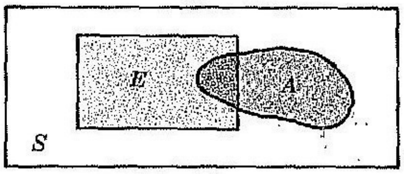
Chapter 4

Conditional Probability and Independence

CONDITIONAL PROBABILITY

Let E be an arbitrary event in a sample space S with $P(E) > 0$. The probability that an event A occurs once E has occurred or, in other words, the *conditional probability* of A given E, written $P(A \mid E)$, is defined as follows:

$$P(A \mid E) = \frac{P(A \cap E)}{P(E)}$$



As seen in the adjoining Venn diagram, $P(A \mid E)$ in a certain sense measures the relative probability of A with respect to the reduced space E.

In particular, if S is a finite equiprobable space and $|A|$ denotes the number of elements in an event A, then

$$P(A \cap E) = \frac{|A \cap E|}{|S|}, \quad P(E) = \frac{|E|}{|S|} \quad \text{and so} \quad P(A \mid E) = \frac{P(A \cap E)}{P(E)} = \frac{|A \cap E|}{|E|}$$

That is,

Theorem 4.1: Let S be a finite equiprobable space with events A and E. Then

$$P(A \mid E) = \frac{\text{number of elements in } A \cap E}{\text{number of elements in } E}$$

or

$$P(A \mid E) = \frac{\text{number of ways } A \text{ and } E \text{ can occur}}{\text{number of ways } E \text{ can occur}}$$

Example 4.1: Let a pair of fair dice be tossed. If the sum is 6, find the probability that one of the dice is a 2. In other words, if

$$E = \{\text{sum is 6}\} = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$$

and

$$A = \{\text{a 2 appears on at least one die}\}$$

find $P(A \mid E)$.

Now E consists of five elements and two of them, $(2,4)$ and $(4,2)$, belong to A: $A \cap E = \{(2,4), (4,2)\}$. Then $P(A \mid E) = \frac{2}{5}$.

On the other hand, since A consists of eleven elements,

$$A = \{(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (1,2), (3,2), (4,2), (5,2), (6,2)\}$$

and S consists of 36 elements, $P(A) = \frac{11}{36}$.

Example 4.2: A couple has two children. Find the probability p that both children are boys if (i) we are given that the younger child is a boy, (ii) we are given that (at least) one of the children is a boy.

The sample space for the sex of two children is $S = \{bb, bg, gb, gg\}$ with probability $\frac{1}{4}$ for each point. (Here the sequence of each point corresponds to the sequence of births.)

(i) The reduced sample space consists of two elements, $\{bb, gb\}$; hence $p = \frac{1}{2}$.

(ii) The reduced sample space consists of three elements, $\{bb, bg, gb\}$; hence $p = \frac{1}{3}$.

MULTIPLICATION THEOREM FOR CONDITIONAL PROBABILITY

If we cross multiply the above equation defining conditional probability and use the fact that $A \cap E = E \cap A$, we obtain the following useful formula.

Theorem 4.2: $P(E \cap A) = P(E)\,P(A \mid E)$

This theorem can be extended by induction as follows:

Corollary 4.3: For any events $A_1, A_2, \ldots, A_n$,

$$P(A_1 \cap A_2 \cap \cdots \cap A_n)$$
$$= P(A_1)\,P(A_2 \mid A_1)\,P(A_3 \mid A_1 \cap A_2) \cdots P(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

We now apply the above theorem which is called, appropriately, the *multiplication theorem.*

Example 4.3: A lot contains 12 items of which 4 are defective. Three items are drawn at random from the lot one after the other. Find the probability p that all three are nondefective.

The probability that the first item is nondefective is $\frac{8}{12}$ since 8 of 12 items are nondefective. If the first item is nondefective, then the probability that the next item is nondefective is $\frac{7}{11}$ since only 7 of the remaining 11 items are nondefective. If the first two items are nondefective, then the probability that the last item is nondefective is $\frac{6}{10}$ since only 6 of the remaining 10 items are now nondefective. Thus by the multiplication theorem,

$$p = \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} = \frac{14}{55}$$

FINITE STOCHASTIC PROCESSES AND TREE DIAGRAMS

A (finite) sequence of experiments in which each experiment has a finite number of outcomes with given probabilities is called a *(finite) stochastic process.* A convenient way of describing such a process and computing the probability of any event is by a *tree diagram* as illustrated below; the multiplication theorem of the previous section is used to compute the probability that the result represented by any given path of the tree does occur.

Example 4.4: We are given three boxes as follows:

Box I has 10 light bulbs of which 4 are defective.

Box II has 6 light bulbs of which 1 is defective.

Box III has 8 light bulbs of which 3 are defective.

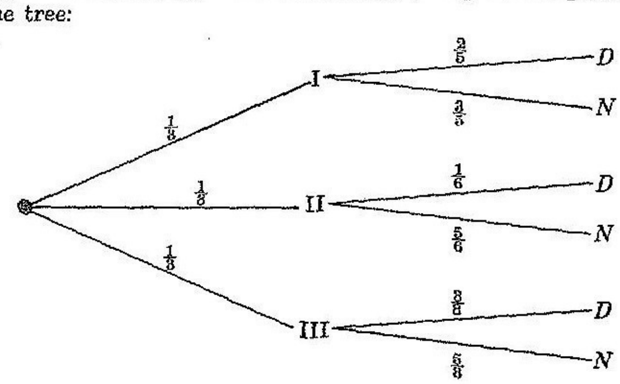
We select a box at random and then draw a bulb at random. What is the probability p that the bulb is defective?

Here we perform a sequence of two experiments:

(i) select one of the three boxes;

(ii) select a bulb which is either defective (D) or nondefective (N).

The following tree diagram describes this process and gives the probability of each branch of the tree:



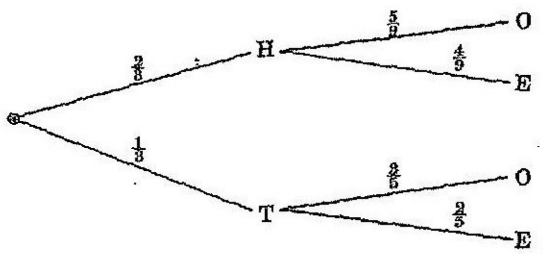
The probability that any particular path of the tree occurs is, by the multiplication theorem, the product of the probabilities of each branch of the path, e.g., the probability of selecting box I and then a defective bulb is $\frac{1}{3}\cdot\frac{2}{5} = \frac{2}{15}$.

Now since there are three mutually exclusive paths which lead to a defective bulb, the sum of the probabilities of these paths is the required probability:

$$p = \frac{1}{3}\cdot\frac{2}{5} + \frac{1}{3}\cdot\frac{1}{6} + \frac{1}{3}\cdot\frac{3}{8} = \frac{113}{360}$$

Example 4.5: A coin, weighted so that $P(\text{H}) = \frac{2}{3}$ and $P(\text{T}) = \frac{1}{3}$, is tossed. If heads appears, then a number is selected at random from the numbers 1 through 9; if tails appears, then a number is selected at random from the numbers 1 through 5. Find the probability p that an even number is selected.

The tree diagram with respective probabilities is



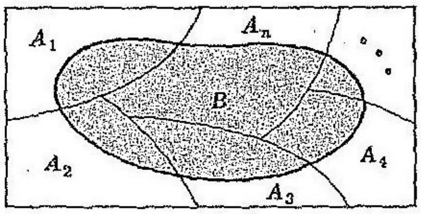
Note that the probability of selecting an even number from the numbers 1 through 9 is $\frac{4}{9}$ since there are 4 even numbers out of the 9 numbers, whereas the probability of selecting an even number from the numbers 1 through 5 is $\frac{2}{5}$ since there are 2 even numbers out of the 5 numbers. Two of the paths lead to an even number: HE and TE. Thus

$$p = P(\text{E}) = \frac{2}{3}\cdot\frac{4}{9} + \frac{1}{3}\cdot\frac{2}{5} = \frac{58}{135}$$

PARTITIONS AND BAYES' THEOREM

Suppose the events $A_1, A_2, \ldots, A_n$ form a partition of a sample space S; that is, the events A_i are mutually exclusive and their union is S. Now let B be any other event. Then

$$B = S \cap B = (A_1 \cup A_2 \cup \cdots \cup A_n) \cap B$$
$$= (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$$



B is shaded.

where the $A_i \cap B$ are also mutually exclusive. Accordingly,

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)$$

Thus by the multiplication theorem,

$$P(B) = P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \cdots + P(A_n)P(B \mid A_n) \qquad (1)$$

On the other hand, for any i, the conditional probability of A_i given B is defined by

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)}$$

In this equation we use (*1*) to replace $P(B)$ and use $P(A_i \cap B) = P(A_i)P(B \mid A_i)$ to replace $P(A_i \cap B)$, thus obtaining

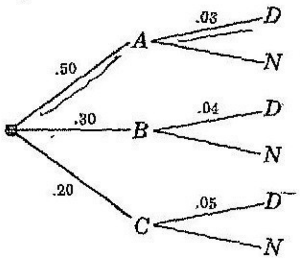
Bayes' Theorem 4.4: Suppose $A_1, A_2, \ldots, A_n$ is a partition of S and B is any event. Then for any i,

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \cdots + P(A_n)P(B \mid A_n)}$$

Example 4.6: Three machines A, B and C produce respectively 50%, 30% and 20% of the total number of items of a factory. The percentages of defective output of these machines are 3%, 4% and 5%. If an item is selected at random, find the probability that the item is defective.

Let X be the event that an item is defective. Then by (1) above,

$$\begin{aligned} P(X) &= P(A)\,P(X \mid A) + P(B)\,P(X \mid B) \\ &\quad + P(C)\,P(X \mid C) \\ &= (.50)(.03) + (.30)(.04) + (.20)(.05) \\ &= .037 \end{aligned}$$



Observe that we can also consider this problem as a stochastic process having the adjoining tree diagram.

Example 4.7: Consider the factory in the preceding example. Suppose an item is selected at random and is found to be defective. Find the probability that the item was produced by machine A; that is, find $P(A \mid X)$.

By Bayes' theorem,

$$\begin{aligned} P(A \mid X) &= \frac{P(A)\,P(X \mid A)}{P(A)\,P(X \mid A) + P(B)\,P(X \mid B) + P(C)\,P(X \mid C)} \\ &= \frac{(.50)(.03)}{(.50)(.03) + (.30)(.04) + (.20)(.05)} = \frac{15}{37} \end{aligned}$$

In other words, we divide the probability of the required path by the probability of the reduced sample space, i.e. those paths which lead to a defective item.

INDEPENDENCE

An event B is said to be *independent* of an event A if the probability that B occurs is not influenced by whether A has or has not occurred. In other words, if the probability of B equals the conditional probility of B given A: $P(B) = P(B \mid A)$. Now substituting $P(B)$ for $P(B \mid A)$ in the multiplications theorem $P(A \cap B) = P(A)\,P(B \mid A)$, we obtain

$$P(A \cap B) = P(A)\,P(B)$$

We use the above equation as our formal definition of independence.

Definition: Events A and B are independent if $P(A \cap B) = P(A)\,P(B)$; otherwise they are dependent.

Example 4.8: Let a fair coin be tossed three times; we obtain the equiprobable space

$$S = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$$

Consider the events

$$A = \{\text{first toss is heads}\}, \qquad B = \{\text{second toss is heads}\}$$
$$C = \{\text{exactly two heads are tossed in a row}\}$$

Clearly A and B are independent events; this fact is verified below. On the other hand, the relationship between A and C or B and C is not obvious. We claim that A and C are independent, but that B and C are dependent. We have

$$P(A) = P(\{\text{HHH, HHT, HTH, HTT}\}) = \tfrac{4}{8} = \tfrac{1}{2}$$
$$P(B) = P(\{\text{HHH, HHT, THH, THT}\}) = \tfrac{4}{8} = \tfrac{1}{2}$$
$$P(C) = P(\{\text{HHT, THH}\}) = \tfrac{2}{8} = \tfrac{1}{4}$$

Then

$$P(A \cap B) = P(\{\text{HHH, HHT}\}) = \tfrac{1}{4}, \qquad P(A \cap C) = P(\{\text{HHT}\}) = \tfrac{1}{8},$$
$$P(B \cap C) = P(\{\text{HHT, THH}\}) = \tfrac{1}{4}$$

Accordingly,

$$P(A)\,P(B) \;=\; \frac{1}{2}\cdot\frac{1}{2} \;=\; \frac{1}{4} \;=\; P(A\cap B), \quad \text{and so } A \text{ and } B \text{ are independent;}$$

$$P(A)\,P(C) \;=\; \frac{1}{2}\cdot\frac{1}{4} \;=\; \frac{1}{8} \;=\; P(A\cap C), \quad \text{and so } A \text{ and } C \text{ are independent;}$$

$$P(B)\,P(C) \;=\; \frac{1}{2}\cdot\frac{1}{4} \;=\; \frac{1}{8} \;\neq\; P(B\cap C), \quad \text{and so } B \text{ and } C \text{ are dependent.}$$

Frequently, we will postulate that two events are independent, or it will be clear from the nature of the experiment that two events are independent.

Example 4.9: The probability that A hits a target is $\frac{1}{4}$ and the probability that B hits it is $\frac{2}{5}$. What is the probability that the target will be hit if A and B each shoot at the target?

We are given that $P(A) = \frac{1}{4}$ and $P(B) = \frac{2}{5}$, and we seek $P(A\cup B)$. Furthermore, the probability that A or B hits the target is not influenced by what the other does; that is, the event that A hits the target is independent of the event that B hits the target: $P(A\cap B) = P(A)\,P(B)$. Thus

$$P(A\cup B) \;=\; P(A) + P(B) - P(A\cap B) \;=\; P(A) + P(B) - P(A)\,P(B)$$
$$= \frac{1}{4} + \frac{2}{5} - \frac{1}{4}\cdot\frac{2}{5} \;=\; \frac{11}{20}.$$

Three events A, B and C are *independent* if:

(i) $P(A\cap B) = P(A)\,P(B)$, $P(A\cap C) = P(A)\,P(C)$ and $P(B\cap C) = P(B)\,P(C)$

i.e. if the events are pairwise independent, and

(ii) $P(A\cap B\cap C) = P(A)\,P(B)\,P(C)$.

The next example shows that condition (ii) does not follow from condition (i); in other words, three events may be pairwise independent but not independent themselves.

Example 4.10: Let a pair of fair coins be tossed; here $S = \{\text{HH, HT, TH, TT}\}$ is an equiprobable space. Consider the events

$$\begin{aligned} A &= \{\text{heads on the first coin}\} &&= \{\text{HH, HT}\}\\ B &= \{\text{heads on the second coin}\} &&= \{\text{HH, TH}\}\\ C &= \{\text{heads on exactly one coin}\} &&= \{\text{HT, TH}\} \end{aligned}$$

Then $P(A) = P(B) = P(C) = \frac{2}{4} = \frac{1}{2}$ and

$$P(A\cap B) = P(\{\text{HH}\}) = \frac{1}{4}, \quad P(A\cap C) = P(\{\text{HT}\}) = \frac{1}{4}, \quad P(B\cap C) = (\{\text{TH}\}) = \frac{1}{4}$$

Thus condition (i) is satisfied, i.e., the events are pairwise independent. However, $A\cap B\cap C = \emptyset$ and so

$$P(A\cap B\cap C) \;=\; P(\emptyset) \;=\; 0 \;\neq\; P(A)\,P(B)\,P(C)$$

In other words, condition (ii) is not satisfied and so the three events are not independent.

INDEPENDENT OR REPEATED TRIALS

We have previously discussed probability spaces which were associated with an experiment repeated a finite number of times, as the tossing of a coin three times. This concept of repetition is formalized as follows:

Definition: Let S be a finite probability space. By *n independent* or *repeated trials*, we mean the probability space T consisting of ordered n-tuples of elements of S with the probability of an n-tuple defined to be the product of the probabilities of its components:

$$P((s_1, s_2, \ldots, s_n)) \;=\; P(s_1)\,P(s_2)\cdots P(s_n)$$

Example 4.11: Whenever three horses a, b and c race together, their respective probabilities of winning are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{6}$. In other words, $S = \{a, b, c\}$ with $P(a) = \frac{1}{2}$, $P(b) = \frac{1}{3}$ and $P(c) = \frac{1}{6}$. If the horses race twice, then the sample space of the 2 repeated trials is

$$T = \{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$$

For notational convenience, we have written ac for the ordered pair (a, c). The probability of each point in T is

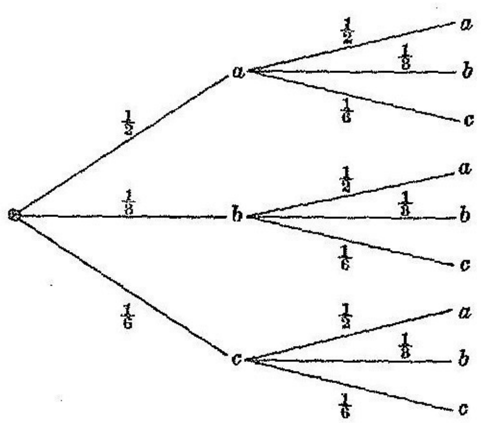
$$P(aa) = P(a)P(a) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} \qquad P(ba) = \frac{1}{6} \qquad P(ca) = \frac{1}{12}$$

$$P(ab) = P(a)P(b) = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6} \qquad P(bb) = \frac{1}{9} \qquad P(cb) = \frac{1}{18}$$

$$P(ac) = P(a)P(c) = \frac{1}{2}\cdot\frac{1}{6} = \frac{1}{12} \qquad P(bc) = \frac{1}{18} \qquad P(cc) = \frac{1}{36}$$

Thus the probability of c winning the first race and a winning the second race is $P(ca) = \frac{1}{12}$.

From another point of view, a repeated trials process is a stochastic process whose tree diagram has the following properties: (i) every branch point has the same outcomes; (ii) the probability is the same for each branch leading to the same outcome. For example, the tree diagram of the repeated trials process of the preceding experiment is as shown in the adjoining figure.



Observe that every branch point has the outcomes a, b and c, and each branch leading to outcome a has probability $\frac{1}{2}$, each branch leading to b has probability $\frac{1}{3}$, and each leading to c has probability $\frac{1}{6}$.

Solved Problems

CONDITIONAL PROBABILITY IN FINITE EQUIPROBABLE SPACES

4.1. A pair of fair dice is thrown. Find the probability p that the sum is 10 or greater if (i) a 5 appears on the first die, (ii) a 5 appears on at least one of the dice.

(i) If a 5 appears on the first die, then the reduced sample space is

$$A = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$$

The sum is 10 or greater on two of the six outcomes: $(5,5)$, $(5,6)$. Hence $p = \frac{2}{6} = \frac{1}{3}$.

(ii) If a 5 appears on at least one of the dice, then the reduced sample space has eleven elements:

$$B = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (1,5), (2,5), (3,5), (4,5), (6,5)\}$$

The sum is 10 or greater on three of the eleven outcomes: $(5,5)$, $(5,6)$, $(6,5)$. Hence $p = \frac{3}{11}$.

4.2. Three fair coins are tossed. Find the probability p that they are all heads if (i) the first coin is heads, (ii) one of the coins is heads.

The sample space has eight elements: $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

(i) If the first coin is heads, the reduced sample space is $A = \{HHH, HHT, HTH, HTT\}$. Since the coins are all heads in 1 of 4 cases, $p = \frac{1}{4}$.

(ii) If one of the coins is heads, the reduced sample space is $B = \{HHH, HHT, HTH, HTT, THH, THT, TTH\}$. Since the coins are all heads in 1 of 7 cases, $p = \frac{1}{7}$.

4.3. A pair of fair dice is thrown. If the two numbers appearing are different, find the probability p that (i) the sum is six, (ii) an ace appears, (iii) the sum is 4 or less.

Of the 36 ways the pair of dice can be thrown, 6 will contain the same numbers: $(1,1), (2,2), \ldots, (6,6)$. Thus the reduced sample space will consist of $36 - 6 = 30$ elements.

(i) The sum 6 can appear in 4 ways: $(1,5), (2,4), (4,2), (5,1)$. (We cannot include $(3,3)$ since the numbers are the same.) Hence $p = \frac{4}{30} = \frac{2}{15}$.

(ii) An ace can appear in 10 ways: $(1,2), (1,3), \ldots, (1,6)$ and $(2,1), (3,1), \ldots, (6,1)$. Hence $p = \frac{10}{30} = \frac{1}{3}$.

(iii) The sum of 4 or less can occur in 4 ways: $(3,1), (1,3), (2,1), (1,2)$. Thus $p = \frac{4}{30} = \frac{2}{15}$.

4.4. Two digits are selected at random from the digits 1 through 9. If the sum is even, find the probability p that both numbers are odd.

The sum is even if both numbers are even or if both numbers are odd. There are 4 even numbers $(2,4,6,8)$; hence there are $\binom{4}{2} = 6$ ways to choose two even numbers. There are 5 odd numbers $(1,3,5,7,9)$; hence there are $\binom{5}{2} = 10$ ways to choose two odd numbers. Thus there are $6 + 10 = 16$ ways to choose two numbers such that their sum is even; since 10 of these ways occur when both numbers are odd, $p = \frac{10}{16} = \frac{5}{8}$.

4.5. A man is dealt 4 spade cards from an ordinary deck of 52 cards. If he is given three more cards, find the probability p that at least one of the additional cards is also a spade.

Since he is dealt 4 spades, there are $52 - 4 = 48$ cards remaining of which $13 - 4 = 9$ are spades. There are $\binom{48}{3} = 17{,}296$ ways in which he can be dealt three more cards. Since there are $48 - 9 = 39$ cards which are not spades, there are $\binom{39}{3} = 9139$ ways he can be dealt three cards which are not spades. Thus the probability q that he is not dealt another spade is $q = \frac{9139}{17{,}296}$; hence $p = 1 - q = \frac{8157}{17{,}296}$.

4.6. Four people, called North, South, East and West, are each dealt 13 cards from an ordinary deck of 52 cards.

(i) If South has no aces, find the probability p that his partner North has exactly two aces.

(ii) If North and South together have nine hearts, find the probability p that East and West each has two hearts.

(i) There are 39 cards, including 4 aces, divided among North, East and West. There are $\binom{39}{13}$ ways that North can be dealt 13 of the 39 cards. There are $\binom{4}{2}$ ways he can be dealt 2 of the four aces, and $\binom{35}{11}$ ways he can be dealt 11 cards from the $39 - 4 = 35$ cards which are not aces. Thus

$$p = \frac{\binom{4}{2}\binom{35}{11}}{\binom{39}{13}} = \frac{6 \cdot 12 \cdot 13 \cdot 25 \cdot 26}{36 \cdot 37 \cdot 38 \cdot 39} = \frac{650}{2109}$$

(ii) There are 26 cards, including 4 hearts, divided among East and West. There are $\binom{26}{13}$ ways that, say, East can be dealt 13 cards. (We need only analyze East's 13 cards since West must have the *remaining* cards.) There are $\binom{4}{2}$ ways East can be dealt 2 hearts from 4 hearts, and $\binom{22}{11}$ ways he can be dealt 11 non-hearts from the $26-4=22$ non-hearts. Thus

$$p = \frac{\binom{4}{2}\binom{22}{11}}{\binom{26}{13}} = \frac{6\cdot 12\cdot 13\cdot 12\cdot 13}{23\cdot 24\cdot 25\cdot 26} = \frac{234}{575}$$

MULTIPLICATION THEOREM

4.7. A class has 12 boys and 4 girls. If three students are selected at random from the class, what is the probability p that they are all boys?

The probability that the first student selected is a boy is 12/16 since there are 12 boys out of 16 students. If the first student is a boy, then the probability that the second is a boy is 11/15 since there are 11 boys left out of 15 students. Finally, if the first two students selected were boys, then the probability that the third student is a boy is 10/14 since there are 10 boys left out of 14 students. Thus, by the multiplication theorem, the probability that all three are boys is

$$p = \frac{12}{16}\cdot\frac{11}{15}\cdot\frac{10}{14} = \frac{11}{28}$$

Another Method. There are $\binom{16}{3} = 560$ ways to select 3 students of the 16 students, and $\binom{12}{3} = 220$ ways to select 3 boys out of 12 boys; hence $p = \frac{220}{560} = \frac{11}{28}$.

A Third Method. If the students are selected one after the other, then there are $16\cdot 15\cdot 14$ ways to select three students, and $12\cdot 11\cdot 10$ ways to select three boys; hence $p = \frac{12\cdot 11\cdot 10}{16\cdot 15\cdot 14} = \frac{11}{28}$.

4.8. A man is dealt 5 cards one after the other from an ordinary deck of 52 cards. What is the probability p that they are all spades?

The probability that the first card is a spade is 13/52, the second is a spade is 12/51, the third is a spade is 11/50, the fourth is a spade is 10/49, and the last is a spade is 9/48. (We assumed in each case that the previous cards were spades.) Thus $p = \frac{13}{52}\cdot\frac{12}{51}\cdot\frac{11}{50}\cdot\frac{10}{49}\cdot\frac{9}{48} = \frac{33}{66,640}$.

4.9. An urn contains 7 red marbles and 3 white marbles. Three marbles are drawn from the urn one after the other. Find the probability p that the first two are red and the third is white.

The probability that the first marble is red is 7/10 since there are 7 red marbles out of 10 marbles. If the first marble is red, then the probability that the second marble is red is 6/9 since there are 6 red marbles remaining out of the 9 marbles. If the first two marbles are red, then the probability that the third marble is white is 3/8 since there are 3 white marbles out of the 8 marbles in the urn. Hence by the multiplication theorem,

$$p = \frac{7}{10}\cdot\frac{6}{9}\cdot\frac{3}{8} = \frac{7}{40}$$

4.10. The students in a class are selected at random, one after the other, for an examination. Find the probability p that the boys and girls in the class alternate if (i) the class consists of 4 boys and 3 girls, (ii) the class consists of 3 boys and 3 girls.

(i) If the boys and girls are to alternate, then the first student examined must be a boy. The probability that the first is a boy is 4/7. If the first is a boy, then the probability that the second is a girl is 3/6 since there are 3 girls out of 6 students left. Continuing in this manner, we obtain the probability that the third is a boy is 3/5, the fourth is a girl is 2/4, the fifth is a boy is 2/3, the sixth is a girl is 1/2, and the last is a boy is 1/1. Thus

$$p = \frac{4}{7}\cdot\frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{35}$$

(ii) There are two mutually exclusive cases: the first pupil is a boy, and the first is a girl. If the first student is a boy, then by the multiplication theorem the probability p_1 that the students alternate is

$$p_1 = \frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{20}$$

If the first student is a girl, then by the multiplication theorem the probability p_2 that the students alternate is

$$p_2 = \frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{20}$$

Thus $p = p_1 + p_2 = \frac{1}{20} + \frac{1}{20} = \frac{1}{10}$.

MISCELLANEOUS PROBLEMS ON CONDITIONAL PROBABILITY

4.11. In a certain college, 25% of the students failed mathematics, 15% of the students failed chemistry, and 10% of the students failed both mathematics and chemistry. A student is selected at random.

(i) If he failed chemistry, what is the probability that he failed mathematics?

(ii) If he failed mathematics, what is the probability that he failed chemistry?

(iii) What is the probability that he failed mathematics or chemistry?

Let M = {students who failed mathematics} and C = {students who failed chemistry}; then

$$P(M) = .25, \quad P(C) = .15, \quad P(M \cap C) = .10$$

(i) The probability that a student failed mathematics, given that he has failed chemistry is

$$P(M \mid C) = \frac{P(M \cap C)}{P(C)} = \frac{.10}{.15} = \frac{2}{3}$$

(ii) The probability that a student failed chemistry, given that he has failed mathematics is

$$P(C \mid M) = \frac{P(C \cap M)}{P(M)} = \frac{.10}{.25} = \frac{2}{5}$$

(iii)
$$P(M \cup C) = P(M) + P(C) - P(M \cap C) = .25 + .15 - .10 = .30 = \frac{3}{10}$$

4.12. Let A and B be events with $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$. Find (i) $P(A \mid B)$, (ii) $P(B \mid A)$, (iii) $P(A \cup B)$, (iv) $P(A^c \mid B^c)$, (v) $P(B^c \mid A^c)$.

(i) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{1}{4}}{\frac{1}{3}} = \dfrac{3}{4}$ (ii) $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \dfrac{1}{2}$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} = \dfrac{7}{12}$

(iv) First compute $P(B^c)$ and $P(A^c \cap B^c)$. $P(B^c) = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3}$. By De Morgan's law, $(A \cup B)^c = A^c \cap B^c$; hence $P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - \frac{7}{12} = \frac{5}{12}$.

Thus $P(A^c \mid B^c) = \dfrac{P(A^c \cap B^c)}{P(B^c)} = \dfrac{\frac{5}{12}}{\frac{2}{3}} = \dfrac{5}{8}$.

(v) $P(A^c) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$. Then $P(B^c \mid A^c) = \dfrac{P(B^c \cap A^c)}{P(A^c)} = \dfrac{\frac{5}{12}}{\frac{1}{2}} = \dfrac{5}{6}$.

4.13. Let A and B be events with $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$ and $P(A \cup B) = \frac{3}{4}$. Find $P(A \mid B)$ and $P(B \mid A)$.

First compute $P(A \cap B)$ using the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$:

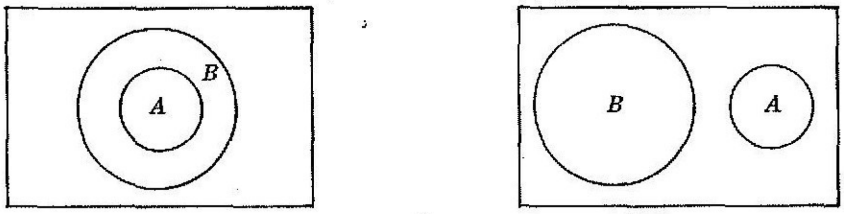
$$\frac{3}{4} = \frac{3}{8} + \frac{5}{8} - P(A \cap B) \quad \text{or} \quad P(A \cap B) = \frac{1}{4}$$

Then $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{1}{4}}{\frac{5}{8}} = \dfrac{2}{5}$ and $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{\frac{1}{4}}{\frac{3}{8}} = \dfrac{2}{3}$.

4.14. Find $P(B \mid A)$ if (i) A is a subset of B, (ii) A and B are mutually exclusive.

(i) If A is a subset of B, then whenever A occurs B must occur; hence $P(B \mid A) = 1$. Alternately, if A is a subset of B then $A \cap B = A$; hence

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1$$



(i) (ii)

(ii) If A and B are mutually exclusive, i.e. disjoint, then whenever A occurs B cannot occur; hence $P(B \mid A) = 0$. Alternately, if A and B are mutually exclusive then $A \cap B = \emptyset$; hence

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(\emptyset)}{P(A)} = \frac{0}{P(A)} = 0$$

4.15. Three machines A, B and C produce respectively 60%, 30% and 10% of the total number of items of a factory. The percentages of defective output of these machines are respectively 2%, 3% and 4%. An item is selected at random and is found defective. Find the probability that the item was produced by machine C.

Let X = {*defective items*}. We seek $P(C \mid X)$, the probability that an item is produced by machine C given that the item is defective. By Bayes' theorem,

$$P(C \mid X) = \frac{P(C)\,P(X \mid C)}{P(A)\,P(X \mid A) + P(B)\,P(X \mid B) + P(C)\,P(X \mid C)}$$

$$= \frac{(.10)(.04)}{(.60)(.02) + (.30)(.03) + (.10)(.04)} = \frac{4}{25}$$

4.16. In a certain college, 4% of the men and 1% of the women are taller than 6 feet. Furthermore, 60% of the students are women. Now if a student is selected at random and is taller than 6 feet, what is the probability that the student is a woman?

Let A = {students taller than 6 feet}. We seek $P(W \mid A)$, the probability that a student is a woman given that the student is taller than 6 feet. By Bayes' theorem,

$$P(W \mid A) = \frac{P(W)\,P(A \mid W)}{P(W)\,P(A \mid W) + P(M)\,P(A \mid M)} = \frac{(.60)(.01)}{(.60)(.01) + (.40)(.04)} = \frac{3}{11}$$

4.17. Let E be an event for which $P(E) > 0$. Show that the conditional probability function $P(* \mid E)$ satisfies the axioms of a probability space; that is,

[P_1] For any event A, $0 \leq P(A \mid E) \leq 1$.

[P_2] For the certain event S, $P(S \mid E) = 1$.

[P_3] If A and B are mutually exclusive, then $P(A \cup B \mid E) = P(A \mid E) + P(B \mid E)$.

[P_4] If $A_1, A_2, \ldots$ is a sequence of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \cdots \mid E) = P(A_1 \mid E) + P(A_2 \mid E) + \cdots$$

(i) We have $A \cap E \subset E$; hence $P(A \cap E) \leq P(E)$. Thus $P(A \mid E) = \frac{P(A \cap E)}{P(E)} \leq 1$ and is also non-negative. That is, $0 \leq P(A \mid E) \leq 1$ and so [P_1] holds.

(ii) We have $S \cap E = E$; hence $P(S \mid E) = \dfrac{P(S \cap E)}{P(E)} = \dfrac{P(E)}{P(E)} = 1$. Thus $[\mathbf{P}_2]$ holds.

(iii) If A and B are mutually exclusive events, then so are $A \cap E$ and $B \cap E$. Furthermore, $(A \cup B) \cap E = (A \cap E) \cup (B \cap E)$. Thus

$$P((A \cup B) \cap E) = P((A \cap E) \cup (B \cap E)) = P(A \cap E) + P(B \cap E)$$

and therefore

$$P(A \cup B \mid E) = \frac{P((A \cup B) \cap E)}{P(E)} = \frac{P(A \cap E) + P(B \cap E)}{P(E)}$$
$$= \frac{P(A \cap E)}{P(E)} + \frac{P(B \cap E)}{P(E)} = P(A \mid E) + P(B \mid E)$$

Hence $[\mathbf{P}_3]$ holds.

(iv) Similarly if $A_1, A_2, \ldots$ are mutually exclusive, then so are $A_1 \cap E, A_2 \cap E, \ldots$. Thus

$$P((A_1 \cup A_2 \cup \cdots) \cap E) = P((A_1 \cap E) \cup (A_2 \cap E) \cup \cdots) = P(A_1 \cap E) + P(A_2 \cap E) + \cdots$$

and therefore

$$P(A_1 \cup A_2 \cup \cdots \mid E) = \frac{P((A_1 \cup A_2 \cup \cdots) \cap E)}{P(E)} = \frac{P(A_1 \cap E) + P(A_2 \cap E) + \cdots}{P(E)}$$
$$= \frac{P(A_1 \cap E)}{P(E)} + \frac{P(A_2 \cap E)}{P(E)} + \cdots = P(A_1 \mid E) + P(A_2 \mid E) + \cdots$$

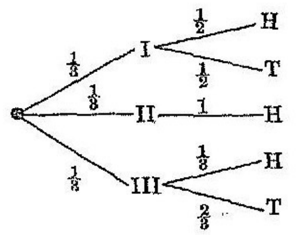
That is, $[\mathbf{P}_4]$ holds.

FINITE STOCHASTIC PROCESSES

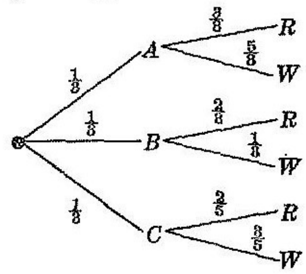
4.18. A box contains three coins; one coin is fair, one coin is two-headed, and one coin is weighted so that the probability of heads appearing is $\frac{1}{3}$. A coin is selected at random and tossed. Find the probability p that heads appears.

Construct the tree diagram as shown in Figure (*a*) below. Note that I refers to the fair coin, II to the two-headed coin, and III to the weighted coin. Now heads appears along three of the paths; hence

$$p = \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{3} = \frac{11}{18}$$



(*a*)



(*b*)

4.19. We are given three urns as follows:

Urn A contains 3 red and 5 white marbles.
Urn B contains 2 red and 1 white marble.
Urn C contains 2 red and 3 white marbles.

An urn is selected at random and a marble is drawn from the urn. If the marble is red, what is the probability that it came from urn A?

Construct the tree diagram as shown in Figure (*b*) above.

We seek the probability that A was selected, given that the marble is red; that is, $P(A \mid R)$. In order to find $P(A \mid R)$, it is necessary first to compute $P(A \cap R)$ and $P(R)$.

The probability that urn A is selected and a red marble drawn is $\frac{1}{3} \cdot \frac{3}{8} = \frac{1}{8}$; that is, $P(A \cap R) = \frac{1}{8}$. Since there are three paths leading to a red marble, $P(R) = \frac{1}{3} \cdot \frac{3}{8} + \frac{1}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{2}{5} = \frac{173}{360}$. Thus

$$P(A \mid R) = \frac{P(A \cap R)}{P(R)} = \frac{\frac{1}{8}}{\frac{173}{360}} = \frac{45}{173}$$

Alternately, by Bayes' theorem,

$$P(A \mid R) = \frac{P(A)\,P(R \mid A)}{P(A)\,P(R \mid A) + P(B)\,P(R \mid B) + P(C)\,P(R \mid C)}$$

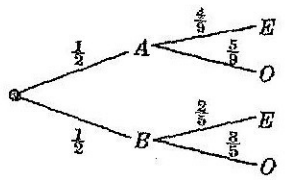
$$= \frac{\frac{1}{3}\cdot\frac{3}{8}}{\frac{1}{3}\cdot\frac{3}{8} + \frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{2}{5}} = \frac{45}{173}$$

4.20. Box A contains nine cards numbered 1 through 9, and box B contains five cards numbered 1 through 5. A box is chosen at random and a card drawn. If the number is even, find the probability that the card came from box A.

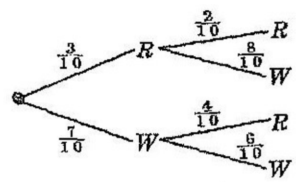
The tree diagram of the process is shown in Figure (*a*) below.

We seek $P(A \mid E)$, the probability that A was selected, given that the number is even. The probability that box A and an even number is drawn is $\frac{1}{2}\cdot\frac{4}{9} = \frac{2}{9}$; that is, $P(A \cap E) = \frac{2}{9}$. Since there are two paths which lead to an even number, $P(E) = \frac{1}{2}\cdot\frac{4}{9} + \frac{1}{2}\cdot\frac{2}{5} = \frac{19}{45}$. Thus

$$P(A \mid E) = \frac{P(A \cap E)}{P(E)} = \frac{\frac{2}{9}}{\frac{19}{45}} = \frac{10}{19}$$



(*a*)



(*b*)

4.21. An urn contains 3 red marbles and 7 white marbles. A marble is drawn from the urn and a marble of the other color is then put into the urn. A second marble is drawn from the urn.

(i) Find the probability p that the second marble is red.

(ii) If both marbles were of the same color, what is the probability p that they were both white?

Construct the tree diagram as shown in Figure (*b*) above.

(i) Two paths of the tree lead to a red marble: $p = \frac{3}{10}\cdot\frac{2}{10} + \frac{7}{10}\cdot\frac{4}{10} = \frac{17}{50}$.

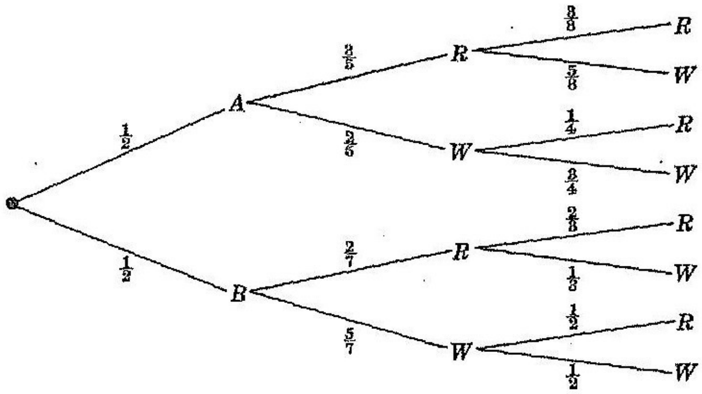
(ii) The probability that both marbles were white is $\frac{7}{10}\cdot\frac{6}{10} = \frac{21}{50}$. The probability that both marbles were of the same color, i.e. the probability of the reduced sample space, is $\frac{3}{10}\cdot\frac{2}{10} + \frac{7}{10}\cdot\frac{6}{10} = \frac{12}{25}$. Hence the conditional probability $p = \frac{21}{50}/\frac{12}{25} = \frac{7}{8}$.

4.22. We are given two urns as follows:

Urn A contains 3 red and 2 white marbles.
Urn B contains 2 red and 5 white marbles.

An urn is selected at random; a marble is drawn and put into the other urn; then a marble is drawn from the second urn. Find the probability p that both marbles drawn are of the same color.

Construct the following tree diagram:



Note that if urn A is selected and a red marble drawn and put into urn B, then urn B has 3 red marbles and 5 white marbles.

Since there are four paths which lead to two marbles of the same color,

$$p = \frac{1}{2}\cdot\frac{3}{5}\cdot\frac{3}{8} + \frac{1}{2}\cdot\frac{2}{5}\cdot\frac{3}{4} + \frac{1}{2}\cdot\frac{2}{7}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{5}{7}\cdot\frac{1}{2} = \frac{901}{1680}$$

INDEPENDENCE

4.23. Let A = event that a family has children of both sexes, and let B = event that a family has at most one boy. (i) Show that A and B are independent events if a family has three children. (ii) Show that A and B are dependent events if a family has two children.

(i) We have the equiprobable space $S = \{bbb, bbg, bgb, bgg, gbb, gbg, ggb, ggg\}$. Here

$$A = \{bbg, bgb, bgg, gbb, gbg, ggb\} \quad \text{and so} \quad P(A) = \frac{6}{8} = \frac{3}{4}$$

$$B = \{bgg, gbg, ggb, ggg\} \quad \text{and so} \quad P(B) = \frac{4}{8} = \frac{1}{2}$$

$$A \cap B = \{bgg, gbg, ggb\} \quad \text{and so} \quad P(A \cap B) = \frac{3}{8}$$

Since $P(A)\,P(B) = \frac{3}{4}\cdot\frac{1}{2} = \frac{3}{8} = P(A \cap B)$, A and B are independent.

(ii) We have the equiprobable space $S = \{bb, bg, gb, gg\}$. Here

$$A = \{bg, gb\} \quad \text{and so} \quad P(A) = \frac{1}{2}$$

$$B = \{bg, gb, gg\} \quad \text{and so} \quad P(B) = \frac{3}{4}$$

$$A \cap B = \{bg, gb\} \quad \text{and so} \quad P(A \cap B) = \frac{1}{2}$$

Since $P(A)\,P(B) \neq P(A \cap B)$, A and B are dependent.

4.24. Prove: If A and B are independent events, then A^c and B^c are independent events.

$$\begin{aligned} P(A^c \cap B^c) &= P((A \cup B)^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B) \\ &= 1 - P(A) - P(B) + P(A)\,P(B) = [1 - P(A)][1 - P(B)] = P(A^c)\,P(B^c) \end{aligned}$$

4.25. The probability that a man will live 10 more years is $\frac{1}{4}$, and the probability that his wife will live 10 more years is $\frac{1}{3}$. Find the probability that (i) both will be alive in 10 years, (ii) at least one will be alive in 10 years, (iii) neither will be alive in 10 years, (iv) only the wife will be alive in 10 years.

Let A = event that the man is alive in 10 years, and B = event that his wife is alive in 10 years; then $P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{3}$.

(i) We seek $P(A \cap B)$. Since A and B are independent, $P(A \cap B) = P(A)\,P(B) = \frac{1}{4}\cdot\frac{1}{3} = \frac{1}{12}$.

(ii) We seek $P(A \cup B)$. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{1}{3} - \frac{1}{12} = \frac{1}{2}$.

(iii) We seek $P(A^c \cap B^c)$. Now $P(A^c) = 1 - P(A) = 1 - \frac{1}{4} = \frac{3}{4}$ and $P(B^c) = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3}$. Furthermore, since A^c and B^c are independent, $P(A^c \cap B^c) = P(A^c)\,P(B^c) = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$.

Alternately, since $(A \cup B)^c = A^c \cap B^c$, $P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - \frac{1}{2} = \frac{1}{2}$.

(iv) We seek $P(A^c \cap B)$. Since $P(A^c) = 1 - P(A) = \frac{3}{4}$ and A^c and B are independent (see Problem 4.56), $P(A^c \cap B) = P(A^c)\,P(B) = \frac{1}{4}$.

4.26. Box A contains 8 items of which 3 are defective, and box B contains 5 items of which 2 are defective. An item is drawn at random from each box.

(i) What is the probability p that both items are nondefective?

(ii) What is the probability p that one item is defective and one not?

(iii) If one item is defective and one is not, what is the probability p that the defective item came from box A?

(i) The probability of choosing a nondefective item from A is $\frac{5}{8}$ and from B is $\frac{3}{5}$. Since the events are independent, $p = \frac{5}{8} \cdot \frac{3}{5} = \frac{3}{8}$.

(ii) **Method 1.** The probability of choosing two defective items is $\frac{3}{8} \cdot \frac{2}{5} = \frac{3}{20}$. From (i) the probability that both are nondefective is $\frac{3}{8}$. Hence $p = 1 - \frac{3}{8} - \frac{3}{20} = \frac{19}{40}$.

Method 2. The probability p_1 of choosing a defective item from A and a nondefective item from B is $\frac{3}{8} \cdot \frac{3}{5} = \frac{9}{40}$. The probability p_2 of choosing a nondefective item from A and a defective item from B is $\frac{5}{8} \cdot \frac{2}{5} = \frac{1}{4}$. Hence $p = p_1 + p_2 = \frac{9}{40} + \frac{1}{4} = \frac{19}{40}$.

(iii) Consider the events $X = \{$defective item from $A\}$ and $Y = \{$one item is defective and one nondefective$\}$. We seek $P(X \mid Y)$. By (ii), $P(X \cap Y) = p_1 = \frac{9}{40}$ and $P(Y) = \frac{19}{40}$. Hence

$$p = P(X \mid Y) = \frac{P(X \cap Y)}{P(Y)} = \frac{\frac{9}{40}}{\frac{19}{40}} = \frac{9}{19}$$

4.27. The probabilities that three men hit a target are respectively $\frac{1}{6}$, $\frac{1}{4}$ and $\frac{1}{3}$. Each shoots once at the target. (i) Find the probability p that exactly one of them hits the target. (ii) If only one hit the target, what is the probability that it was the first man?

Consider the events $A = \{$first man hits the target$\}$, $B = \{$second man hits the target$\}$, and $C = \{$third man hits the target$\}$; then $P(A) = \frac{1}{6}$, $P(B) = \frac{1}{4}$ and $P(C) = \frac{1}{3}$. The three events are independent, and $P(A^c) = \frac{5}{6}$, $P(B^c) = \frac{3}{4}$, $P(C^c) = \frac{2}{3}$.

(i) Let $E = \{$exactly one man hits the target$\}$. Then

$$E = (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$$

In other words, if only one hit the target, then it was either only the first man, $A \cap B^c \cap C^c$, or only the second man, $A^c \cap B \cap C^c$, or only the third man, $A^c \cap B^c \cap C$. Since the three events are mutually exclusive, we obtain (using Problem 4.62)

$$\begin{aligned} p = P(E) &= P(A \cap B^c \cap C^c) + P(A^c \cap B \cap C^c) + P(A^c \cap B^c \cap C) \\ &= P(A)\,P(B^c)\,P(C^c) + P(A^c)\,P(B)\,P(C^c) + P(A^c)\,P(B^c)\,P(C) \\ &= \frac{1}{6} \cdot \frac{3}{4} \cdot \frac{2}{3} + \frac{5}{6} \cdot \frac{1}{4} \cdot \frac{2}{3} + \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{12} + \frac{5}{36} + \frac{5}{24} = \frac{31}{72} \end{aligned}$$

(ii) We seek $P(A \mid E)$, the probability that the first man hit the target given that only one man hit the target. Now $A \cap E = A \cap B^c \cap C^c$ is the event that only the first man hit the target. By (i), $P(A \cap E) = P(A \cap B^c \cap C^c) = \frac{1}{12}$ and $P(E) = \frac{31}{72}$; hence

$$P(A \mid E) = \frac{P(A \cap E)}{P(E)} = \frac{\frac{1}{12}}{\frac{31}{72}} = \frac{6}{31}$$

INDEPENDENT TRIALS

4.28. A certain type of missile hits its target with probability .3. How many missiles should be fired so that there is at least an 80% probability of hitting a target?

The probability of a missile missing its target is .7; hence the probability that n missiles miss a target is $(.7)^n$. Thus we seek the smallest n for which

$$1 - (.7)^n > .8 \quad \text{or equivalently} \quad (.7)^n < .2$$

Compute: $(.7)^1 = .7$, $(.7)^2 = .49$, $(.7)^3 = .343$, $(.7)^4 = .2401$, $(.7)^5 = .16807$. Thus at least 5 missiles should be fired.

4.29. A certain soccer team wins (W) with probability .6, loses (L) with probability .3 and ties (T) with probability .1. The team plays three games over the weekend. (i) Determine the elements of the event A that the team wins at least twice and doesn't lose; and find $P(A)$. (ii) Determine the elements of the event B that the team wins, loses and ties; and find $P(B)$.

(i) A consists of all ordered triples with at least 2 W's and no L's. Thus

$$A = \{\text{WWW, WWT, WTW, TWW}\}$$

Furthermore,
$$\begin{aligned} P(A) &= P(\text{WWW}) + P(\text{WWT}) + P(\text{WTW}) + P(\text{TWW}) \\ &= (.6)(.6)(.6) + (.6)(.6)(.1) + (.6)(.1)(.6) + (.1)(.6)(.6) \\ &= .216 + .036 + .036 + .036 = .324 \end{aligned}$$

(ii) Here $B = \{\text{WLT, WTL, LWT, LTW, TWL, TLW}\}$. Since each element of B has probability $(.6)(.3)(.1) = .018$, $P(B) = 6(.018) = .108$.

4.30. Let S be a finite probability space and let T be the probability space of n independent trials in S. Show that T is well defined; that is, show (i) the probability of each element of T is nonnegative and (ii) the sum of their probabilities is 1.

If $S = \{a_1, \ldots, a_r\}$, then T can be represented by

$$T = \{a_{i_1} \cdots a_{i_n} : i_1, \ldots, i_n = 1, \ldots, r\}$$

Since $P(a_i) \geqq 0$, we have

$$P(a_{i_1} \cdots a_{i_n}) = P(a_{i_1}) \cdots P(a_{i_n}) \geqq 0$$

for a typical element $a_{i_1} \cdots a_{i_n}$ in T, which proves (i).

We prove (ii) by induction on n. It is obviously true for $n = 1$. Therefore we consider $n > 1$ and assume (ii) has been proved for $n - 1$. Then

$$\begin{aligned} \sum_{i_1, \ldots, i_n = 1}^{r} P(a_{i_1} \cdots a_{i_n}) &= \sum_{i_1, \ldots, i_n = 1}^{r} P(a_{i_1}) \cdots P(a_{i_n}) = \sum_{i_1, \ldots, i_{n-1} = 1}^{r} P(a_{i_1}) \cdots P(a_{i_{n-1}}) \sum_{i_n = 1}^{r} P(a_{i_n}) \\ &= \sum_{i_1, \ldots, i_{n-1} = 1}^{r} P(a_{i_1}) \cdots P(a_{i_{n-1}}) = \sum_{i_1, \ldots, i_{n-1} = 1}^{r} P(a_{i_1} \cdots a_{i_{n-1}}) = 1 \end{aligned}$$

by the inductive hypothesis, which proves (ii) for n.