1.2. Factoring polynomials

The axiom of distribution enables us to write a product as a sum. For example,

$$(x+2)(x+5) = x^2+7x+10.$$

Distribution also enables us to write a sum as a product. For example,

$$x^2+9x+20 = (x+4)(x+9).$$

While it is usually easy to write a product as a sum, it can be far from obvious how to rewrite a sum as a product. The process of rewriting a sum as a product is called "factoring".

1.2.1. Factoring polynomials extended

So far, the polynomials we have considered have been factorable over the integers. This means the factored form involved only integers. Some polynomials that are not factorable over the integers are factorable over the rational numbers.

Example 1.9

Factor $\frac{1}{6}x^2-\frac{1}{6}x+1$.

Solution

The easiest way to accomplish this is to rewrite

$$\frac{1}{6}x^2-\frac{1}{6}x+1$$

as

$$\frac{1}{6}(x^2-x+6).$$

Then,

$$\begin{aligned}\frac{1}{6}x^2-\frac{1}{6}x+1 &= \frac{1}{6}(x^2-x+6)\\ &= \frac{1}{6}(x-3)(x+2).\end{aligned}$$

Example 1.10

Factor $x^2-\frac{5}{6}x+\frac{1}{6}$.

Solution

$$x^2 - \frac{5}{6}x + \frac{1}{6} = \frac{1}{6}(6x^2 - 5x + 1)$$

$$= \frac{1}{6}(3x-1)(2x-1) \tag{1.1}$$

$$= \left(x - \frac{1}{3}\right)\left(x - \frac{1}{2}\right). \tag{1.2}$$

Either Equation (1.1) or Equation (1.2) is an acceptable form. ■

There are polynomials that are not factorable over the rational numbers, but that can be factored using irrational numbers.

Example 1.11

Factor $x^2 - 2\sqrt{3} + 3$.

Solution

Since $3 = (\sqrt{3})^2$, we can write 3 as the square of the number $\sqrt{3}$. Then,

$$x^2 - 2\sqrt{3} + 3 = (x - \sqrt{3})^2.$$

Example 1.12

Factor $x^2 - \sqrt{3}x - \sqrt{2}x + \sqrt{6}$.

Solution

The technique of factoring by grouping comes to our rescue.

$$x^2 - \sqrt{3}x - \sqrt{2}x + \sqrt{6} = x(x - \sqrt{3}) - \sqrt{2}(x - \sqrt{3})$$

$$= (x - \sqrt{2})(x - \sqrt{3}).$$

Exercise 1.1

Factor each of the following in the real numbers.

1. $\frac{1}{6}x^2 - \frac{2}{3}x + \frac{1}{2}$
2. $x^2 + \frac{5}{3}x + \frac{2}{3}$
3. $x^2 + \frac{1}{4}x - \frac{1}{8}$
4. $x^2 + x + \frac{1}{4}$
5. $x^2 - \frac{1}{25}$
6. $\frac{x^2}{3} + \frac{x}{3} + \frac{1}{12}$
7. $x^2 + \sqrt{3}x - \sqrt{2}x - \sqrt{6}$
8. $x^2 - \sqrt{5}x - \sqrt{2}x + \sqrt{10}$
9. $x^2 - \sqrt{2}x - x + \sqrt{2}$
10. $x^2 - 2\sqrt{2}x + 2$
11. $x^2 + 2\sqrt{5}x + 5$
12. $x^2 - 5$
13. $x^2 - 7$
14. $x^2 - \sqrt{11}x + \sqrt{7}x - \sqrt{77}$
15. $x^2 - \sqrt{7}x - \sqrt{2}x + \sqrt{14}$
16. $x^2 - 2\sqrt{2}x - 2x + 4\sqrt{2}$

Appendix A

Answers to Exercises

Answers to Exercise 1.1

(1) $\frac{1}{6}(x-1)(x-3)$ **(2)** $\left(x+\frac{2}{3}\right)(x+1)$ **(3)** $\left(x-\frac{1}{4}\right)\left(x+\frac{1}{2}\right)$

(4) $\left(x+\frac{1}{2}\right)^2$ **(5)** $\left(x-\frac{1}{5}\right)\left(x+\frac{1}{5}\right)$ **(6)** $\frac{1}{3}\left(x+\frac{1}{2}\right)^2$

(7) $(x-\sqrt{2})(x+\sqrt{3}$ **(8)** $(x-\sqrt{5})(x-\sqrt{2})$ **(9)** $(x-\sqrt{2})(x-1)$

(10) $(x-\sqrt{2})^2$ **(11)** $(x+\sqrt{5})^2$ **(12)** $(x-\sqrt{5})(x+\sqrt{5})$

(13) $(x-\sqrt{7})(x+\sqrt{7})$ **(14)** $(x-\sqrt{11})(x+\sqrt{7})$ **(15)** $(x-\sqrt{2})(x-\sqrt{7})$

(16) $(x-2)(x-\sqrt{8})$

Answers to Exercise 2.1

(1) $x^2+14x+49=(x+7)^2$ **(2)** $x^2-20x+100=(x-10)^2$

(3) $x^2-16x+64=(x-8)^2$ **(4)** $x^2+7x+\frac{49}{4}=\left(x+\frac{7}{2}\right)^2$

(5) $x^2-9x+\frac{81}{4}=\left(x-\frac{9}{2}\right)^2$ **(6)** $x^2+\frac{2x}{3}x+\frac{1}{9}=\left(x+\frac{1}{3}\right)^2$