$$\cos\beta = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\beta = \frac{b}{\sqrt{a^2+b^2}}.$$

So, equation (2.12) may be rewritten

$$a\sin kx + b\cos kx = \sqrt{a^2+b^2}(\cos\beta\sin kx + \sin\beta\cos kx).$$

So that,

$$a\sin kx + b\cos kx = \sqrt{a^2+b^2}\ [\sin(\beta + kx)].$$

Or, equivalently,

$$a\sin kx + b\cos kx = \sqrt{a^2+b^2}\ \left(\sin k(x + \frac{\beta}{k})\right).$$

Evidently, the sum of the sine and cosine functions with matching frequencies and amplitudes a and b respectively is a sine function amplitude $\sqrt{a^2+b^2}$ shifted $\dfrac{\beta}{k}$ left.