1.1 Finding a suitable δ

1.1.1 Simple cases

Prove that the limit at $x = 3$ of $\dfrac{2x^2 - 5x - 3}{x - 3}$ is 7; i.e. prove

$$\lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 3} = 7.$$

We must show the definition of limit is satisfied. For every value ϵ takes, we must compute a positive value for δ that guarantees $f(x)$ is within distance ϵ of 7 whenever x is within distance δ of 3, but not at 3. Obviously, the way we compute the value of δ must take into account the value of ϵ. The key to the proof is discovering how δ should depend on ϵ.

Preliminary analysis. The goal is to figure out how to compute δ, given ϵ, so that $0 < |x - 3| < \delta \implies |f(x) - 7| < \epsilon$. We reason as follows.

$$\begin{aligned} |f(x) - 7| &= \left| \frac{2x^2 - 5x - 3}{x - 3} - 7 \right| \\ &= \left| \frac{(2x+1)(x-3)}{x-3} - 7 \right| \end{aligned}$$

since $x \neq 3$ (remember, $0 < |x - 3|$)

$$\begin{aligned} &= |(2x + 1) - 7| \\ &= |2x - 6| \\ &= |2(x - 3)| \\ &= 2\,|x - 3|\,. \end{aligned} \tag{1.1}$$

Since $|f(x) - 7| = 2\,|x - 3|\,,\ |f(x) - 7| < \epsilon \iff 2\,|x - 3| < \epsilon$. *And*

$$2\,|x - 3| < \epsilon \quad \text{whenever} \quad |x - 3| < \frac{\epsilon}{2}.$$

If for every value of ϵ, we choose $\delta = \dfrac{\epsilon}{2}$,

$$2\,|x - 3| < \epsilon \quad \text{whenever} \quad |x - 3| < \delta.$$

Proof. Fix $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. Suppose $0 < |x-3| < \delta$. Then

$$\begin{aligned} |f(x) - 7| &= \left| \frac{2x^2 - 5x - 3}{x - 3} - 7 \right| \\ &= \left| \frac{(2x+1)(x-3)}{x-3} - 7 \right| \\ &= |(2x+1) - 7| \\ &= |2x - 6| \\ &= |2(x-3)| \\ &= 2\,|x-3| \\ &< 2\delta \\ &= 2\frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore

$$\lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 3} = 7,$$

by definition of limit. □

Remark 1.1.

1. Be sure you can find the point in the proof at which we used the supposition $0 < |x - 3| < \delta$.
2. The crucial step in the preliminary analysis is at equation 1.1. You will succeed at finding how to compute δ only if you make the expression that is to be bounded by δ appear.