Chapter 6

Linear Function

6.1. Introduction

We begin recalling what is by now a familiar object, the linear equation. An equation is linear if it can be written in the form

$$ax+b=c$$

where x, a, b and c are any numbers and $a \neq 0$.

As you know, when a, b and c are given, there is exactly one value for the unknown x that makes a linear equation true. For example: if $3x+7=5$, then only $x=\frac{-2}{3}$ makes the equation a true statement.

We now wish to investigate the equation

$$ax+b=y$$

where a and b are any constants with $a \neq 0$ and x and y are variables. The meanings of "constants" and "variables" will become clear to you in this chapter. Here is an example of such an equation:

$$y=2x+7, \tag{6.1}$$

where x and y are any numbers that make the equation true.

Instead of discussing Equation (6.1), let us take the simpler equation

$$y=2x \tag{6.2}$$

Equation (6.2) is unusual compared to equations you have seen, because it appears to contain two unknowns instead of just one. Equation (6.2) means that $2x$ and y are names for the same number. Maybe you are wondering "How do I solve it?"

Substituting $x=3$ and $y=6$, the RHS is 6 and the LHS is 6. Since LHS = RHS, the pair of numbers $(x=3, y=6)$ make Equation (6.2) a true statement. Therefore, the pair of numbers $(x=3, y=6)$ is a solution of Equation (6.2).

But that is not all! The pairs

$$(x = 2, y = 4),$$
$$(x = 5, y = 10),$$
$$(x = 6, y = 12),$$
$$(x = \frac{1}{2}, y = 1),$$
$$(x = \frac{2}{3}, y = \frac{4}{3}),$$
$$(x = \frac{3}{7}, y = \frac{6}{7}),$$

are all solutions too.

Perhaps by now you are thinking that there is no end to the pairs of numbers that make Equation (6.2) true. Correct! There are infinitely many pairs of numbers that solve Equation (6.2).

One cannot help but notice how easy it is to to find solutions to Equation (6.2). Just substitute a particular number for x, then multiply by 2 and the corresponding value for y appears.

6.1.1. Seriously boring

The author is the first to admit that solving Equation (6.2) is not interesting. Worse, the task is unending. Find a million solutions for Equation (6.2) and there will remain infinitely more solutions for you to find.

6.1.2. So what is interesting?

Equation (6.2) shows how two numbers are related and it shows how to find all and only those pairs of numbers that have that relationship.

6.2. The idea of a function: it's about the relationship!

Equation (6.2) is interesting when our attention shifts from the pairs of numbers that satisfy it to the relationship of each pair's numbers. The relationship, when one looks at Equation (6.2) is crystal clear: the value of y is double the value or x.

It is easy to find pairs of numbers that satisfy Equation (6.2), because the recipe for making them is right in front of our eyes. Just substitute a particular number for x and multiply by 2.

Finding pairs of numbers that make $y = 2x$ true just amounts to picking any old number for x, then following the rule *multiply by 2*. Now, you know how your calculator feels!

For any value of x, the rule produces the one value of y that makes $y = 2x$ true. To express this idea, we say $y = 2x$ gives "y as a *function* of x." We call

$y = 2x$ a *function*; we call y the *dependent variable* and x the *independent variable*. The "2" is called a constant, because value of "2" does not vary.

For the time being, this is a good enough idea of what a function is that we use it to define the idea of a function.

Definition 6.1 (Function)
A **function** is a rule that shows how the value of one variable, called the **dependent variable**, is uniquely determined by the value of another variable, called the **independent variable**.

Remark 6.1
The independent variable is often called the "argument of the function" and the dependent variable is often called the "value of the function".

Remark 6.2
Not every equation in which two variables appear is a function. Only equations in which the dependent variable is *uniquely* determined by the independent variable are of the special kind called functions. For example, the equation $x + y^2 = 5$ does *not* determine a unique value of y for every value of x. When $x = 1$, y can equal either 2 or -2 and equation $x + y^2 = 5$ will be true. So in $x + y^2 = 5$, y is not a function of x.

Definition 6.2 (Linear function)
A function that can be written in the form

$$y = ax + b, a \neq 0,$$

is a **linear function.** ■

6.3. Ordered pair

On page 147, we listed several pairs of values for x and y that made the equation $y = 2x$ true. We wrote, for example, $(x = 2,\ y = 4)$ and $(x = 5,\ y = 10)$. If we agree that within each parenthesis, the value of the independent variable will be first and the value of the dependent variable second, then there is no need to keep writing "$x = \ ,\ y =$ ". Writing "$(2,4)$" is enough, because the order indicates that the value of the dependent variable is equal to 4 when the value of the independent variable is 2. We call pairs such as $(2,4)$ "ordered" pairs.

Definition 6.3 (Ordered pair)
A pair of numbers within parenthesis such as (a,b) is called an **ordered pair**. In (a,b), a is called the **first element** of the pair and b is called the **second element** of the pair. The number b is the value of the dependent variable corresponding to the value a of the independent variable. ■

Remark 6.3
The order of the elements in an ordered pair matters. The ordered pairs $(2,7)$ and $(7,2)$ are not the same. The ordered pair $(2,7)$ satisfies $3x+1=y$, but the ordered pair $(7,2)$ does not.

Example 6.1

Suppose y is a function of x such that $y=2x$. Write any five ordered pairs whose elements satisfy $y=2x$.

Solution

Five such pairs are $(-2,-4),(0,0),(3,6),(\frac{9}{5},\frac{18}{5}),(100,200)$.

Example 6.2

Suppose t is a function of s such that $t=2+s$. Write any five ordered pairs whose elements satisfy $t=2+s$.

Solution

Five such pairs are $(-5,-3),(-1,1),(8,10),(\frac{1}{2},\frac{5}{2}),(\frac{3}{4},\frac{11}{4})$.

Example 6.3

Suppose y is a function of x such that $y=5+x$. Write the pairing of x and y as ordered pairs for $x=-2,-1,0,\frac{2}{5},4$.

Solution

$(-2,3),(-1,4),(0,5),(\frac{2}{5},\frac{27}{5}),(4,9)$.

Example 6.4

Suppose the values of x and y are paired by a certain function as follows: $(1,3),(2,6),(3,9),(4,12)$. Write y as a function of x.

Solution

Notice that the second element (value of dependent variable) of each pair is 3 times the first element (value of independent variable). So the function is $y=3x$.

Example 6.5

Given the function $t = 5s$, which of the following ordered pairs is not (are not) produced by this function? The candidates are: $(-3,-15),(-1,-5)$, $(0,0),(2,11),(4,20),(\frac{3}{7},\frac{15}{7}),(6,32)$.

Solution

The function $t = 5s$ requires that the second element be 5 times the first element. But, $11 \neq 2 \cdot 5$ and $32 \neq 6 \cdot 5$. So (2,11) and (6,32) are not produced by the function $t = 5s$.

Exercise 6.1

1. Suppose y is a function of x such that $y = 4x$. Write the pairing of x and y as ordered pairs for $x = -3,-2,0,1,\frac{3}{8},5$.
2. Suppose t is a function of s such that $t = s+3$. Write the pairing of s and t as ordered pairs for $s = 0,1,2,3,\frac{10}{3}$.
3. Suppose z is a function of w such that $z = 3w$. Write the pairing of w and z as ordered pairs for $w = \frac{-2}{3},-1,0,1,2$
4. Could the ordered pair $(2,9)$ be produced by the function $y = 3x$?
5. Could the ordered pair $(-1,5)$ be produced by the function $y = x-2$?
6. Suppose the values of x and y are paired by a certain function as follows: $(-2,-10),(0,0),(2,10),(3,15)$. Write y as a function of x.
7. Suppose the values of x and y are paired by a certain function as follows: $(1,5),(2,6),(3,7),(4,8)$. Write y as a function of x.
8. Suppose the values of s and t are paired by a certain function as follows: $(-2,10),(-1,5),(0,0),(1,-5),(2,-10)$. Write t as a function of s.
9. Show that the ordered pairs $(2,6)$ and $(6,2)$ are not the same.

6.4. Domain and Range

There are many ideas associated with functions. Two of these ideas are "domain" and "range".

Definition 6.4 (Domain)
The set of numbers whose members serve as the arguments of the function is called the **domain** of the function.

Definition 6.5 (Range)
The set of numbers that the function takes as values is called the **range** of the function.

Remark 6.4
The domain of the function is not always stated. If it is not, then the domain is taken to be the largest set of numbers for which the function makes mathematical sense. If a number would cause a division by zero, that number is excluded from the domain.

Example 6.6

Suppose $y = 2x$ and the domain of the function is $\{0, 1, 3, 5, 7\}$. What is the range of the function?

Solution

When $x = 0, y = 0$. When $x = 1, y = 2$. When $x = 3, y = 6$. When $x = 5, y = 10$. When $x = 7, y = 14$. So the range is $\{0, 2, 6, 10, 14\}$.

Example 6.7

Suppose $y = 2x$ and the domain of the function is $\{0, 2, 4, 6, 8\}$. What is the range of the function?

Solution

When $x = 0, y = 0$. When $x = 2, y = 4$. When $x = 4, y = 8$. When $x = 6, y = 12$. When $x = 8, y = 16$. So the range is $\{0, 4, 8, 12, 16\}$.

Example 6.8

Suppose $y = 2x$ and the domain of the function is all numbers between 0 and 5. What is the range of the function?

Solution

When $x = 0, y = 0$. When $x = 5, y = 10$. Since x takes all values between 0 and 5, y takes all values between 0 and 10. So, the range is all numbers between 0 and 10 including 0 and 10. ■

If we know the ordered pairs produced by a function, we know the domain and the range. Remember, the domain is the set of values the argument takes and these will be the first elements of the ordered pairs. The range is the set of values the function produces and these will be the second elements of the ordered pairs.

Example 6.9

Suppose a function produces these, and only these, ordered pairs:

$$(-1,-3),(0,0),(2,6),(10,30).$$

What is the domain and what is the range of this function?

Solution

The domain is $\{-1,0,2,10\}$ and the range is $\{-3,0,6,30\}$.

Exercise 6.2

1. Suppose that $y = x - 5$ domain $\{0,4,8,11\}$. What is the range of $y = x - 5$?
2. If $z = 10w$ domain $\{-3,-1,0,2,9\}$, what is the range?
3. If $t = s + 5$ domain $\{0, \frac{1}{3}, \frac{1}{2}, 1, 3\}$, what is the range?
4. What is the range of the function $y = 3x + 1$ when the domain is $\{-2,-1,0,1,2\}$?
5. What is the range of the function $y = 3x + 1$ when the domain is $\{3,4,5,6\}$?
6. What is the range of the function $y = 3x + 1$ when the domain is $\{7,8,9,10,11\}$?
7. A function produces all and only the following pairs
$$(1,\frac{2}{3}),(2,\frac{3}{4}),(3,7),(5,11),(12,27).$$
What is the domain and range of the function?

6.5. Idea of function as correspondence

We have said that a function shows how a number in a set called “the range of the function” is determined by another number in a set called “the domain of the function”. The correspondence of numbers in the domain with numbers in the range is shown in Figure (6.1) for the function $y = 2x$. The picture is drawn for only a few of the numbers in each set.

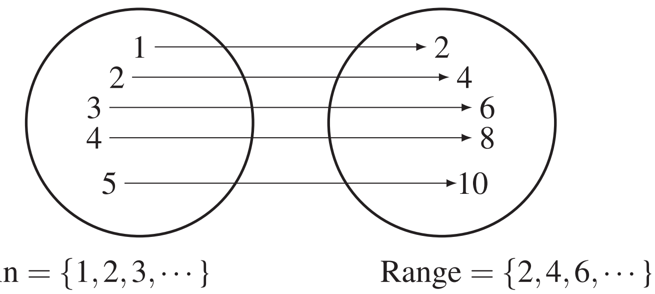


FIGURE 6.1. Domain and Range of $y = 2x$

Mathematicians often find it useful to emphasize that a function shows the correspondence of members of sets. You too will find this idea useful in subsequent mathematics courses. To write “y corresponds to x”, we use an arrow:

$$x \longrightarrow y.$$

We can write the correspondences shown in Figure (6.1) like this:

$$\begin{aligned} 1 &\longrightarrow 2 \\ 2 &\longrightarrow 4 \\ 3 &\longrightarrow 6 \\ 4 &\longrightarrow 8 \\ 5 &\longrightarrow 10 \end{aligned}$$

When we write, for example, $1 \longrightarrow 2$, we should realize that the function *pairs* 1 with 2, 2 with 4, and so on. Writing $(1,2)$ expresses the same idea as writing $1 \longrightarrow 2$. Therefore,

$$\begin{aligned} 1 \longrightarrow 2 &\quad \text{is equivalent to} \quad (1,2) \\ 2 \longrightarrow 4 &\quad \text{is equivalent to} \quad (2,4) \\ 3 \longrightarrow 6 &\quad \text{is equivalent to} \quad (3,6) \\ 4 \longrightarrow 8 &\quad \text{is equivalent to} \quad (4,8) \\ 5 \longrightarrow 10 &\quad \text{is equivalent to} \quad (5,10) \end{aligned}$$

Remark 6.5
The idea of function was refined over a period of decades as people acquired deeper and deeper insight into what a function is. As a consequence of that history, the language used in discussing functions is rich (read "confusing to a beginner"). You might as well know now that "$x \longrightarrow y$" is also read, "the function sends x to y" or "the function maps x to y". For example, the function $y = 2x$ maps 1 to 2. Or, equally correct, the function $y = 2x$ sends 1 to 2.

Example 6.10

For the function $y = 3x + 4$ domain $\mathbb{Z}$, use the "$\longrightarrow$" notation to write $x \longrightarrow y$ for $x = -8, -5, -1, 0, 1, 4$.

Solution

$$\begin{aligned} -8 &\longrightarrow -20 \\ -5 &\longrightarrow -11 \\ -1 &\longrightarrow 1 \\ 0 &\longrightarrow 4 \\ 1 &\longrightarrow 7 \\ 4 &\longrightarrow 16 \end{aligned}$$

Example 6.11

Suppose that the domain of a function is $\{-3, -2, -1, 0, 2, 5\}$ and that the function maps -3 to -12, -2 to -8, -1 to -4, 0 to 0, 2 to 8, 5 to 20. Write the function and state its range.

Solution

We are given that

$$\begin{aligned} -3 &\longrightarrow -12 \\ -2 &\longrightarrow -8 \\ -1 &\longrightarrow -4 \\ 0 &\longrightarrow 0 \\ 2 &\longrightarrow 8 \\ 5 &\longrightarrow 20 \end{aligned}$$

Noticing that -12 is 4 times -3, -8 is 4 times -2 and so on, the function is $y = 4x$. The range is $\{-12, -8, -4, 0, 8, 20\}$.

Exercise 6.3

1. For the function $y = 5x$ with domain $\{1, 2, 3, \cdots\}$, draw a picture like Figure (6.1) for elements $\{1, 2, 3, 4\}$ of the domain.
2. For the function $y = 5 - x$ with domain $\{\cdots, -3 - 2, -1, 0, 1, 2, 3, \cdots\}$, draw a picture like Figure (6.1) for elements $\{-5, -1, 0, 1, 7\}$ of the domain.
3. Use the "$\longrightarrow$" notation to express the idea that 2 corresponds to 7, 3 corresponds to 10, and 5 corresponds to 16.
4. For the function $y = 2x + 3$ domain $\mathbb{Z}$, use the "$\longrightarrow$" notation to write $x \longrightarrow y$ for $x = -7, -2, -1, 0, 1, 3$. [See Example (6.10).]
5. Suppose a function has domain $\{1, 2, 3, 4, 5\}$ and maps 1 to 10, 2 to 20, 3 to 30, 4 to 40, and 5 to 50. Write the function and state its range. [See Example (6.11).]

6.6. Romeo and Juliet

Suppose Romeo is exactly two years younger than Juliet. You would have no trouble telling Romeo how old Juliet will be when he is 13 years old. If Romeo wanted Juliet's age when he is 14, you would say she will be 16. And if he asks, "How old will Juliet be when I am 15?", you would say, "17". If Romeo asks for her age when he is 16, you might begin to get fed up and make a chart for him so that he could just look up Juliet's age without bothering you. You provide the handy pocket chart shown below.

If you object that in fact Romeo was older than Juliet, you are thinking of the wrong Romeo and Juliet. The Romeo and Juliet referred to here live in the Bronx.

Romeo's age in years	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Juliet's age in years	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**

TABLE 6.1. Romeo and Juliet Ages

All is well with Romeo, until one night you get a call because he wishes to know how old Juliet will be when he is 31 years old. Instead of continuing the table, you might simply tell him a rule for finding Juliet's age. "Just add 2 to your age in years to get Juliet's age in years."

Rule for finding Juliet's age, j (years), given Romeo's age, r (years):

(6.3) $$j = r + 2.$$

Table (6.1), Equation (6.3), and the instruction "Just add 2 to Romeo's age in years to get Juliet's age in years," all express the same idea. Each provides Juliet's age, given Romeo's age. In mathematics, we say each tells Juliet's age *as a function of* Romeo's age.

6.6.1. Cute story, but what's the point?

First point. The author was hoping that you would look at Table (6.6) and realize it is just another way of expressing correspondence or pairing. If you did not think of Table (6.6) that way, look at it again and think

$$\begin{array}{lcl} 10 \longrightarrow 12 & \ldots \text{or} \ldots & (10,12) \\ 11 \longrightarrow 13 & \ldots \text{or} \ldots & (11,13) \\ 12 \longrightarrow 14 & \ldots \text{or} \ldots & (12,14) \end{array}$$

and so on up to

$$20 \longrightarrow 22 \quad \ldots \text{or} \ldots \quad (20,22).$$

Second point. We often use a function to accomplish a practical end. Just as Romeo used a function to know Juliet's age, given his age, so also do engineers use functions to predict the deflection of a bridge as a train passes over it, and resource biologists to predict the population of wolves in a region over a period of time. Those are somewhat complicated functions that you will get to know in subsequent mathematics courses.

Example 6.12

Suppose Jane leaves work at 5:00 PM and drives home at a constant speed of 40 mph arriving home at 5:30 PM. Write the distance Jane covers as a function of time. Also, state the domain and range of the function.

Solution

Let d represent distance in miles and t represent time in hours. Then $d = 40t$. The domain is all numbers between 0 and $\frac{1}{2}$. When $t = 0$, $d = 0$ and when $t = \frac{1}{2}$, $d = 20$, so the range is all numbers between 0 and 20.

Example 6.13

John drives at a constant speed of 60 mph. Write the distance d miles he covers as a function of time t hours.

Solution

$d = 60t$.

Remark 6.6
The author knows from experience that the phrases “distance d miles” and “time t hours” confuse beginning algebra students. All that “distance d miles” means is “when you write the function, use d to represent distance and the units are miles”. All that “time t hours” means is “use the letter t to represent time and the units are hours”. In other words, you are just being told what letters and units to use. That’s all there is to it.

Example 6.14

A tank, initially empty, is being filled with water at the rate of 5 gallons per minute. Write the volume V gallons of water in the tank as a function of time t minutes.

Solution

$V = 5t$.

Example 6.15

A tank that initially contained 800 gallons of water is being drained at the rate of 4 gallons per minute. Write the volume V gallons of water in the tank as a function of time t minutes. Also, state the domain and range of the function.

Solution

$V = 800 - 4t$. At time $t = 0$, the tank contains 800 gallons. It will be empty when $4t = 800$. So, the tank will be empty when $t = \frac{800}{4}$ minutes which is 200 minutes. Therefore the domain is all numbers between 0 and 200. The range is all numbers between 0 and 800.

Example 6.16

Make a table that shows the quantity of water in the tank from Example (6.15) at 20 minute intervals.

Solution

t (min)	0	20	40	60	80	100	120	140	160	180	200
V (gal)	800	720	640	560	480	400	320	240	160	80	0

Example 6.17

Barb saves \$400 per month. Write her total saved T dollars as a function of n months.

Solution

$T = 400n.$

Example 6.18

A construction supervisor wishes to know how many lineal feet of sidewalk a crew paves as a function of time. The crew paved 36 feet in 12 hours, 45 feet in 15 hours, and 93 in 31 hours.

Solution

Noting that

$$12 \longrightarrow 36$$
$$15 \longrightarrow 45$$
$$31 \longrightarrow 93$$

The supervisor realizes that in general

$$t \longrightarrow 3t.$$

The supervisor concludes that the function is $L = 3t$ where L represents lineal feet of sidewalk and t represents time in hours.

Example 6.19

The supervisor from Example (6.18) must tell her boss how long the crew will take to pave 306 lineal feet of sidewalk. What should she tell her boss?

Solution

She knows that the quantity of sidewalk paved as a function of time is $L = 3t$ where L is in lineal feet and t is in hours. Substituting 306 for L she obtains the equation $306 = 3t$. After she solves this for t, she tells her boss it will take 102 hours to complete 306 lineal feet of sidewalk.

Exercise 6.4

1. Write the function that gives d miles as a function of t hours, if the constant speed is 60 $^{\text{mi}}/_{\text{hr}}$.

2. A pail that initially contains 2 gallons of water is filled at a rate of 3 $^{\text{gal}}/_{\text{min}}$.
   a) Write the volume V gallons as a function of time t minutes.
   b) Make a table that shows the volume at increments of 2 minutes from 0 minutes up to 12 minutes.
3. Gary's combine harvests soybeans at the rate of 7 $^{\text{acres}}/_{\text{hr}}$.
   a) Write the number of acres A harvested as a function of time t hours.
   b) Find the value of this function at $t = 7$.
4. Jill walks to school at a constant rate of 4 $^{\text{mi}}/_{\text{hr}}$. The distance from her house to school is 0.75 mi.
   a) Write the time remaining, t, in Jill's walk to school as a function of the distance, d, she has walked.
   b) State the domain of the function.
   c) State the range of the function.
5. Suppose that the correspondence between two variables, x and y, is as shown below.

$$\begin{aligned} x &\longrightarrow y \\ 0 &\longrightarrow 0 \\ 1 &\longrightarrow 5 \\ 2 &\longrightarrow 10 \\ 3 &\longrightarrow 15 \\ 4 &\longrightarrow 20 \end{aligned}$$

   a) Write the function as a rule that shows how the value of y is determined by the value of x.
   b) State the domain of the function.
   c) State the range of the function.
   d) Find the value of this function when x is 70.

6.7. Graphs

One of the best ways to acquire an understanding of a particular function is to create a picture of it. That picture is referred to as the graph of the function. We will return to our discussion of functions in a little while. But for now, the topic is graphs.

6.7.1. The coordinate plane

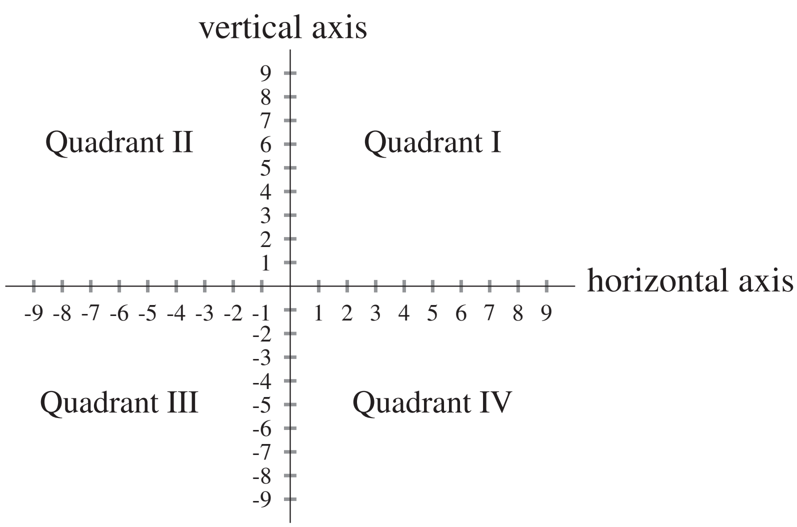


FIGURE 6.2. The coordinate plane

Figure (6.2) shows the coordinate plane. The point at which the axes intersect is called the *origin*. The axes intersect at a right angle. It is understood that each axis continues in both directions in spite of the fact that in this book we do not put arrows on the ends of the axes. The two axes divide the plane into four regions called quadrants. The quadrants are named with roman numerals as in the figure.

Other names for the coordinate plane are the "Cartesian plane", the "Cartesian coordinate system", or the "rectangular coordinate system".

Yes, there are many coordinate systems that are not rectangular.

The exact location of any point in the plane can be given by an ordered pair of numbers that are always written in parentheses. These numbers are called the *coordinates* of the point. The first number of the pair is the horizontal coordinate of the point, the second number the vertical coordinate.

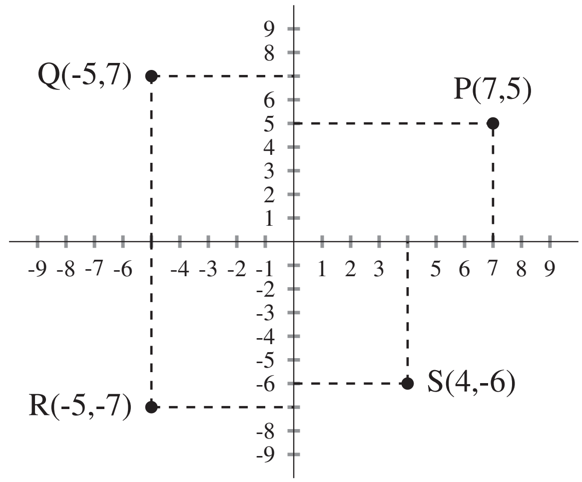


FIGURE 6.3. The coordinate plane

Figure (6.3) shows several points in the plane labeled with their coordinates. The dashed lines show how the coordinates of the points are obtained.

It is worth noting that although we rarely write the numbers on the axes as ordered pairs, we could do so. Figure (6.4) shows how this would look.

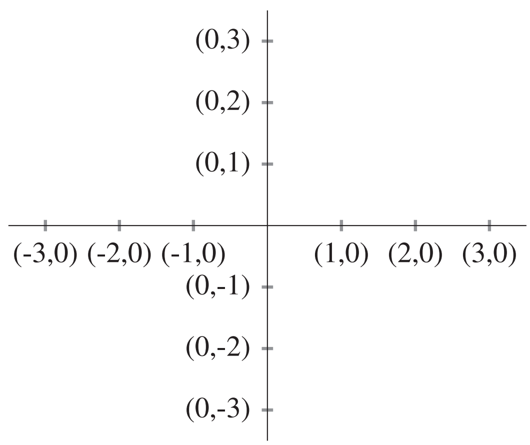


FIGURE 6.4. The coordinates of points on the axes.

The first coordinate of every point on the vertical axis is 0. The second coordinate of every point on the horizontal axis is 0.

Exercise 6.5

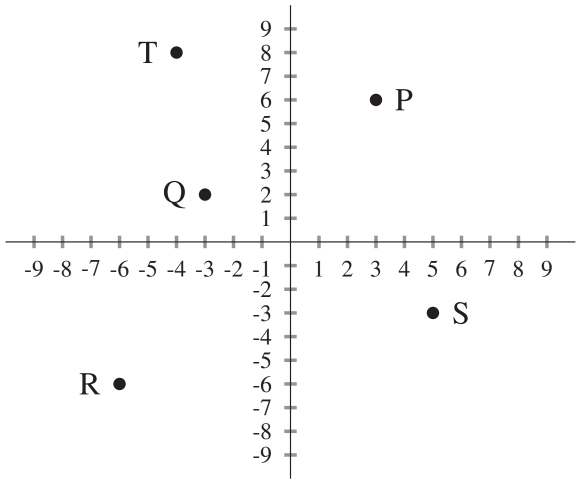


FIGURE 6.5.

1. Based on the appearance of Figure (6.5), state the coordinates of each point. (The first question is answered as an example.)
   a) T. Answer: $(-4,8)$.
   b) P.
   c) Q.
   d) R.
   e) S.
2. In which quadrant does each point in Figure (6.5) lie?
   a) T.
   b) P.
   c) Q.
   d) R.
   e) S.
3. Plot each of the following points.
   a) $A(3,5)$.
   b) $B(3,-5)$.
   c) $C(-3,5)$.
   d) $D(-3,-5)$.
4. Draw the axes. Label the points $-3,-2,-1,1,2,3$ on the horizontal axis and $-3,-2,-1,1,2,3$ on the vertical axis, using ordered pairs.

6.8. The graph of *y = 2x*

Figure (6.6), shows the function

$$y = 2x, \quad \text{domain} = \{-3, -2, -1, 0, 1, 2, 3, 4\}$$

as a correspondence and as a graph.

$$\begin{aligned} x &\longrightarrow y \\ -3 &\longrightarrow -6 \\ -2 &\longrightarrow -4 \\ -1 &\longrightarrow -2 \\ 0 &\longrightarrow 0 \\ 1 &\longrightarrow 2 \\ 2 &\longrightarrow 4 \\ 3 &\longrightarrow 6 \\ 4 &\longrightarrow 8 \end{aligned}$$

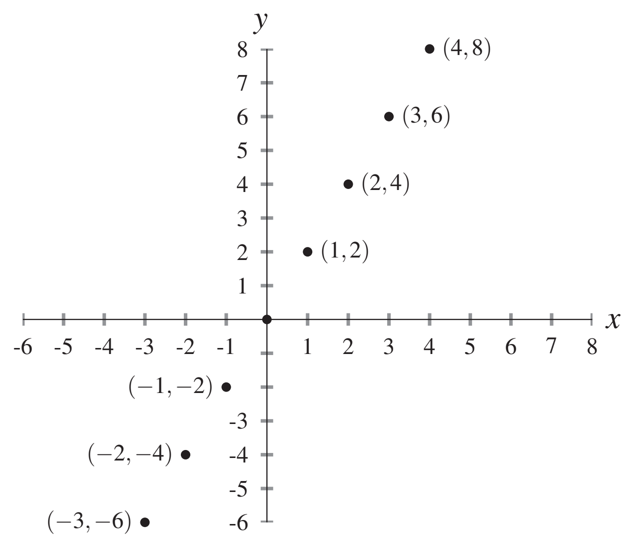


FIGURE 6.6. $y = 2x$, domain $\{-3, -2, -1, 0, 1, 2, 3, 4\}$

Who can help but think, "The points make a straight line!"?

So it seems. But, looks can deceive.

We might try to get a finer view of how the function behaves at values of x between, for example, 0 and 1. We consider the same rule, but with a different domain.

$$y = 2x, \text{ domain} = \{-0.5, -0.375, -0.25, \cdots, 1\}.$$

Figure (6.7) shows the graph. Note that the horizontal (x-axis) is marked every 0.125, the vertical (y-axis) every 0.25.

The evidence that graph of the function $y = 2x$ is a straight line of points is somewhat convincing. But we can do better. We can be certain. But to achieve such certainty, we need a few more ideas. For now we will leave open the question *Do the points produced by the function* $y = 2x$ *all lie on the same straight line?*

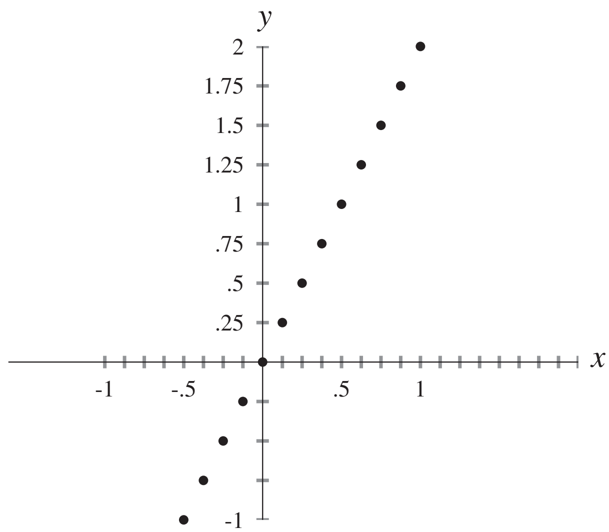


FIGURE 6.7. $y = 2x$, domain $= \{-0.5, -0.375, \cdots, 1\}$

6.8.1. Rise and run

If you study Figure (6.8), you will learn what is meant when we say “rise” and “run”.

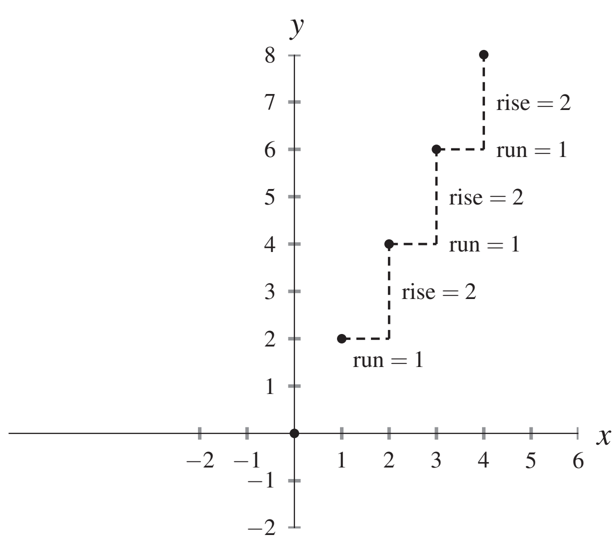


FIGURE 6.8. Graph of $y = 2x$, domain $= \{0, 1, 2, 3, 4\}$

If you have the idea that the run is “how far to the right” and the rise is “how far up”, then you are on the correct track. There is an additional detail that you will discover when you study Figure (6.9).

The additional idea is that rise can be “how far down” in which case it is a negative number. The ratio of rise to run, $\frac{\text{rise}}{\text{run}}$, is a number that plays an

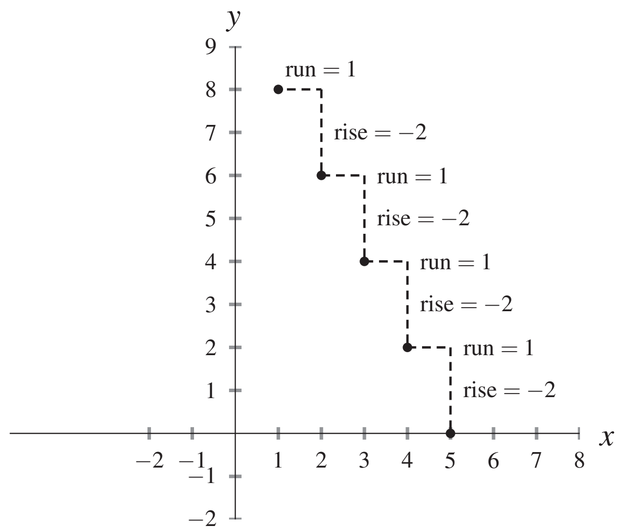


FIGURE 6.9. Graph of $y = 2x$, domain $= \{1, 2, 3, 4, 5\}$

important part in the study of functions. If you examine Figure (6.8), you will see that the $\frac{\text{rise}}{\text{run}} = \frac{2}{1} = 2$. In Figure (6.9), $\frac{\text{rise}}{\text{run}} = \frac{-2}{1} = -2$.

The phrases "How far up" and "How far right" are valuable, because they express our sense of what "rise" and "run" should mean. Let us see if we can say more precisely, though, what our ideas of rise and run are.

Let P and Q be two points whose coordinates are $P(x_1, y_2)$ and $Q(x_2, y_2)$ as shown in Figure (6.10).

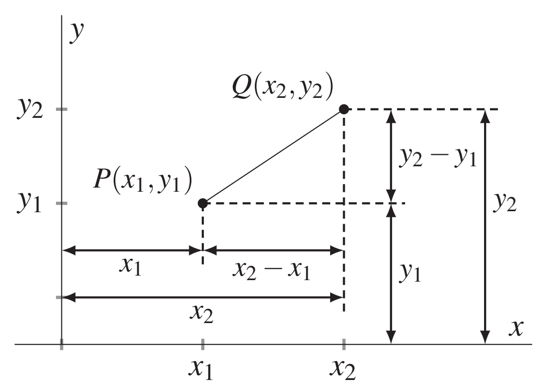


FIGURE 6.10. Computing $\frac{\text{rise}}{\text{run}}$ ratio.

Guided by Figure (6.10), the answer to "How far is Q up from P?" is $y_2 - y_1$. And the answer to "How far right is Q from P?" is $x_2 - x_1$.

Not only have we made our ideas about rise and run more precise, we have even found a way to measure “How far up” and “How far right”. The computations are shown in Equation (6.4) and Equation (6.5).

$$\text{rise} = y_2 - y_1 \tag{6.4}$$

and

$$\text{run} = x_2 - x_1. \tag{6.5}$$

Example 6.20

Find the rise, run, and $\frac{\text{rise}}{\text{run}}$ ratio for the points $P(2,1), Q(7,5)$.

Solution

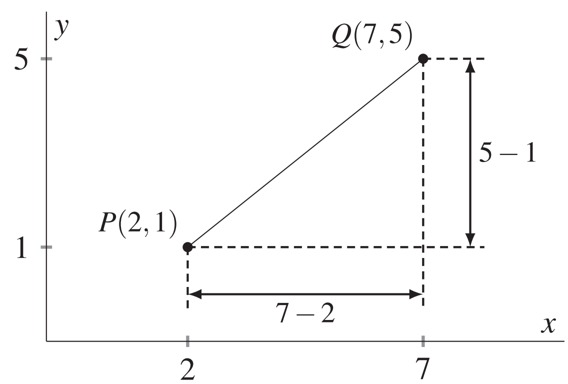


FIGURE 6.11. Example (6.20)

The rise is 4, the run is 5, and $\frac{\text{rise}}{\text{run}} = \frac{4}{5}$. ■

Another way to think through Example (6.20) would be like this: From point P, point Q is 5 right then 4 up, so the run is 5, the rise is 4, and $\frac{\text{rise}}{\text{run}} = \frac{4}{5}$. Figure (6.12) illustrates this.

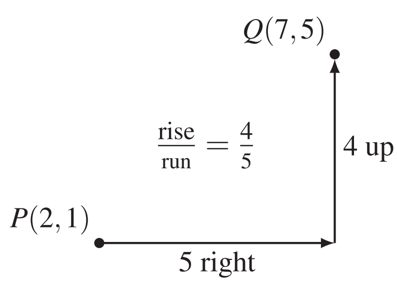


FIGURE 6.12. Another good way to think about Example (6.20)

Example 6.21

Find the ratio $\frac{\text{rise}}{\text{run}}$ for each of the following pairs of points.

(a) $P(-3,-5)$ and $Q(9,2)$
(b) $P(-2,7)$ and $Q(4,-5)$
(c) $P(3,-1)$ and $Q(-4,7)$

Solution

Both ways of thinking will be used for each problem.

(a) Q is 12 right and 7 up from P, so $\frac{\text{rise}}{\text{run}} = \frac{7}{12}$.
(a) $y_2 - y_1 = 2 - (-5) = 7$ and $x_2 - x_1 = 9 - (-3) = 12$. So, $\frac{\text{rise}}{\text{run}} = \frac{7}{12}$.

(b) Q is 6 right and 12 down from P, so $\frac{\text{rise}}{\text{run}} = \frac{-12}{6} = -2$.
(b) $y_2 - y_1 = -5 - 7 = -12$ and $x_2 - x_1 = 4 - (-2) = 6$. So, $\frac{\text{rise}}{\text{run}} = \frac{-12}{6} = -2$.

(c) Q is 7 left and 8 up from P, so $\frac{\text{rise}}{\text{run}} = \frac{8}{-7} = \frac{-8}{7}$.
(c) $y_2 - y_1 = 7 - (-1) = 8$ and $x_2 - x_1 = -4 - 3 = -7$. So, $\frac{\text{rise}}{\text{run}} = \frac{8}{-7} = \frac{-8}{7}$.

Exercise 6.6

[Part 1] *Find the ratio $\frac{rise}{run}$ for each pair of points.*

1. $P(2,5)$ and $Q(6,8)$
2. $P(1,4)$ and $Q(7,2)$
3. $P(-5,8)$ and $Q(3,10)$
4. $P(-2,-5)$ and $Q(2,3)$
5. $P(-1,-7)$ and $Q(-3,-11)$
6. $P(-4,-3)$ and $Q(-2,6)$
7. $P(1,7)$ and $Q(-3,-8)$
8. $P(0,0)$ and $Q(-3,4)$

[Part 2]

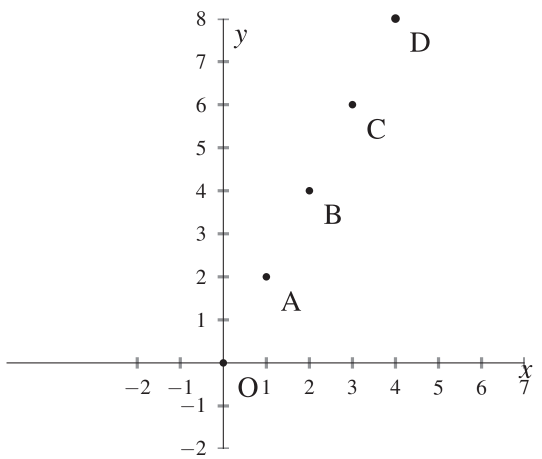


FIGURE 6.13. Graph of $y = 2x$, domain $= \{0,1,2,3,4\}$

1. Based on the appearance of Figure (6.13), complete the following table of the ratio $\frac{\text{rise}}{\text{run}}$ for each pair of points. (The first row is completed as an example.)

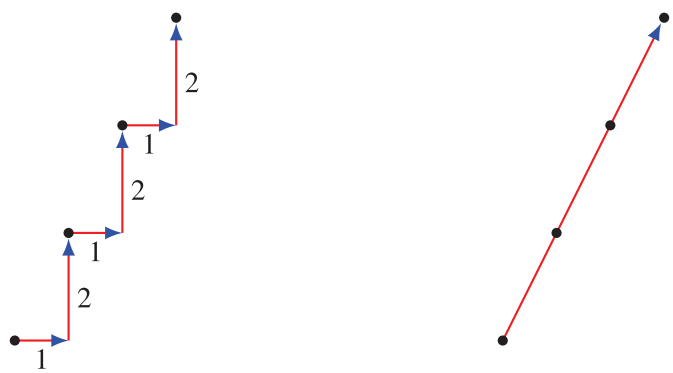
	0	**A**	**B**	**C**	**D**
0	undefined	$\frac{2}{1} = 2$	$\frac{4}{2} = 2$	$\frac{6}{3} = 2$	$\frac{8}{4} = 2$
A		undefined			
B			undefined		
C				undefined	
D					undefined

2. What is notable about the $\frac{\text{rise}}{\text{run}}$ that you filled in?
3. Why does the word "undefined" appear in certain cells of the table?

6.8.2. Collinear points

Think of the map to buried treasure so popular among the pirates of old.

In Figure (6.14), the path from one point to the next point is always the same, "1 right, 2 up". We can say more.The path from *every point* to *every other point* is in the *ratio* $\frac{\text{2 up}}{\text{1 right}}$. The consequence is that the four points lie on the same line. This idea is general and we state it in Definition (6.6).



(a) Direction constant. (b) Straight path.

FIGURE 6.14. $\frac{rise}{run}$ shows direction.

Definition 6.6 (Collinear points)
For a given collection of points, if the ratio $\frac{rise}{run}$ is the same for every pair of points in the collection, then all the points of the collection are **collinear** (lie on the same straight line). ■

6.8.3. The graph of *y = 2x*

On page 164 we wondered *Will the points produced by the function* $y = 2x$ *always lie on the same straight line?* The answer is "Yes". And we can be certain of this. All we need to show is that for *every* pair of points produced by $y = 2x$, the ratio $\frac{\text{rise}}{\text{run}}$ is identical. Here goes.

Let points $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points produced by $y = 2x$ where $x_1 \neq x_2$. Then $y_1 = 2x_1$ and $y_2 = 2x_2$. The rise is $2x_2 - 2x_1$. The run is $x_2 - x_1$. So

$$\begin{aligned}\frac{\text{rise}}{\text{run}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2x_2 - 2x_1}{x_2 - x_1} \\ &= 2\left(\frac{x_2 - x_1}{x_2 - x_1}\right) \\ &= 2.\end{aligned}$$

Since $\frac{\text{rise}}{\text{run}}$ is identical for *every* pair of points produced by $y = 2x$, the points produced by $y = 2x$ are collinear.

6.8.4. The function $y = ax$ unbounded

We have just discovered in Section (6.8.2) that the graph of $y = 2x$ is a set of collinear points for every possible domain. If we allow the domain to be all the numbers, then the points of the graph fill in to produce a solid continuous straight line. The graph is in Figure (6.15). It is understood that the graph, like the axes, continues in both directions.

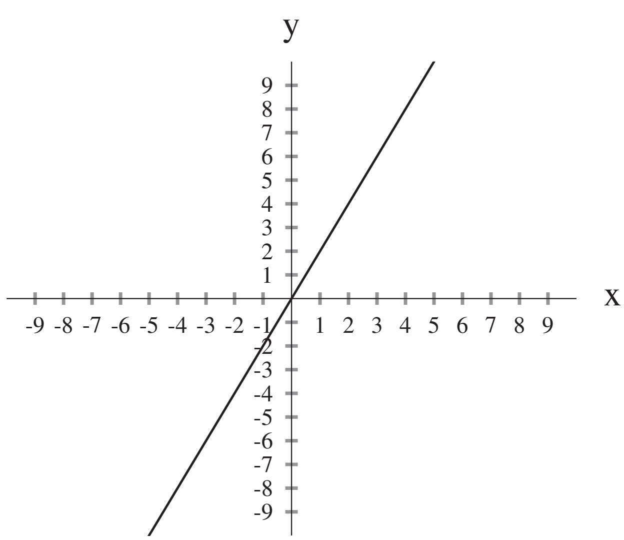


FIGURE 6.15. Graph of $y = 2x$, domain all numbers.

Remark 6.7

There is a difference between plotting points and graphing. We might plot points in science class to make a chart of, for example, temperature and number of cricket chips per hour. The chart would consist of accurately plotted points on carefully laid out axes. When we graph a function in mathematics, we plot only a few points. The process by which we arrived at the graph of $y = 2x$ in Figure (6.15) is a good example of how we graph in mathematics. We knew to draw a straight line, because we understood that the constant $\frac{\text{rise}}{\text{run}}$ guaranteed the graph would be a straight line.

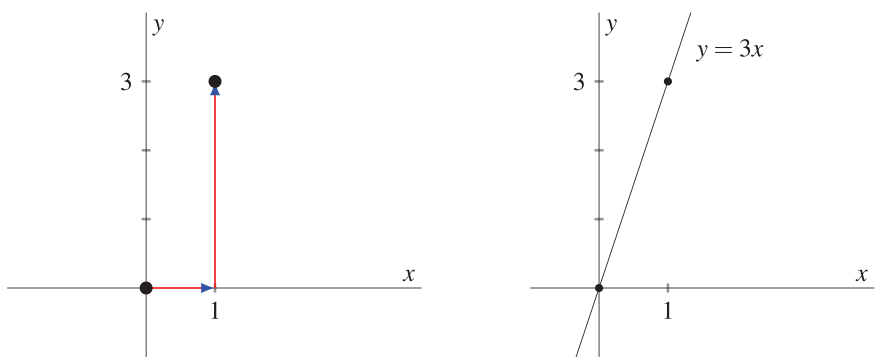
6.9. Graphing the function *y* = *ax*

The function $y = 2x$ is but one example of $y = ax$ where a is a constant. By now we know that the number a is the $\frac{\text{rise}}{\text{run}}$ ratio. We could write $y = \left(\frac{\text{rise}}{\text{run}}\right)x$, but letting "$a$" represent $\frac{\text{rise}}{\text{run}}$ makes a better looking equation. In fact, since ratio $\frac{\text{rise}}{\text{run}}$ turns up so often, it is convenient to give it a name. We agree to call this ratio the **slope**.

Example 6.22

Graph the function $y = 3x$, domain all numbers.

Solution



(a) Plot $(0,0)$. Then use slope. (b) Draw the line.

FIGURE 6.16. Graphing $y = 3x$

Figure (6.16) shows the steps. (a) The point $(0,0)$ is on the graph, because $0 = 3 \cdot 0$. The slope is $3 = \frac{3}{1}$ and we use it to locate another point on the line thinking "1 right, 3 up,". (b) Then draw the straight line through points $(0,0)$ and $(1,3)$. ■

Figure (6.16)(a), shows what we thought as we made Figure (6.16)(b). You would not show Part (a) when graphing.

The several examples that follow should answer some of your questions about graphing.

Example 6.23

Graph $y = x$.

Solution

The graph goes through the origin. The slope is 1, think "$y = 1 \cdot x$". So another point on the line is $(1,1)$.

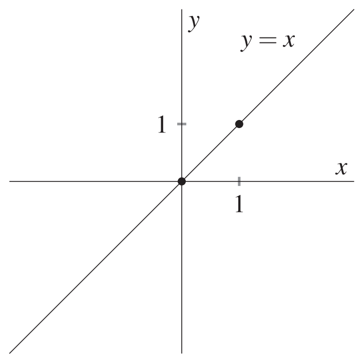


FIGURE 6.17. Example (6.23)

Example 6.24

Graph $y = \dfrac{2}{3}x$.

Solution

The graph goes through the origin. The slope is $\frac{2}{3}$, think "three right, two up". So another point on the line is $(3,2)$.

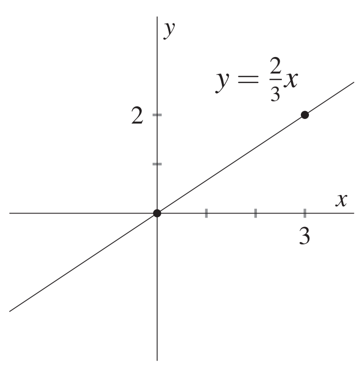


FIGURE 6.18. Example (6.24)

Example 6.25

Graph $y = \dfrac{-2}{3}x$.

Solution

The graph goes through the origin. The slope is $\frac{-2}{3}$, think "three right, two down". So another point on the line is $(3, -2)$.

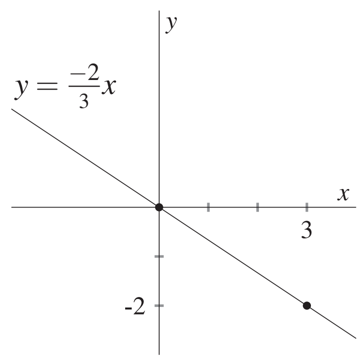


FIGURE 6.19. Example (6.25)

Example 6.26

Graph $y = \dfrac{-1}{4}x$.

Solution

The graph goes through the origin. The slope is $\frac{-1}{4}$, think "four right, one down". So another point on the line is $(-1, 4)$.

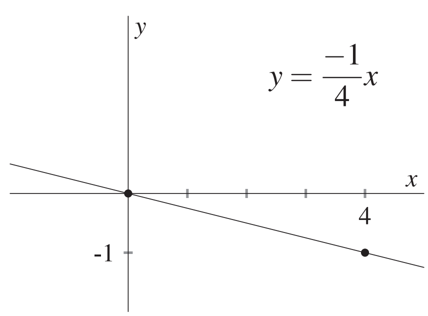


FIGURE 6.20. Example (6.26)

Example 6.27

Graph $y = 3x$.

Solution

The graph goes through the origin. The slope is 3, so another point on the line is $(1,3)$.

In Figure (6.21), we write the coordinates of the point $(1,3)$ beside the point, since there are several numbers shown on the axes.

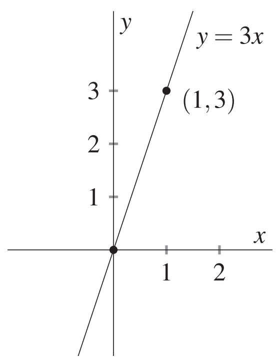


FIGURE 6.21. Example (6.27)

Exercise 6.7

1. Graph the function $y = 4x$.
2. Graph the function $y = \frac{1}{4}x$.
3. Graph the function $y = -2x$.
4. Graph the function $y = -3x$.
5. Graph the function $y = \dfrac{-2}{3}x$.
6. Graph the function $y = 8x$.

6.9.1. Graphs to equations

Sometimes we wish to write the function, given its graph. For example, consider the graph of Figure (6.22) which is of a straight line through the origin $(0,0)$.

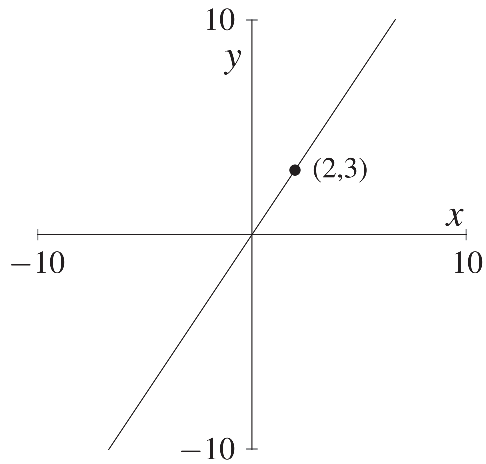


FIGURE 6.22. A straight line through points $(0,0)$ and $(2,3)$.

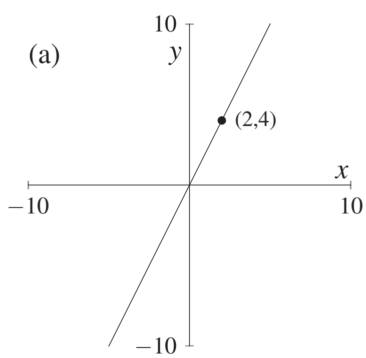
Since the graph is a straight line through the origin, it must have an equation of the form $y = ax$ where a is the slope of the line. The point $(2,3)$ on the line is 2 right of the $(0,0)$ and 3 up from $(0,0)$, so the slope equals $\frac{3}{2}$. Therefore the function must be $y = \frac{3}{2}x$.

Exercise 6.8

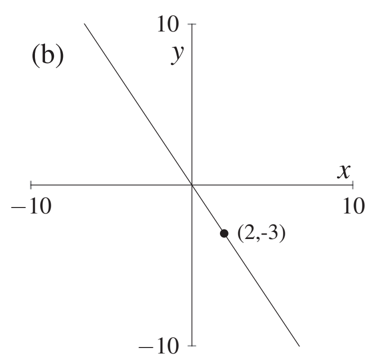
For each of the following graphs, write the function.

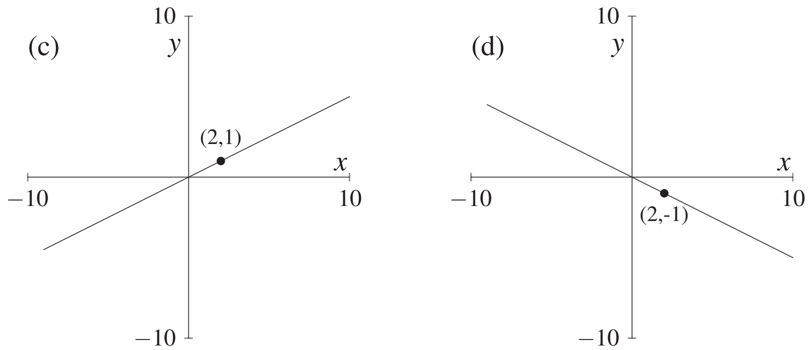
1.

(a)



(b)





6.10. Slope

We have been using the words “rise” and “run” quite a bit. But, it is unlikely that you will hear them used in more advanced mathematics courses. What we have been calling the “rise” is the change in the value of the dependent variable. Similarly, the run is the change in the value of the independent variable. So,

$$\frac{\text{rise}}{\text{run}} = \frac{\text{change in the value of the dependent variable}}{\text{change in the value of the independent variable}}. \tag{6.6}$$

Nobody really wants to say “the change in the value of the dependent variable *over* the change in the value of the independent variable”. Usually we do not have to, because usually we are talking about a particular function. If, for example, $y = ax$, then we just say “the change in y over the change in x”.

Improved, but admittedly, not so much. We can do better yet. In mathematics, the uppercase Greek letter Δ means “change”. To express the idea *change in* y, we just write Δy. Similarly for the change in x, just write Δx. We can express the idea of Equation (6.6) by writing Equation (6.7)

Δ is pronounced “delta”.

$$\frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}. \tag{6.7}$$

We now define “slope”.

Definition 6.7 (Slope)

If $y = ax$, then the **slope** a is $\dfrac{\Delta y}{\Delta x}$. ■

Remark 6.8

If the function were written $t = as$, we would write the slope as $\dfrac{\Delta t}{\Delta s}$. If the function were written $z = aw$, we would write the slope as $\dfrac{\Delta z}{\Delta w}$.

Example 6.28

Suppose $z = \frac{7}{6}w$. Using the Δ notation, write the rise, the run, and the slope.

Solution

$$\Delta z = 7, \quad \Delta w = 6, \quad \frac{\Delta z}{\Delta w} = \frac{7}{6}.$$

Example 6.29

For the graph in Figure (6.23), use the Δ notation to write the rise, the run, and the slope.

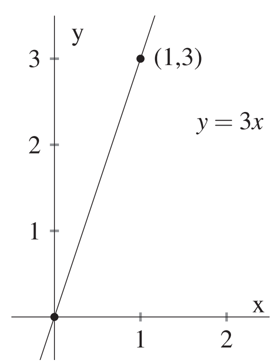


FIGURE 6.23. Example (6.29)

Solution

$$\Delta y = 3, \quad \Delta x = 1, \quad \frac{\Delta y}{\Delta x} = \frac{3}{1} = 3.$$

Example 6.30

For the graph in Figure (6.24), use the Δ notation to write the rise, the run, and the slope.

Solution

$$\Delta y = -3, \quad \Delta x = 2, \quad \frac{\Delta y}{\Delta x} = \frac{-3}{2}. \quad \blacksquare$$

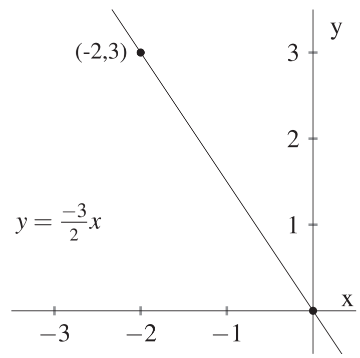


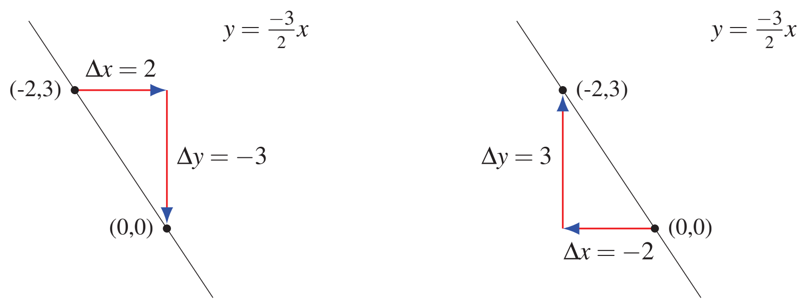
FIGURE 6.24. Example (6.30)

Someone is bound to point out that Example (6.30) could also be worked like this:

Alternate solution to Example (6.30)

$$\begin{aligned}\Delta y &= 3,\\ \Delta x &= -2,\\ \frac{\Delta y}{\Delta x} &= \frac{3}{-2}\\ &= \frac{-3}{2}. \quad \blacksquare\end{aligned}$$

Both the solution and the alternate solution provide the correct answer. So which solution is better? Figure (6.25) shows a picture for each solution.



(a) Solution. (b) Alternate solution.

FIGURE 6.25. Alternate solutions for Example (6.30)

When studying the solution (a) shown in Figure (6.25), you should think:

> As the values of x increase from -2 to 0, the values of y decrease from 3 to 0.

When studying the solution (b) shown in Figure (6.25), you should think:

As the values of x decrease from 0 to -2, the values of y increase from 0 to 3.

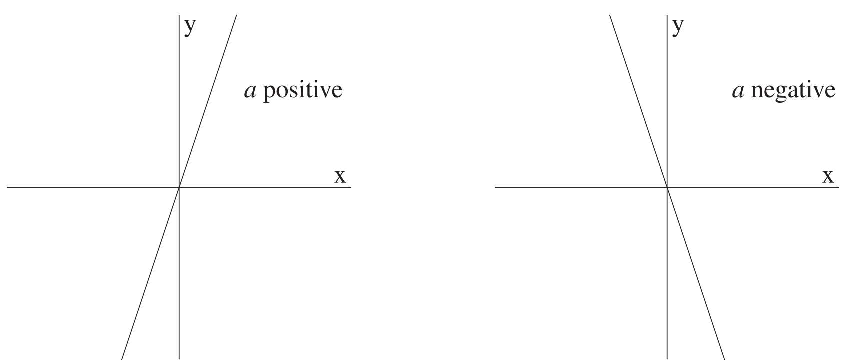
"Which correct solution is better?" The original solution which is shown in Figure (6.25)(a) is better. The thinking that accompanies it, "As the values of x increase from -2 to 0, the values of y decrease from 3 to 0", is well suited to some mathematics you will use a few years from now. We prefer to think about what happens to y as x *increases*, and "increases" means to the right on the x-axis.

Since we have the ideas "increasing" and "decreasing", we may as well define two ideas that you will use in subsequent courses.

Definition 6.8 (Increasing (decreasing) function)
Let y be a function of x. If the value of y increases as the value of x increases, we say y is an **increasing function** of x. Alternatively, if the value of y decreases as the value of x increases, we say y is a **decreasing function** of x.

Theorem 6.1
Let y be a function of x of the form $y = ax$. If a is positive, y is an increasing function of x. If a is negative, y is a decreasing function of x. ■

Figure (6.26) illustrates these ideas.



(a) y is an increasing function of x. (b) y is a decreasing function of x.

FIGURE 6.26. Graphing $y = ax$

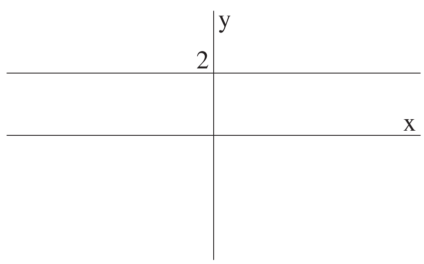
6.10.1. Slope - extreme cases

Horizontal line

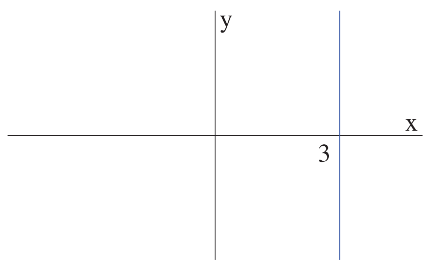
What is the slope of a horizontal line? Since the rise, Δy, of a horizontal line is 0, the slope is $\frac{0}{\Delta x} = 0$. Therefore, the slope of a horizontal line is 0.

Vertical line What is the slope of a vertical line?
Since the run, Δx, of a vertical line is 0, the slope would be $\frac{\Delta y}{0}$. But this is undefined. Therefore, the slope of a vertical line is undefined.

Figure (6.27) illustrates these ideas.



(a) Slope is 0.



(b) Slope is undefined.

FIGURE 6.27. Slope: extreme cases

You might be surprised to know that in each of these cases, we can still write an equation for the line. In Figure (6.27) (a), the equation is $y = 2$. In (b), the equation is $x = 3$.

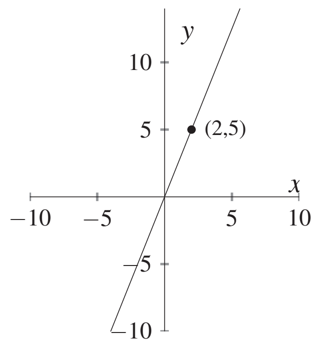
Exercise 6.9

[Part 1] *Use the Δ notation to write the rise, run, and slope. State whether the function is increasing or decreasing.*

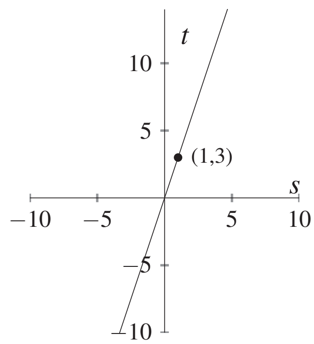
1. $y = \frac{2}{5}x$
2. $z = \frac{3}{11}w$
3. $t = \frac{-1}{4}s$
4. $y = 3x$
5. $y = \frac{5}{2}x$
6. $t = -9s$

[Part 2] *Use the Δ notation to write the rise, run, and slope. State whether the function is increasing or decreasing.*

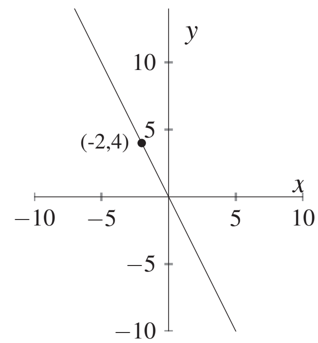
1.



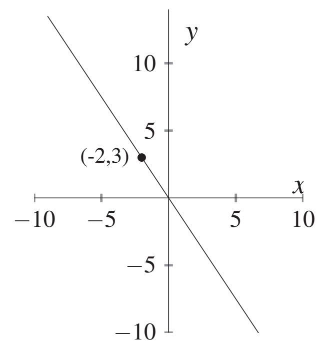
2.



3.



4.



6.11. Linear functions and rate of change

Suppose that

$$y = 2x. \tag{6.8}$$

Equation (6.8) says that whatever number x is, y is double that number. It is not quite so obvious that Equation (6.8) also says that whatever the change in x, the change in y is twice that. Table (6.2) provides some evidence for this claim. The first row of Table (6.2) shows that when $x = 1, y = 2$. These values are called "old x and old y". When $x = 3, y = 6$. These are called "new x and new y". The change in x, Δx, is the old value minus the new value which is $3 - 1 = 2$. The change in y, Δy, is "new y" subtract "old y" which is $6 - 2 = 4$. The last column of the table points out that the change in y is 2 times the change in x, $\Delta y = 2\Delta x$; in this example, the computation is $4 = 2 \cdot 2$.

We can do better than merely provide evidence that the change in y is double the change in x when $y = 2x$. We can prove that this must be so.

old x	new x	Δx	old y	new y	Δy	result
1	3	2	2	6	4	4 is 2 times 2
3	9	6	6	18	12	12 is 2 times 6
2	13	11	4	26	22	22 is 2 times 11
9	109	100	18	218	200	200 is 2 times 100
1/2	11/16	3/16	1	11/8	3/8	3/8 is 2 times 3/16

TABLE 6.2. Several computations showing effect on y of increasing x when $y = 2x$.

Proof. Let $y = 2x$. Suppose that x changes by an amount Δx and y by an amount Δy. Then

$$\begin{aligned} y + \Delta y &= 2(x + \Delta x) \\ y + \Delta y &= 2x + 2\Delta x \\ y + \Delta y &= y + 2\Delta x, \qquad \text{[WHY?]} \\ \Delta y &= 2\Delta x \end{aligned}$$

Therefore, the change in y is 2 times the change in x. ■

Another way to express the idea that the change in y is 2 times the change in x is to say that "the rate of change in y with respect to x is 2". This is true for all non-zero values of a. Theorem (6.2) says just that.

Theorem 6.2
If $y = ax, a \neq 0$, then the number a is the rate of change of y with respect to x. ■

It may be reassuring to know that Theorem (6.2) says what we have known, though perhaps not so generally, since 6th grade or earlier.

For example, you know that distance traveled equals the speed times the time of traveling at that speed. That is,

$$\text{distance} = (\text{speed}) \times (\text{time}), \text{ provided the speed is constant.} \tag{6.9}$$

Now, Equation (6.9) has the form $y = ax$, with speed playing the role of a. Theorem (6.2) tells us that a must be the rate of change in y with respect to x. And speed is exactly that: *the rate of change in distance with respect to time.*

Example 6.31

At what rate does the circumference of a circle change with respect to its diameter?

Solution

The circumference, C, of a circle is a function of its diameter, D. The function is $C = \pi D$. This equation has the form $y = ax$ with a constant, C and D variables. Therefore, the rate of change in circumference with respect to diameter, $\frac{\Delta C}{\Delta D}$, is π.

Exercise 6.10

1. A tank is being filled with water. The volume, V liters, of water in the tank is a function of time, t minutes. In fact, $V = 23\frac{\text{L}}{\text{min}}t$. Find $\frac{\Delta V}{\Delta t}$.
2. For a triangle whose base is 10 feet, at what rate is the triangle's area changing with respect to the triangle's height in feet?
3. Sally runs at a rate $1\frac{1}{2}$ that of Peter. If Peter covers 800 feet in a certain period of time, how many feet will Sally cover in that same period of time?
4. Tank A is being filled according to the function $V = 40t$ while tank B is being filled according to the function $V = 60t$, where V is in liters and t is in minutes. How much has the volume in tank B increased when the volume of tank A has increased from 30L to 60L?
5. * Tank A is being filled according to the function $V_A = 40t$ while tank B is being filled according to the function $V_B = 60t$, where V_A and V_B are in liters and t is in minutes. How much has the volume in tank B increased when the volume of tank A has increased an amount ΔV_A liters? Answer in terms of ΔV_A.

6.12. Graphs and rate of change

Consider the graph in Figure (6.28).

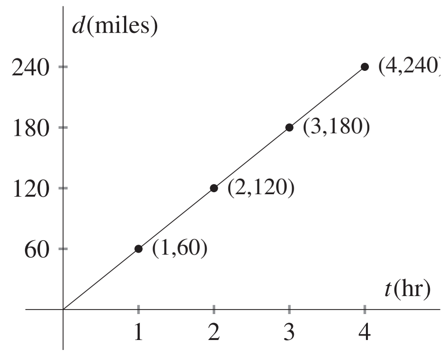


FIGURE 6.28. A four hour trip at a constant speed.

Time is shown on the horizontal axis. Distance is shown on the vertical axis. What feature of the graph shows the speed?

Well, speed is the rate at which d changes with respect to t. And, that is the slope of the line! Or, what is equivalent, it is the ratio $\frac{\text{rise}}{\text{run}}$. Or, what is equivalent, the ratio $\frac{\Delta d}{\Delta t}$.

To find the speed, think as illustrated in Figure (6.29).

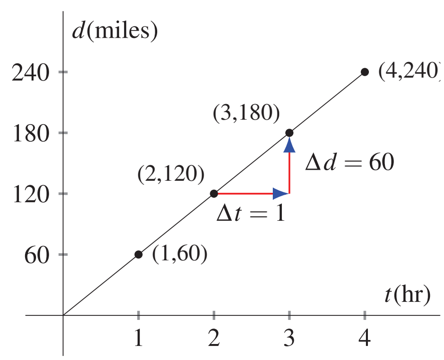


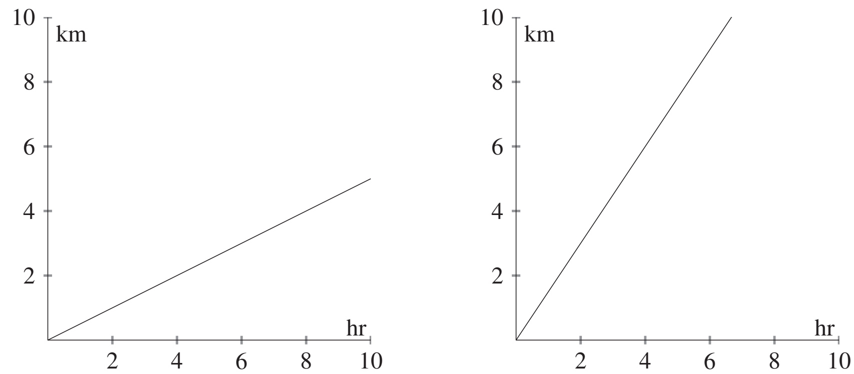
FIGURE 6.29. Same four hour trip.

Since speed is rate of change in distance with respect to time,

$$\text{the speed } = \frac{\Delta d}{\Delta t} = \frac{60 \text{ mile}}{1 \text{ hour}} = 60\frac{\text{mi}}{\text{hr}}.$$

Example 6.32

Figure (6.30) shows the distance Tom traveled as a function of time and the distance Sue traveled as a function of time. Which person's speed was greater?



(a) Tom's journey. (b) Sue's journey.

FIGURE 6.30. Tom and Sue, Example (6.32)

Solution

The slope is greater for Sue than for Tom. The slope of the line is $\frac{\Delta\text{distance}}{\Delta\text{time}}$. So, the rate of change in distance with respect to time is greater for Sue than for Tom. Sue traveled at the greater speed.

Exercise 6.11

1. Alice drove at a constant speed of 40 mph from Town A to Town B. She completed the trip in 3 hours. Write the distance she covered, d miles, as a function of time t hours. Be sure to state the domain of the function.
2. A bucket of ice, temperature $0\,°C$, was heated on a stove. The water boiled ($100\,°C$) after 20 minutes of heating. Assuming that the rate of increase of water temperature was constant, write the temperature $T\,°C$, as a function of time t minutes.

3. The graph showing distance and time of a journey by automobile is shown in Figure (6.31).

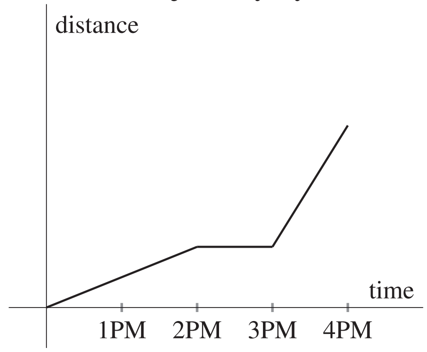


FIGURE 6.31. Four hour car ride.

a) During what time did the car travel the fastest?
b) The driver stopped for lunch. When was that?
c) How long did the lunch stop take?

6.13. $y = ax + b$

We have discussed the function $y = ax$, a constant. It is the special case of $y = ax + b$ when $b = 0$. Now we consider $y = ax + b, a \neq 0, b \neq 0$.

6.13.1. The role of b

To appreciate the effect of b in $y = ax + b$, think how you would compute $y = \frac{1}{2}x + 3$ when, for example, $x = 2$. First you would determine the value of $\frac{1}{2}x$. That is $\frac{1}{2} \cdot 2 = 1$. Then you would add 3. When $x = 4$, you would compute $\frac{1}{2} \cdot 4 = 2$, then you would add 3. The "+3" tacked onto "$\frac{1}{2}x$" just causes y to be 3 greater than $\frac{1}{2}x$ for every value of x. Figure (6.32) illustrates this idea.

The lines $y = ax$ and $y = ax + b$ must be parallel, because they have identical slope. As Figure (6.32) shows, the line $y = 2x + 3$ is the line $y = 2x$ shifted 3 units in the vertical direction. When a line is shifted without any rotation, as in Figure (6.32), we say the line is "translated". Every point on the line $y = ax + b$ is a point on $y = ax$ translated b units vertically.

The graph of $y = ax + b$ where b is a negative number, is the graph of $y = ax$ translated b units in the negative vertical direction (otherwise known as "down"). Figure (6.33) shows the graphs of $y = \frac{1}{2}x$ and $y = \frac{1}{2}x - 3$.

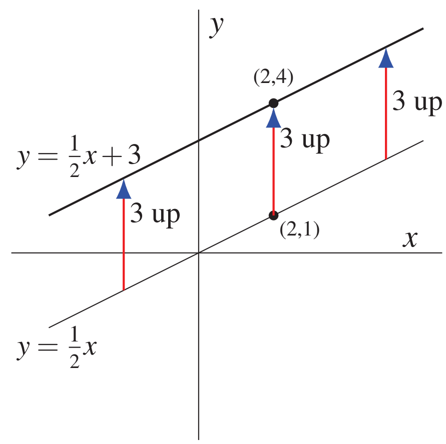


FIGURE 6.32. $y = 2x$ compared to $y = 2x + 3$.

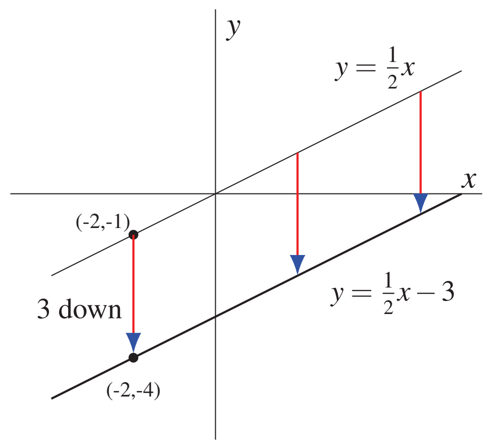


FIGURE 6.33. $y = 2x$ compared to $y = 2x - 3$.

6.13.2. Intercepts

The points at which a graph crosses the axes are landmarks. They are called intercepts.

Definition 6.9 (Vertical intercept)
The second coordinate of the point at which a graph crosses the vertical axis is called the **vertical intercept**.

Definition 6.10 (Horizontal intercept)
The first coordinate of the point at which a graph crosses the vertical axis is called the **horizontal intercept**.

Remark 6.9
If the function is written $y = ax + b$, we often call the vertical axis the "y axis" and speak of the "y intercept". If the function is written $t = as + b$,

the vertical axis may be called the "t axis" and the vertical intercept the "t intercept".

Example 6.33

State the intercepts of the function $y = 2x + 6$ shown in Figure (6.34).

Solution

The line $y = 2x + 6$ crosses the y axis at the point $(0, 6)$ where the second coordinate is 6. It crosses the x axis at $(-3, 0)$.Therefore, the y intercept is 6 and the x intercept is -3.

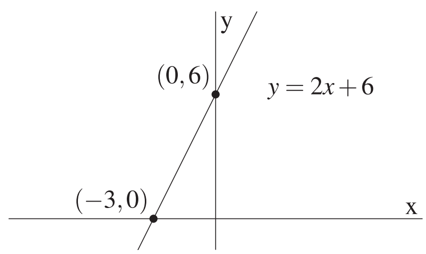


FIGURE 6.34

■

Was it necessary to make a graph? No. Recall from page 162 that the first coordinate of every point on the vertical axis is 0 and that the second coordinate of every point on the horizontal axis is zero. These facts are used in Example (6.34).

Example 6.34

Find the intercepts of $y = 8x + 5$.

Solution

Find the y intercept by setting $x = 0$. Then $y = 8(0) + 5$. So the y intercept is 5.
Find the x intercept by setting $y = 0$. $0 = 8x + 5 \implies x = \frac{-5}{8}$. So the x intercept is $\frac{-5}{8}$.

6.13.3. Solving problems just by looking at them

When a linear function is written $y = ax + b$, all one needs to do is look! The coefficient of x must be the slope, the constant b must be the y intercept. When we answer by merely looking at $y = ax + b$, we say that we found the slope and y intercept "by inspection".

Example 6.35

Determine the slope and y intercept of $y = 19x + 100$.

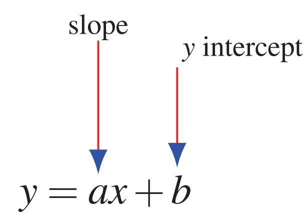
Solution

By inspection, the slope is 19 and the y intercept is 100.

6.13.4. Slope-intercept form

We call the form $y = ax + b$ the "slope-intercept" form of a linear function.

Slope-intercept form



Example 6.36

Determine the slope and t intercept of $t = \frac{2}{3}x - 5$.

Solution

The function written in slope-intercept form is $t = \frac{2}{3}x + (-5)$. By inspection, the slope is $\frac{2}{3}$ and the t intercept is -5.

Example 6.37

Determine the slope and x intercept of $3x + 5y - 7 = 9$.

Solution

Begin by writing $3x + 5y - 7 = 9$ in slope-intercept form.

$$\begin{aligned} 3x + 5y - 7 &= 9 \\ 5y &= -3x + 16. \\ y &= \frac{-3}{5}x + \frac{16}{5} \end{aligned} \tag{6.10}$$

Equation (6.10) is in slope-intercept form. By inspection, the slope is $\frac{-3}{5}$ and the y intercept is $\frac{16}{5}$.

Exercise 6.12

Write the slope and both intercepts for each of the following.

1. $y = 3x + 7$
2. $t = 5s + \frac{1}{2}$
3. $t = \frac{5}{8}s + \frac{3}{2}$
4. $y = \frac{-5}{9} + 6$
5. $y = \frac{-3}{4} - 1$
6. $2y + x = 6$
7. $3x - y + 2 = 0$
8. $2(3x - y) - 7 = 10$
9. $12 - (x + 7) = 5y$
10. $\frac{x}{3} + \frac{y}{2} = 1$

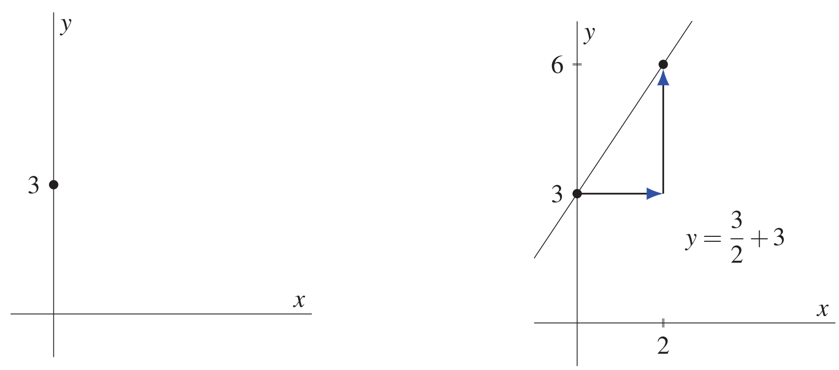
Example 6.38

Graph the function $y = \frac{3}{2}x + 3$.

Solution

The y intercept is 3, so the point $(0, 3)$ is on the line.

Since the slope is $\frac{3}{2}$, 2 units right then 3 units up produces another point on the line.



(a) Plot $(0, 3)$.

(b) Draw the line.

FIGURE 6.35. $y = \dfrac{3}{2} + 3$

An alternative method for graphing a function is to find the intercepts, then draw a line though them.

Example 6.39

Graph the function $y = \frac{2}{5}x + 2$.

Solution

The y intercept is 2, so the point $(0,2)$ is on the line. The x intercept is -5, so the point $(-5,0)$ is on the line.

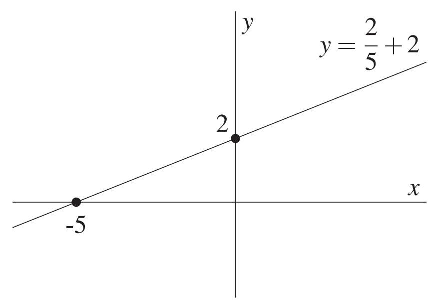


FIGURE 6.36. $y = \frac{2}{5}x + 2$

Exercise 6.13

Graph each of the following functions. Label the intercepts.

1. $y = \frac{1}{2}x + 3$
2. $y = x + 3$
3. $y = \frac{2}{3}x + 2$
4. $y = \frac{-3}{5}x + 3$
5. $t = -6s + 6$
6. $y = \frac{3}{4}x + 3$
7. $y = \frac{2}{7}x - 2$
8. $z = \frac{1}{3}w - 1$

Example 6.40

Places A and B are 1000 meters apart. Sue walks 200 meters per minute from place A to Place B.

(a) Write Sue's distance from Place A, d meters, as a function of time t minutes. State the domain of the function. Graph the function.

(b) Write Sue's distance from Place B, d meters, as a function of time t minutes. State the domain of the function. Graph the function.

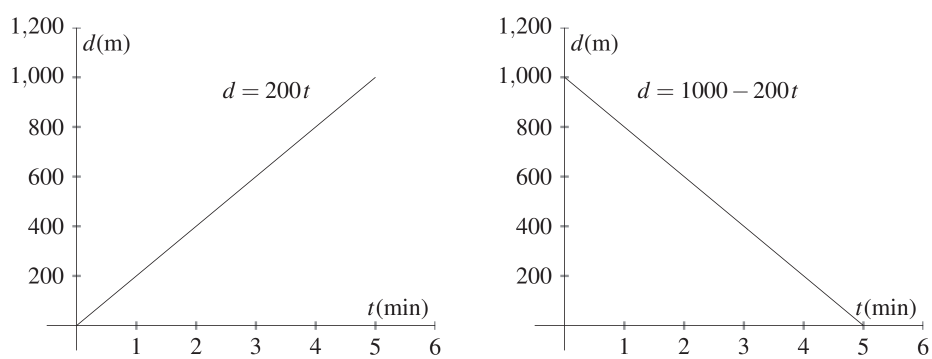
Solution

(a) The function is $d = 200t$. Sue will complete her walk when d is 1000 m. Solving $1000 = 200t$, for t, we find that $t = 5$ when $d = 1000$. So the domain of the function is all numbers between 0 and 5.

(b) The function is $d = 1000 - 200t$. Sue will complete her walk when d is 0 m. Solving $0 = 1000 - 200t$, for t, we find that $t = 5$ when $d = 0$. So the domain of the function is all numbers between 0 and 5.

The graphs of both functions are shown side by side in Figure (6.37).

(a) **(b)**



(a) Distance from Place A. (b) Distance to Place B.

FIGURE 6.37. Sue's walk, Example (6.40)

Remark 6.10
Example (6.40), notice that Sue's distance from Place A is an increasing function of time and her distance from Place B is a decreasing function of time.

Example 6.41

A tank of capacity 1600 gallons initially contains 200 gallons of water. Then water is added to the tank at the rate of 50 gallons per minute until the tank is full.

(a) Write the volume of water in the tank, V gallons, as a function of time t minutes. Graph the function.

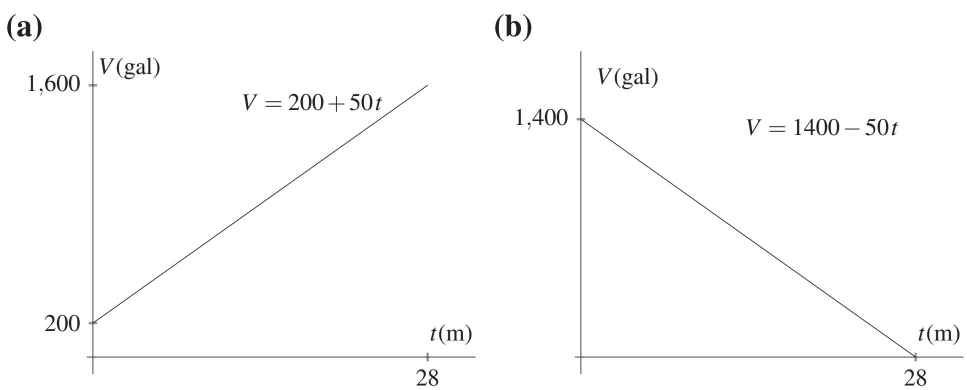
(b) Write the volume of water that must be added to fill the tank, V gallons, as a function of time t minutes. Graph the function.

Solution

(a) The function is $V = 200 + 50t$. The tank will be full when V is 1600 gal. Solving $1600 = 200 + 50t$, for t, we find that $t = 28$ when $V = 1600$. So the domain of the function is all numbers between 0 and 28.

(b) The function is $V = 1400 - 50t$. The tank will be full when V is 0 gal. Solving $0 = 1400 - 50t$, for t, we find that $t = 28$ when $V = 0$. So the domain of the function is all numbers between 0 and 28.

The graphs of both functions are shown side by side in Figure (6.38).



(a) Volume water in tank. (b) Volume water to add.

FIGURE 6.38. Tank filling, Example (6.41)

■

Part (a) of Example (6.41) suggests a helpful way to think of a linear function.

Slope-intercept form

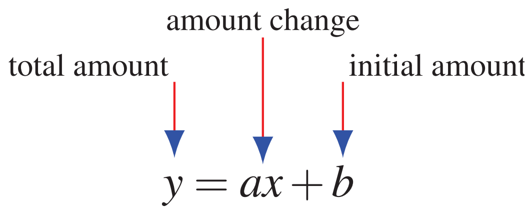


Figure (6.39) illustrates this idea.

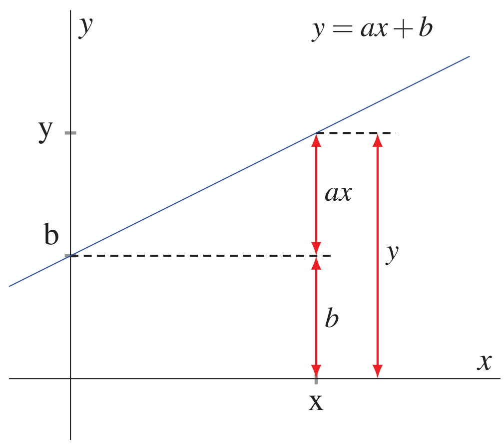


FIGURE 6.39. A helpful way to think about $y = ax + b$

Exercise 6.14

1. Station B is 900 miles east of Station A. Station C is 300 miles east of Station A. A train travels at 60 mph from Station C to Station B.
   a) Write the distance of the train from Station A, y miles, as a function of time t hours. Graph the function. State whether the distance is an increasing or decreasing function of time.
   b) Write the distance of the train from Station B, y miles, as a function of time t hours. Graph the function. State whether the distance is an increasing or decreasing function of time.

2. Alex walks at 1.5 meters per second (m/s) and Barb runs at 4 m/s. Al and Barb each start at the same time, but Al has a 10 meter head start. How long does it take for Barb to catch Alex? Write Alex's distance, y_A meters (m) as a function of time t seconds (s) and and Barb's distance y_B m as a function of time t s. Then graph both functions on the same axes and estimate the time at which Barb catches Alex.
3. Find the exact answer to Question (2).
4. On page 183, we showed that if $y = 2x$, then the change in y is 2 times the change in x. Is it also true that if $y = 2x + 5$, then the change in y is 2 times the change in x? Provide the reasoning for your answer.
5. Draw a graph like the one in Figure (6.39) for the function $y = 2x + 5$.

Answers to Exercise 5.4

(**1**) $120\frac{m}{h}$ (**2**) $72\frac{km}{h}$ (**3**) $1320\frac{m}{min}$ (**4**) $818\frac{m}{h}$ (**5**) $375\frac{m}{h}$

Answers to Exercise 5.5

(**1**) 260 miles. (**2**) $3\frac{1}{2}$h or 3h 30m. (**3**) 6.5 miles. (**4**) $5\frac{1}{3}$h or 5 h 20 m (**5**) 26 km/h. (**6**) 72 (**7**) 64 mph. (**8**) 40 mph. (**9**) 30 mph. (**10**) $2\frac{1}{2}$ hours. (**11**) 20. (**12**) 12:09 PM. (**13**) 1:15 PM. (**14**) 1320 miles.

Answers to Exercise 5.6

(**1**) 6 (**2**) 3 inches (**3**) 151, 152, 153 (**4**) 17 (**5**) 100 (**6**) \$200000 (**7**) 4:00 PM (**8**) $3\frac{1}{5}$ hours or 3h 12m (**9**) 136.5 miles (**10**) 1:06 PM

Answers to Exercise 6.1

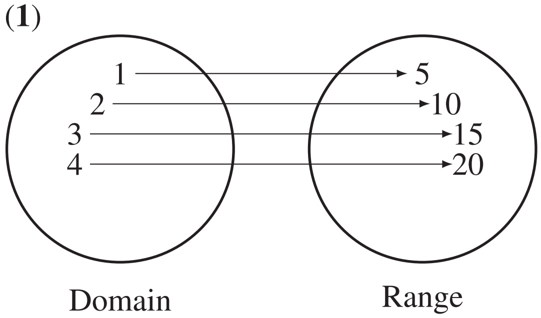
(**1**) $(-4,-12),(-2,-8),(0,0),(1,-4),(\frac{3}{8},\frac{3}{2}),(5,20)$.
(**2**) $(0,3),(1,4),(2,5),(3,15),(\frac{10}{3},\frac{19}{3})$. (**3**) $(\frac{-1}{3},-2),(-1,-3),(0,0),(1,3),(2,6)$.
(**4**) No, because $3\cdot 2 = 6 \neq 9$. (**5**) No, because $-1-2=-3\neq 5$. (**6**) $y = 5x$. (**7**) $y = x+4$. (**8**) $t = -5s$. (**9**) To see that $(2,6)$ and $(6,2)$ are not the same, note that $(2,6)$ is a solution of $y = 3x$ but $(6,2)$ is not.

Answers to Exercise 6.2

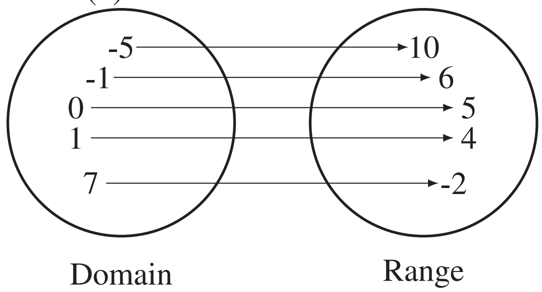
(**1**) $\{-5,-1,3,6\}$ (**2**) $\{-30,-10,0,20,90\}$ (**3**) $\{5,\frac{16}{3},\frac{11}{2},6,8\}$
(**4**) $\{-5,-2,1,4,7\}$ (**5**) $\{10,13,16,19\}$ (**6**) $\{22,25,28,31,34\}$
(**7**) Domain $= \{1,2,3,5,12\}$. Range $= \{\frac{2}{3},\frac{3}{4},7,11,27\}$.

Answers to Exercise 6.3

(**1**)



(**2**)



(**3**) $2 \longrightarrow 7, 3 \longrightarrow 10, 5 \longrightarrow 16$.
(**4**) $-7 \longrightarrow -11, -2 \longrightarrow -1, -1 \longrightarrow 1, 0 \longrightarrow 3, 1 \longrightarrow 5, 3 \longrightarrow 9$.
(**5**) $y = 10x$, range $= \{10, 20, 30, 40, 50\}$.

Answers to Exercise 6.4

(**1**) $d = 60t$. (**2**) (**a**) $V = 2 + 3t$.

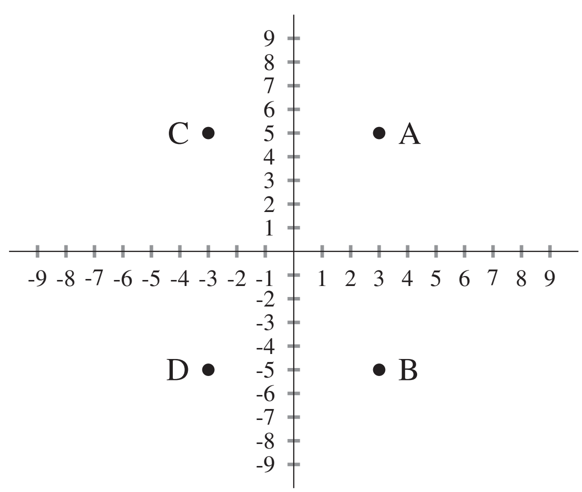
(**b**)

t (min)	0	2	4	6	8	10	12
V (gal)	2	8	14	20	26	32	38

(**3**) (**a**) $A = 7t$. (**b**) 49. (**4**) (**a**) $t = \frac{0.75-d}{4}$. (**b**) All numbers between 0 and 0.75. (**c**) All numbers between 0 and 0.1875.
(**5**) (**a**) $y = 5x$. (**b**) $\{0, 1, 2, 3, 4\}$. (**c**) $\{0, 5, 10, 15, 20\}$. (**d**) 350.

Answers to Exercise 6.5

(**1**) (**a**) Done as example. (**b**) $(3, 6)$. (**c**) $(-3, 2)$. (**d**) $(-6, -6)$. (**e**) $(5, -3)$.
(**2**) (**a**) II. (**b**) I. (**c**) II. (**d**) III. (**e**) IV.
(**3**)



(**4**) Answer should look the same as Figure (6.4) page 162.

Answers to Exercise 6.6

[Part 1] (**1**) $\frac{3}{4}$ (**2**) $\frac{-2}{6} = \frac{-1}{3}$ (**3**) $\frac{2}{8} = \frac{1}{4}$ (**4**) $\frac{8}{4} = 2$ (**5**) $\frac{-4}{-2} = 2$
(**6**) $\frac{9}{2}$ (**7**) $\frac{-15}{-4}$ (**8**) $\frac{4}{-3}$

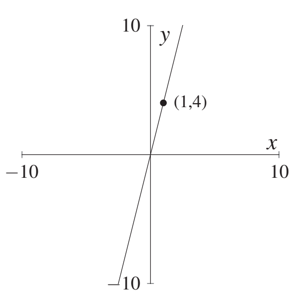
[Part 2]

(**1**)

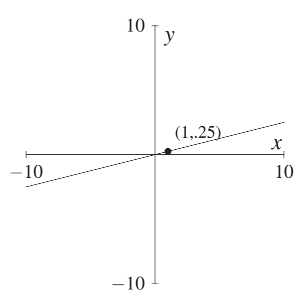
	0	**A**	**B**	**C**	**D**
0	undefined	$\frac{2}{1} = 2$	$\frac{4}{2} = 2$	$\frac{6}{3} = 2$	$\frac{8}{4} = 2$
A		undefined	$\frac{2}{1} = 2$	$\frac{4}{2} = 2$	$\frac{6}{3} = 2$
B			undefined	$\frac{2}{1} = 2$	$\frac{4}{2} = 2$
C				undefined	$\frac{2}{1} = 2$
D					undefined

(**2**) The ratio $\frac{\text{rise}}{\text{run}}$ equals 2 whenever it is defined. (**3**) The run is 0, so the ratio $\frac{\text{rise}}{\text{run}}$ is undefined because division by zero is undefined.

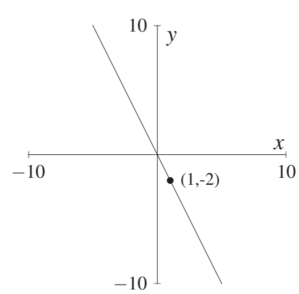
Answers to Exercise 6.7



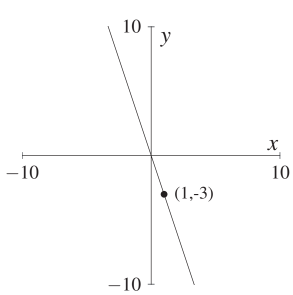
(**1**)



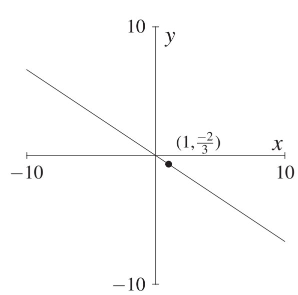
(**2**)



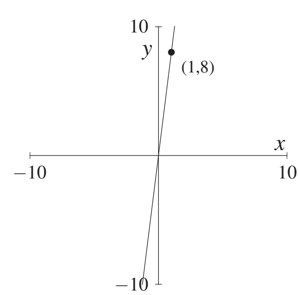
(**3**)



(**4**)



(**5**)



(**6**)

Answers to Exercise 6.8

(**1**) (a) $y = 2x$. (b) $y = \frac{-3}{2}x$. (c) $y = \frac{1}{2}x$. (d) $y = \frac{-1}{2}x$.

Answers to Exercise 6.9

[Part 1] (**1**) $\Delta y = 2, \Delta x = 5, \frac{\Delta y}{\Delta x} = \frac{2}{5}$, increasing (**2**) $\Delta z = 3, \Delta w = 11, \frac{\Delta z}{\Delta w} = \frac{3}{11}$, increasing (**3**) $\Delta t = -1, \Delta s = 4, \frac{\Delta t}{\Delta s} = \frac{-1}{4}$, decreasing
(**4**) $\Delta y = 3, \Delta x = 1, \frac{\Delta y}{\Delta x} = \frac{3}{1} = 3$, increasing (**5**) $\Delta y = 5, \Delta x = 2, \frac{\Delta y}{\Delta x} = \frac{5}{2}$, increasing (**6**) $\Delta t = -9, \Delta s = 1, \frac{\Delta t}{\Delta s} = \frac{-9}{1} = 9$, decreasing
[Part 2] (**1**) $\Delta y = 5, \Delta x = 2, \frac{\Delta y}{\Delta x} = \frac{5}{2}$, increasing (**2**) $\Delta t = 3, \Delta s = 1, \frac{\Delta t}{\Delta s} = \frac{3}{1} = 3$, increasing (**3**) $\Delta y = -4, \Delta x = 2, \frac{\Delta y}{\Delta x} = \frac{-4}{2} = -2$, decreasing
(**4**) $\Delta y = -3, \Delta x = 2, \frac{\Delta y}{\Delta x} = \frac{-3}{2}$, decreasing

Answers to Exercise 6.10

(**1**) $23\frac{\text{L}}{\text{min}}$ (**2**) 5 feet (**3**) 1200 feet (**4**) 30 L (**5**) $\frac{3}{2}b$

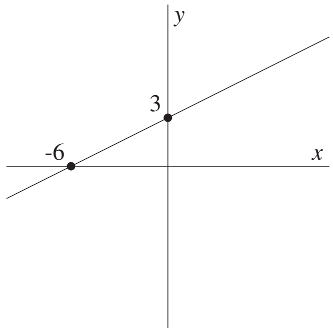
Answers to Exercise 6.11

(**1**) $d = 40t$, domain is all numbers from 0 to 3. (**2**) $T = 5t$, domain is all numbers from 0 to 20. (**3**) (**a**) 3PM-4PM (**b**) 2PM (**c**) 1 hr.

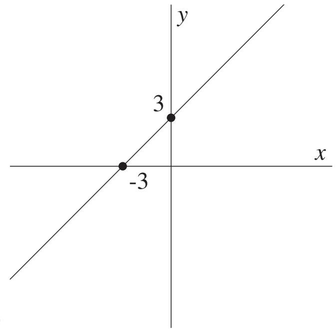
Answers to Exercise 6.12

(**1**) slope $= 3$, y intercept $= 7$, x intercept $= \frac{-7}{3}$
(**2**) slope $= 5$, t intercept $= \frac{1}{2}$, s intercept $= \frac{-1}{10}$
(**3**) slope $= \frac{5}{8}$, t intercept $= \frac{3}{2}$, s intercept $= \frac{-12}{5}$
(**4**) slope $= \frac{-5}{9}$, y intercept $= 6$, x intercept $= \frac{54}{5}$
(**5**) slope $= \frac{-3}{4}$, y intercept $= -1$, x intercept $= \frac{-4}{3}$
(**6**) slope $= \frac{-1}{2}$, y intercept $= 3$, x intercept $= 6$
(**7**) slope $= 3$, y intercept $= 2$, x intercept $= \frac{-2}{3}$
(**8**) slope $= 3$, y intercept $= \frac{-17}{2}$, x intercept $= \frac{17}{6}$
(**9**) slope $= \frac{-1}{5}$, y intercept $= 1$, x intercept $= 5$
(**10**) slope $= \frac{-2}{3}$, y intercept $= 2$, x intercept $= 3$

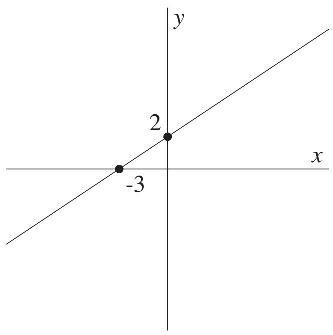
Answers to Exercise 6.13



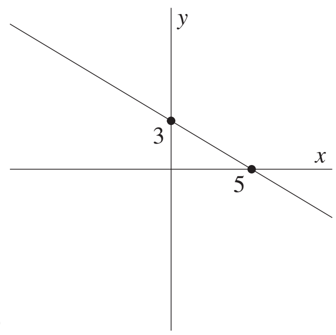
(**1**)



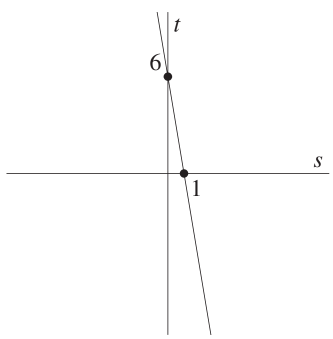
(**2**)



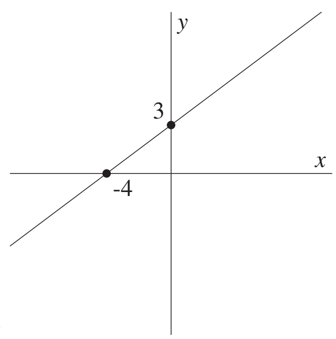
(**3**)



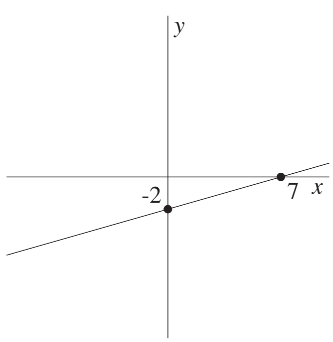
(**4**)



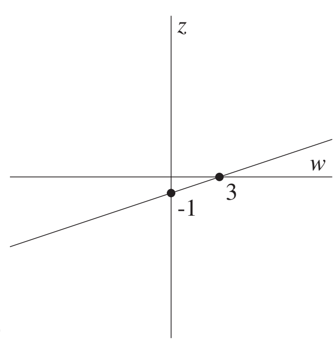
(**5**)



(**6**)



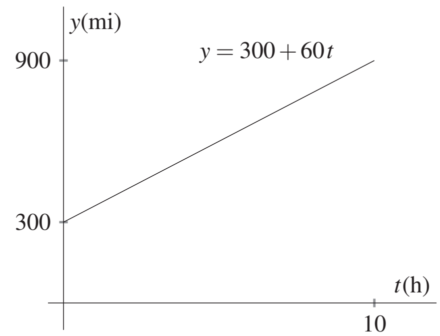
(**7**)



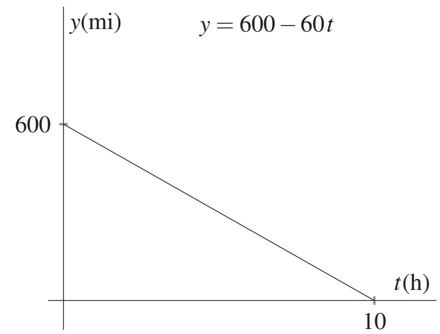
(**8**)

Answers to Exercise 6.14

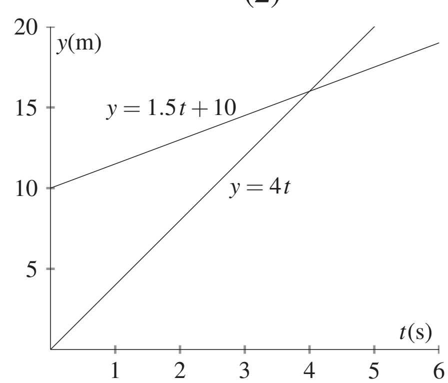
(**1**) (**a**) $y = 300 + 60t$, Domain = all numbers from 0 to 10. Increasing.



(**b**) $y = 600 - 60t$, Domain = all numbers from 0 to 10. Decreasing.



(**2**)



It appears that Barb catches Alex at about 4 seconds.

(**3**) Barb catches Alex at time 4 seconds.

(**4**) True.

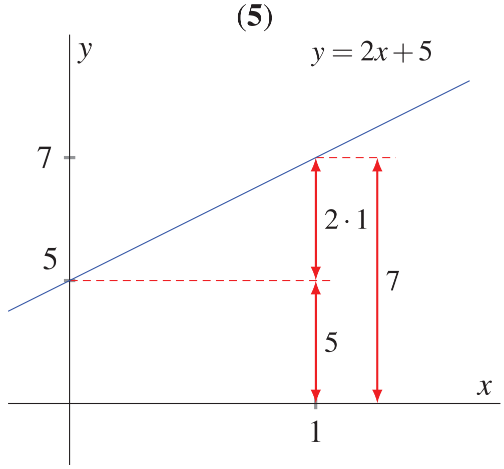
Proof. Let $y = 2x + 5$. Suppose that x changes by an amount Δx and y by an amount Δy. Then

$$\begin{aligned} y + \Delta y &= 2(x + \Delta x) + 5 \\ y + \Delta y &= 2x + 2\Delta x + 5 \\ y + \Delta y &= y + 2\Delta x \\ \Delta y &= 2\Delta x \end{aligned}$$

Therefore, the change in y is 2 times the change in x.

■

(**5**)



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