1.3 The sum sine and cosine

In general, the sum of the sine and cosine functions is difficult to compute. But when the frequency of the functions match, the sum is not too hard to find. Though a little ingenuity is required. Let's see if we can find the following sum.

$$y = a\sin kx + b\cos kx. \tag{1.5}$$

We multiply by 1 in a form that will initially seem arbitrary, but really isn't.

$$\begin{aligned} a\sin kx + b\cos kx &= \frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}}\left(a\sin kx + b\cos kx\right) \\ &= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin kx + \frac{b}{\sqrt{a^2+b^2}}\cos kx\right). \end{aligned} \tag{1.6}$$

Now, notice that

$$\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 = 1$$

This means that for some number (angle) β,

$$\cos\beta = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\beta = \frac{b}{\sqrt{a^2+b^2}}.$$

So, equation (1.6) may be rewritten

$$a\sin kx + b\cos kx = \sqrt{a^2+b^2}(\cos\beta\sin kx + \sin\beta\cos kx).$$

So that,

$$a\sin kx + b\cos kx = \sqrt{a^2+b^2}\;\;[\sin(\beta + kx)].$$

Or, equivalently,

$$a\sin kx + b\cos kx = \sqrt{a^2+b^2}\;\;\left(\sin k(x+\frac{\beta}{k})\right).$$

Evidently, the sum of the sine and cosine functions with matching frequencies and amplitudes a and b respectively is a sine function amplitude $\sqrt{a^2+b^2}$ shifted $\dfrac{\beta}{k}$ left.