Algebra Second Year

First Edition

Ray Tenebruso

ABSTRACT. NEED ABSTRACT

Algebra Second Year, First Edition

ISBN NEEDED

For updates to this book, visit http://mnrt.net/Publications/

Typeset using LaTeX

Contents

Chapter 2

Polynomials

2.1. Product of polynomials

You have often found the product of polynomials. Your computations were limited to the product of a monomial and a binomial. For example, $3(x+5) = 3x+15$. We now wish to multiply polynomials having more terms than one and two.

Example 2.1

Find the product $(x+2)(x+3)$.

Solution.

$$\begin{aligned}(x+2)(x+3) &= (x+2)x+(x+2)3\\ &= x^2+2x+3x+6\\ &= x^2+5x+6.\end{aligned}$$

Instead of saying "find the product" or "multiply" when working with polynomials, we often say "expand".

Example 2.2

Expand $(2x+5)(x+7)$.

Solution.

$$\begin{aligned}(2x+5)(x+7) &= (2x+5)x+(2x+5)7\\ &= 2x^2+5x+14x+35\\ &= 2x^2+19x+35.\end{aligned}$$

Example 2.3

Expand $(2x+3)(5x+7)$.

Solution.

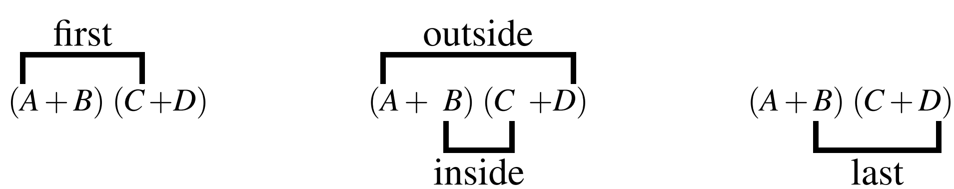
$$\begin{aligned}(2x+3)(5x+7) &= (2x+3)5x+(2x+3)7\\ &= 10x^2+15x+14x+35\\ &= 10x^2+29x+35. \quad \blacksquare\end{aligned}$$

If you study examples 2.1, 2.2, 2.3, you might notice a pattern. To better see the pattern, we replace the first and second terms of one binomial with "A", "B", and the terms of the second binomial with "C" and "D".

$$\begin{aligned}(A+B)(C+D) &= (A+B)C+(A+B)D\\ &= AC+BC+AD+BD. \quad \blacksquare\end{aligned}$$

We call

AC**:** the product of the first terms,
BC**:** the product of the inside terms,
AD**:** the product of the outside terms,
BD**:** the product of the last terms.



With a little practice, you will apply this pattern mentally, instead of writing out the double distribution.

Example 2.4

Expand $(3x+5)(2x+3)$.

Solution.

$$(3x+5)(2x+3) = 6x^2+10x+9x+15$$

combining like terms

$$= 6x^2+19x+15. \quad \blacksquare$$

Since distribution is used, watch out for sign mistakes.

Example 2.5

Expand $(x-4)(x+5)$.

Solution.

$$\begin{aligned}(x-4)(x+5) &= x^2-4x+5x-20\\ &= x^2+x-20.\end{aligned}$$

Example 2.6

Expand $(x-5)(2x-3)$.

Solution.

$$\begin{aligned}(x-5)(2x-3) &= 2x^2-10x-3x+15\\ &= 2x^2-13x+15. \quad \blacksquare\end{aligned}$$

When there are like terms, they are usually combined mentally as in the following examples.

Example 2.7

Expand $(x+2)(7x+5)$.

Solution.

$$(x+2)(7x+5) = 7x^2+19x+10.$$

Example 2.8

Expand $(x-6)(2x+3)$.

Solution.

$$(x-6)(2x+3) = 2x^2-9x-18. \quad \blacksquare$$

2.2. Factoring polynomials

The axiom of distribution enables us to write a product as a sum. For example,

$$(x+2)(x+5) = x^2 + 7x + 10.$$

Distribution also enables us to write a sum as a product. For example,

$$x^2 + 9x + 20 = (x+4)(x+9).$$

While it is usually easy to write a product as a sum, it can be far from obvious how to rewrite a sum as a product. The process of rewriting a sum as a product is called "factoring".

2.2.1. Factoring polynomials extended

So far, the polynomials we have considered have been factorable over the integers. This means the factored form involved only integers. Some polynomials that are not factorable over the integers are factorable over the rational numbers.

Example 2.9

Factor $\frac{1}{6}x^2 - \frac{1}{6}x + 1$.

Solution

The easiest way to accomplish this is to rewrite

$$\frac{1}{6}x^2 - \frac{1}{6}x + 1$$

as

$$\frac{1}{6}(x^2 - x + 6).$$

Then,

$$\begin{aligned}\frac{1}{6}x^2 - \frac{1}{6}x + 1 &= \frac{1}{6}(x^2 - x + 6)\\ &= \frac{1}{6}(x-3)(x+2).\end{aligned}$$

Example 2.10

Factor $x^2 - \frac{5}{6}x + \frac{1}{6}$.

Solution

$$x^2 - \frac{5}{6}x + \frac{1}{6} = \frac{1}{6}(6x^2 - 5x + 1)$$

$$= \frac{1}{6}(3x-1)(2x-1) \tag{2.1}$$

$$= \left(x - \frac{1}{3}\right)\left(x - \frac{1}{2}\right). \tag{2.2}$$

Either Equation (2.1) or Equation (2.2) is an acceptable form. ■

There are polynomials that are not factorable over the rational numbers, but that can be factored using irrational numbers.

Example 2.11

Factor $x^2 - 2\sqrt{3} + 3$.

Solution

Since $3 = (\sqrt{3})^2$, we can write 3 as the square of the number $\sqrt{3}$. Then,

$$x^2 - 2\sqrt{3} + 3 = (x - \sqrt{3})^2.$$

Example 2.12

Factor $x^2 - \sqrt{3}x - \sqrt{2}x + \sqrt{6}$.

Solution

The technique of factoring by grouping comes to our rescue.

$$x^2 - \sqrt{3}x - \sqrt{2}x + \sqrt{6} = x(x - \sqrt{3}) - \sqrt{2}(x - \sqrt{3})$$
$$= (x - \sqrt{2})(x - \sqrt{3}).$$

Exercise 2.1

Factor each of the following in the real numbers.

1. $\frac{1}{6}x^2 - \frac{2}{3}x + \frac{1}{2}$
2. $x^2 + \frac{5}{3}x + \frac{2}{3}$
3. $x^2 + \frac{1}{4}x - \frac{1}{8}$
4. $x^2 + x + \frac{1}{4}$
5. $x^2 - \frac{1}{25}$
6. $\frac{x^2}{3} + \frac{x}{3} + \frac{1}{12}$
7. $x^2 + \sqrt{3}x - \sqrt{2}x - \sqrt{6}$
8. $x^2 - \sqrt{5}x - \sqrt{2}x + \sqrt{10}$
9. $x^2 - \sqrt{2}x - x + \sqrt{2}$
10. $x^2 - 2\sqrt{2}x + 2$
11. $x^2 + 2\sqrt{5}x + 5$
12. $x^2 - 5$
13. $x^2 - 7$
14. $x^2 - \sqrt{11}x + \sqrt{7}x - \sqrt{77}$
15. $x^2 - \sqrt{7}x - \sqrt{2}x + \sqrt{14}$
16. $x^2 - 2\sqrt{2}x - 2x + 4\sqrt{2}$

Answers to Exercise 2.1

(1) $\frac{1}{6}(x-1)(x-3)$ **(2)** $\left(x+\frac{2}{3}\right)(x+1)$ **(3)** $\left(x-\frac{1}{4}\right)\left(x+\frac{1}{2}\right)$

(4) $\left(x+\frac{1}{2}\right)^2$ **(5)** $\left(x-\frac{1}{5}\right)\left(x+\frac{1}{5}\right)$ **(6)** $\frac{1}{3}\left(x+\frac{1}{2}\right)^2$

(7) $(x-\sqrt{2})(x+\sqrt{3}$ **(8)** $(x-\sqrt{5})(x-\sqrt{2})$ **(9)** $(x-\sqrt{2})(x-1)$

(10) $(x-\sqrt{2})^2$ **(11)** $(x+\sqrt{5})^2$ **(12)** $(x-\sqrt{5})(x+\sqrt{5})$

(13) $(x-\sqrt{7})(x+\sqrt{7})$ **(14)** $(x-\sqrt{11})(x+\sqrt{7})$ **(15)** $(x-\sqrt{2})(x-\sqrt{7})$

(16) $(x-2)(x-\sqrt{8})$

Chapter 3

Quadratic Equations

A polynomial equation has a polynomial on one side and a constant on the other. For example, the linear equations $2x+3=5$ and $5x-8=0$ are each polynomial equations. The equations $2x^2+7x+6=0$ and $x^3-x^2+5x+7=0$ are also polynomial equations. The degree of a polynomial equation that contains only one unknown is equal to the greatest exponent that occurs on the unknown. Table (3.1) shows some polynomial equations and their degree.

Equation	Degree	Common Name
$3x+7=0$	1	linear
$5x^2+3x-17=0$	2	quadratic
$5x^3+3x^2-8x+1=0$	3	cubic
$x^4+4x^3-9x^2+x-7=0$	4	quartic
$7x^5-3x^4+7x^3-2x^2-5x+17=0$	5	quintic

TABLE 3.1. Polynomial equations in one unknown

Our study in this chapter is limited to quadratic equations in one unknown.

Definition 3.1 (Quadratic equation)
A quadratic equation in one unknown is any equation that can be written in the form

$$ax^2+bx+c=0,$$

where a, b and c are any numbers and $a \neq 0$. ■

There are several methods of solving a quadratic equation. All methods are based on the following fact.

Theorem 3.1
Let a and b represent any numbers. If $ab = 0$ then $a = 0$ or $b = 0$. ■

3.1. Solve by factorization

Several examples illustrate using Theorem (3.1) to solve quadratic equations by factoring.

Example 3.1

Solve $x^2 + 5x + 6 = 0$.

Solution.

$$\begin{aligned} x^2 + 5x + 6 &= 0 \\ (x+2)(x+3) &= 0. \end{aligned}$$

Using Theorem (3.1),

$$\begin{aligned} x + 2 = 0 &\quad \text{or} \quad x + 3 = 0. \\ \therefore\ x = -2 &\quad \text{or} \quad x = -3. \end{aligned}$$

Example 3.2

Solve $x^2 + 2x - 15 = 0$.

Solution.

$$\begin{aligned} x^2 + 2x - 15 &= 0 \\ (x-3)(x+5) &= 0. \end{aligned}$$

Using Theorem (3.1),

$$\begin{aligned} x - 3 = 0 &\quad \text{or} \quad x + 5 = 0. \\ \therefore\ x = 3 &\quad \text{or} \quad x = -5. \end{aligned}$$

Example 3.3

Solve $x^2 - 25 = 0$.

Solution.

$$\begin{aligned} x^2 - 25 &= 0 \\ (x-5)(x+5) &= 0. \end{aligned}$$

Then,

$$\begin{aligned} x-5 = 0 \quad &\text{or} \quad x+5 = 0. \\ \therefore\ x = 5 \quad &\text{or} \quad x = -5. \quad \blacksquare \end{aligned}$$

The next example requires the equation be put in the form required by Theorem (3.1).

Example 3.4

Solve $-8x = -x^2 - 12$.

Solution.

$$\begin{aligned} -8x &= -x^2 - 12 \\ x^2 - 8x + 12 &= 0 \\ (x-2)(x-6) &= 0. \end{aligned}$$

Then,

$$\begin{aligned} x-2 = 0 \quad &\text{or} \quad x-6 = 0. \\ \therefore\ x = 2 \quad &\text{or} \quad x = 6. \quad \blacksquare \end{aligned}$$

The values of the unknown that make a polynomial equation true are usually called the "**solutions**" or the "**roots**" of the equation.

Example 3.5

Find all solutions of $2x^2 - 7x + 3 = 0$.

Solution.

$$\begin{aligned} 2x^2 - 7x + 3 &= 0 \\ (2x-1)(x-3) &= 0. \end{aligned}$$

Then,

$$2x-1=0 \quad \text{or} \quad x-3=0.$$
$$\therefore\ x=\tfrac{1}{2} \quad \text{or} \quad x=3.$$

Example 3.6

Find all roots of $x^2-4x+4=0$.

Solution.

$$x^2-4x+4=0$$
$$(x-2)^2=0.$$

Then,

$$x-2=0.$$
$$\therefore\ x=2.$$

Example (3.6) shows that sometimes both solutions are identical. In such a case we say the equation has "**one root multiplicity two**."

Example 3.7

Find all roots of $x^2=7$.

Solution.

$$x^2=7$$
$$x^2-7=0$$
$$(x+\sqrt{7})(x-\sqrt{7})=0$$

Then,

$$x+\sqrt{7}=0 \quad \text{or} \quad x-\sqrt{7}.$$
$$\therefore\ x=-\sqrt{7} \quad \text{or} \quad x=\sqrt{7}.$$

3.1.1. Equations quadratic in form

Some polynomial equations of degree greater than 2 are quadratic in form. You have already practiced factoring polynomials that are quadratic in form.

So, solving equations of higher degree that are quadratic in form by factoring will seem natural to you. Before we do this, however, we must extend Theorem (3.1) to cover any number of factors.

Theorem 3.2
Let $a_1, a_2, \cdots, a_n$ represent n-number of factors. If $a_1 \cdot a_2 \cdot \ldots \cdot a_n = 0$ then $a_1 = 0$ or $a_2 = 0$ or $\ldots$ or $a_n = 0$.

Example 3.8

Find all solutions of $x^4 - 16 = 0$.

Solution.

$$\begin{aligned} x^4 - 16 &= 0 \\ (x^2-4)(x^2+4) &= 0. \\ (x-2)(x+2)(x^2+4) &= 0. \end{aligned}$$

Then,

$x - 2 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x^2 + 4 = 0.$

So,

$$x = 2 \quad \text{or} \quad x = -2 \quad \text{or} \quad \{\,\} \quad (x^2+4 \text{ does not factor in the real numbers}).$$

The equation has two roots in the real numbers.

$$x = -2 \quad \text{or} \quad x = 2.$$

Example 3.9

Find all roots of $(x+2)^2 + 5(x+2) + 6 = 0$.

Solution.

$$\begin{aligned} (x+2)^2 + 5(x+2) + 6 &= 0 \\ ((x+2)+2)((x+2)+3) &= 0. \\ (x+4)(x+5) &= 0. \end{aligned}$$

Then,

$$x+4=0 \quad \text{or} \quad x+5=0.$$
$$\therefore\ x=-4 \quad \text{or} \quad x=-5.$$

3.2. Solve by completing the square

Our success solving quadratic equations has depended on our discovering the factorization of a polynomial. But, some polynomials are inconvenient to factor. Others are just plain hard to factor. For example, the equation

$$x^2-5x-6=0 \tag{3.1}$$

is easy to solve, because

$$x^2-5x-6$$

is simple to factor.

$$x^2-5x-6=0$$
$$(x-3)(x-2)=0$$
$$\therefore\ x=3 \quad \text{or} \quad x=2.$$

On the other hand, merely changing the last term of Equation (3.1) from 6 to 7 produces

$$x^2-5x-7=0 \tag{3.2}$$

which factors as

$$\left(x-\frac{5-3\sqrt{53}}{2}\right)\left(x-\frac{5+3\sqrt{53}}{2}\right)=0.$$

Hardly anyone's idea of a good time. Despair not. The procedure called "completing the square" makes solving an equation such as Equation (3.2) a routine matter.

3.2.1. Producing a square trinomial

The point of this subsection is to practice one of the several steps of completing the square. We shall then return to the technique of completing the square.

The trinomial x^2+6x+9 is a square trinomial. Notice that the constant term 9 is $(\text{one half of } 6)^2$. Another square trinomial is $x^2+12x+36$. The constant 36 is $\left(\frac{1}{2}\cdot 12\right)^2$.

What we observe in these two examples is true for *all* square trinomials whose leading coefficient is 1. That is

$$x^2 + bx + \left(\frac{b}{2}\right)^2.$$

This is no surprise when we think of producing a square trinomial:

$$(x+b)^2 = x^2 + 2bx + b^2.$$

The constant term is $(\text{one half of } 2b)^2$.

Example 3.10

For each of the following, supply the constant term that results in a square trinomial. Then write the trinomial in its factored form.

(1) $x^2 + 8x + \_\_\_\_$

(2) $x^2 + 10x + \_\_\_\_$

(3) $x^2 - 6x + \_\_\_\_$

(4) $x^2 - 8x + \_\_\_\_$

(5) $x^2 - 12x + \_\_\_\_$

(6) $x^2 + 5x + \_\_\_\_$

(7) $x^2 - x + \_\_\_\_$

(8) $x^2 - \frac{x}{2} + \_\_\_\_$

Solution

(1) $x^2 + 8x + \mathbf{16} = (x+4)^2$

(2) $x^2 + 10x + \mathbf{25} = (x+5)^2$

(3) $x^2 - 6x + \mathbf{9} = (x-3)^2$

(4) $x^2 - 8x + \mathbf{16} = (x-4)^2$

(5) $x^2 - 12x + \mathbf{36} = (x-6)^2$

(6) $x^2 + 5x + \mathbf{\frac{25}{4}} = \left(x + \frac{5}{2}\right)^2$

(7) $x^2 - x + \mathbf{\frac{1}{4}} = \left(x - \frac{1}{2}\right)^2$

(8) $x^2 - \frac{x}{2} + \mathbf{\frac{1}{16}} = \left(x - \frac{1}{4}\right)^2$

Exercise 3.1

For each of the following, supply the constant term that results in a square trinomial. Then write the trinomial in its factored form.

1. $x^2 + 14x + \_\_\_\_$
2. $x^2 - 20x + \_\_\_\_$
3. $x^2 - 16x + \_\_\_\_$
4. $x^2 + 7x + \_\_\_\_$
5. $x^2 - 9x + \_\_\_\_$
6. $x^2 + \frac{2x}{3}x + \_\_\_\_$

Answers to Exercise 3.1

(**1**) $x^2 + 14x + 49 = (x+7)^2$ (**2**) $x^2 - 20x + 100 = (x-10)^2$

(**3**) $x^2 - 16x + 64 = (x-8)^2$ (**4**) $x^2 + 7x + \frac{49}{4} = \left(x + \frac{7}{2}\right)^2$

(**5**) $x^2 - 9x + \frac{81}{4} = \left(x - \frac{9}{2}\right)^2$ (**6**) $x^2 + \frac{2x}{3}x + \frac{1}{9} = \left(x + \frac{1}{3}\right)^2$

3.2.2. Factor by completing the square

Knowing how to select the constant term to create a square trinomial as in Example (3.10) and Exercise (3.1), is a valuable tool for factoring an expression that appears otherwise hopeless.

Before we try our new tool on a seemingly hopeless case, we will practice using it to factor several expressions whose factorization we already know. We know $x^2 - 6x - 16 = (x-8)(x+2)$.

Example 3.11

Factor $x^2 - 6x - 16$ by completing the square.

Solution

Adding 9 to $x^2 - 6x$ would produce the square trinomial $x^2 - 6x + 9$. We add and subtract 9.

$$x^2 - 6x - 16 = x^2 - 6x + 9 - 9 - 16 \tag{3.3}$$

The square trinomial $x^2 - 6x + 9$ factors into $(x-3)^2$. Also, $-9 - 16 = -25$.

$$= (x-3)^2 - 25$$

This is the difference of squares. So,

$$= ((x-3)-5)((x-3)+5)$$
$$= (x-8)(x+2),$$

as expected. ■

We added and subtracted 9 in Equation (3.3). You have on many occasions made good use of multiplying by 1, the multiplicative identity. This example used "the ol' add and subtract trick". No trick, really, we added 0, the additive identity element.

We know that $x^2+7x+12 = (x+3)(x+4)$.

Example 3.12

Factor $x^2+7x+12$ by completing the square.

Solution

$$x^2+7x+12 = x^2+7x+\frac{49}{4}-\frac{49}{4}+12 \qquad \left(\text{note: } \frac{49}{4} = \left(\frac{7}{2}\right)^2\right)$$
$$= \left(x+\frac{7}{2}\right)^2-\frac{49}{4}+12$$
$$= \left(x+\frac{7}{2}\right)^2-\frac{1}{4}$$
$$= \left(\left(x+\frac{7}{2}\right)-\sqrt{\frac{1}{4}}\right)\left(\left(x+\frac{7}{2}\right)+\sqrt{\frac{1}{4}}\right)$$
$$= \left(\left(x+\frac{7}{2}\right)-\frac{1}{2}\right)\left(\left(x+\frac{7}{2}\right)+\frac{1}{2}\right)$$
$$= (x+3)(x+4) \quad ■$$

Example (3.12) has probably convinced you that you would need to be desperate before turning to completing the square. But if you need to factor

$$x^2-3x-1,$$

you will soon be desperate enough to use completing the square.

Example 3.13

Factor $x^2 - 3x - 1$.

Solution

$$\begin{aligned}
x^2 - 3x - 1 &= x^2 - 3x + \tfrac{9}{4} - \tfrac{9}{4} - 1 \qquad \left(\text{note: } \tfrac{9}{4} = \left(\tfrac{3}{2}\right)^2\right) \\
&= (x - \tfrac{3}{2})^2 - \tfrac{5}{4} \\
&= \left(\left(x - \tfrac{3}{2}\right) - \sqrt{\tfrac{5}{4}}\right)\left(\left(x - \tfrac{3}{2}\right) + \sqrt{\tfrac{5}{4}}\right) \\
&= \left(\left(x - \tfrac{3}{2}\right) - \tfrac{\sqrt{5}}{2}\right)\left(\left(x - \tfrac{3}{2}\right) + \tfrac{\sqrt{5}}{2}\right) \\
&= \left(x - \tfrac{3+\sqrt{5}}{2}\right)\left(x - \tfrac{3-\sqrt{5}}{2}\right).
\end{aligned}$$

We got started on completing the square when we considered solving Equation (3.2) on page 14. We solve this equation in Example (3.14).

Example 3.14

Solve $x^2 - 5x - 7 = 0$.

Solution

$$\begin{aligned}
x^2 - 5x - 7 &= 0 \\
x^2 - 5x + \tfrac{25}{4} - \tfrac{25}{4} - 7 &= 0 \qquad \left(\text{note: } \tfrac{25}{4} = \left(\tfrac{5}{2}\right)^2\right) \\
\left(x - \tfrac{5}{2}\right)^2 - \tfrac{53}{4} &= 0 \\
\left(x - \tfrac{5}{2} - \tfrac{\sqrt{53}}{2}\right)\left(x - \tfrac{5}{2} + \tfrac{\sqrt{53}}{2}\right) &= 0 \\
\left(x - \tfrac{5+\sqrt{53}}{2}\right)\left(x - \tfrac{5-\sqrt{53}}{2}\right) &= 0 \\
x - \tfrac{5+\sqrt{53}}{2} = 0 \quad \text{or} \quad x - \tfrac{5-\sqrt{53}}{2} &= 0 \\
\therefore\ x = \tfrac{5+\sqrt{53}}{2} \quad \text{or} \quad x = \tfrac{5-\sqrt{53}}{2}&.
\end{aligned}$$

Index