Beginning Algebra at Madison Country Day School

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Preface

> *An equation means nothing to me unless it expresses a thought of God.*
> —Srinivasa Ramanujan

Note that throughout the text the symbol *indicates a problem at the boundary of what a student at this level might be able to work. The symbol ** identifies a problem just beyond that boundary.

Ray Tenebruso

Chapter 1

Preliminary ideas

1.1. Obvious ideas

Some of the ideas you will encounter in mathematics will seem obvious to you. It is my responsibility to convince you that, sometimes, obvious ideas lead to spectacularly non-obvious conclusions.

Some new notation

When we write $1, 2, 3, \cdots, 70$, we mean the "$\cdots$" to include the numbers from 4 to 69. When we write $1, 2, 3, \cdots$, we mean that this pattern continues without end.

The early bird gets the worm

Case 1. More birds than worms:

$$\begin{array}{cccccccc} bird & bird & bird & bird & bird & bird & bird & bird \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ worm & worm & worm & worm & worm & worm & worm & \end{array}$$

Case 2. More worms than birds:

$$\begin{array}{cccccccc} bird & bird & bird & bird & bird & bird & bird & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ worm & worm & worm & worm & worm & worm & worm & worm \end{array}$$

Case 3. Exactly the same number of birds and worms!:

$$\begin{array}{cccccccc} bird & bird & bird & bird & bird & bird & bird & bird \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ worm & worm & worm & worm & worm & worm & worm & worm \end{array}$$

In this last case, there is a 1-1 correspondence of birds and worms which was lacking in the first two cases. Seeing that there is a 1-1 correspondence is all that is needed to know there are exactly as many birds as worms. We did not have to count either birds or worms to know this.

Let us commemorate this obvious idea by making it our first theorem.

Theorem 1.1 (1-1 correspondence). *Two collections have exactly the same number of members if and only if there is a 1-1 correspondence between the members of the two collections.*

Numbers and even numbers

Case 1. Stop at 10:

$$\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\
 & 2 & & 4 & & 6 & & 8 & & 10
\end{array}$$

Half as many even numbers as numbers.

Case 2. Stop at 100:

$$\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & 100 \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots & \updownarrow \\
 & 2 & & 4 & & 6 & & 8 & \cdots & 100
\end{array}$$

Half as many even numbers as numbers.

Case 3. Stop at 10000:

$$\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & 10000 \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots & \updownarrow \\
 & 2 & & 4 & & 6 & & 8 & \cdots & 10000
\end{array}$$

Yup, half as many!

Case 4. Never stop:

$$\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots \\
2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & \cdots
\end{array}$$

There are *exactly the same number of even numbers as there are numbers!* We know this, because there is a 1-1 correspondence between the even numbers and the numbers.

Moral of story

Even the most obvious of ideas can sometimes have non-obvious, in this example, bizarre (but true), consequences. So, it may be wise to show patience for obvious ideas.

Exercise 1.1.

(1) Using $\cdots$, write the numbers from 0 to 1000.
(2) Using $\cdots$, write the numbers from 7 to 93.
(3) Using $\cdots$, write the numbers from 5 on.
(4) Are there the same number of odd numbers as even numbers? Show why your answer is correct.
(5) Are there the same number of odd numbers as numbers? Show why your answer is correct.
(6) Are there the same number of multiples of 5 as numbers? Show why your answer is correct.

1.2. Letters

1.2.1. Letters are a convenience

There is nothing magical or mysterious about the use of letters in mathematics. The letter is merely a very brief name that we use instead of a long name or descriptive phrase.

Some school textbooks call letters "variables". This is unfortunate, because not every letter is a variable. We will call letters "letters".

Example 1.1. You know that

$$\begin{aligned} 1+2&=2+1,\\ 1+3&=3+1,\\ 2+5&=5+2,\\ 4+4&=4+4. \end{aligned}$$

You know much more than this, because you know this is true for every pair of numbers. How can we communicate this idea? Here is a convenient way.

Let a and b represent any numbers. Then $a+b=b+a$. □

The symbol □ signals the end of an example, proof, definition, or theorem.

Example 1.2. The following statements are true.

$$(1+2)+3=1+(2+3).$$
$$(3+4)+5=3+(4+5).$$
$$(3+7)+8=3+(7+8).$$

Assert that this idea holds true for *every* triplet of numbers.

Solution.

Let a, b, c represent any numbers. Then $a+(b+c)=(a+b)+c$.

□

In previous grades, you learned that the product of two fractions is a fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators, provided that no denominator is zero. For instance, $\frac{2}{3}\times\frac{5}{7}=\frac{2\times 5}{3\times 7}$. That was a mouthful. Using letters, the idea is communicated with simplicity.

Example 1.3. Write the rule for finding the product of fractions.

Solution. For any numbers a, b, c, d with b and d different from zero, $\dfrac{a}{b}\times\dfrac{c}{d}=\dfrac{a\times c}{b\times d}$.

□

Example 1.4. Write the rule for canceling common factors in fractions.

Solution. For any numbers a, b, c with b and c different from zero, $\dfrac{a\times c}{b\times c}=\dfrac{a}{b}$.

□

Remark 1.1. The choice of letters is of no consequence. The statements "$a+b=b+a$" and "$x+y=y+x$" have exactly the same meaning. Each statement expresses the idea that the terms of a sum may be added in any order.

Exercise 1.2.

1. Using the letter a, express the idea that a number multiplied by 1 is that same number.
2. Using the letter a, express the idea that a fraction in which the numerator and the denominator are identical is equal to 1.
3. Using the letter b, express the idea that zero added to a number results in the number.
4. Using the letters a, b, and c, show the addition of two fractions that have a common denominator.

(5) Using the letters a and b, express the idea that the order in which two numbers are multiplied does not affect the product.
(6) * Using the letter y, write all the even numbers.

1.3. Equality

1.3.1. The meaning of equality

The symbol "=" is very familiar. Even when you were little, you wrote "$2+1=3$" and "$2 \times 3 = 6$". Have you ever wondered what it means when we write "$2+1=3$"? The author of this book recalls once having been told that "$2+1=3$" means that "$2+1$" and "3" are the same. This answer did not seem right, because, for starters, "$2+1$" and "3" don't even look the same. When he mentioned this, he was told "Well, you don't understand, it means that "$2+1$" and "3" are the same number!" But he was still confused, because as near as he could tell, the numbers 2 and 1 appeared on the left side of "$2+1=3$" but only the one number, 3, appeared on the right side.

Later, many years later, the author received an explanation of what "$2+1=3$" means that actually did make sense. It went like this, "$2+1$" is the name for a number. "3" is the name for a number. The statement "$2+1=3$" says that "$2+1$" and "3" name the same number. The statement "$2+1=3$" is true, because "$2+1$" and "3" do name the same number. The statement "$2+1=4$" is false, because "$2+1$" and "4" do not name the same number. Let's make a special note of this fact about equality.

Definition 1.1 (Equality). $a = b$ means that a and b are names for the same object.

Two names for one person

The statement "Mark Twain wrote the book *Adventures of Huckleberry Finn*" is true. Many people know this. There are not quite as many people who know that "Samuel Longhorn Clemens wrote *Adventures of Huckleberry Finn*". But this statement is true, too. This is because "Mark Twain" and "Samuel Longhorn Clemens" are both names for the same person, who did write *Adventures of Huckleberry Finn.* This brings up an important idea. If "Mark Twain" and "Samuel Longhorn Clemens" both name the same person, then the statements "Mark Twain wrote the book *Adventures of Huckleberry Finn*" and "Samuel Longhorn Clemens wrote *Adventures of Huckleberry Finn*" are both true, regardless of which name is used for the author. You may use either name, as you wish. This idea is so important in mathematics, that it is given a special name. It is called the "Principle of Substitution".

Principle of substitution.

If two expressions name the same object, one expression may be substituted for the other in any statement without changing the truth of the statement.

Theorem 1.2. *Let a, b and c be any numbers.*

1. *If $a = b$ then $a + c = b + c$, and*
2. *If $a = b$ then $a \times c = b \times c$.*

Proof. Let a, b and c be any numbers. Suppose that $a = b$.

$$\begin{aligned} a + c &= a + c \\ a + c &= b + c, && \text{by substitution. We supposed } a = b. \end{aligned}$$

Therefore, if $a = b$ then $a + c = b + c$. ☐

The proof for multiplication is similar, so it is left for an exercise.

1.3.2. The nature of equality

However rich the idea of equality may be, only three properties of equality are needed for mathematics. Let a, b, and c represent any numbers.

1. Reflexive. $a = a$.
2. Symmetric. $a = b$ implies $b = a$.
3. Transitive. $a = b$ and $b = c$ implies $a = c$.

Example 1.5. The following illustrate the reflexive nature of equality.

1. $5 = 5$.
2. $3 + 4 = 3 + 4$.
3. $(3 + 7) + 8 = (3 + 7) + 8$.
4. $x + 9 = x + 9$. ☐

Example 1.6. The following illustrate the symmetric nature of equality.

1. $5 + 3 = 8$ implies $8 = 5 + 3$.
2. $5 + 2 = 4 + 3$ implies $4 + 3 = 5 + 2$.
3. $a + 5 = b$ implies $b = a + 5$. ☐

Example 1.7. The following illustrate the transitive nature of equality.

(1) $5+3=8$ and $8=6+2$ implies $5+3=6+2$.
(2) $7+5\times 2=7+10$ and $7+10=17$ implies $7+5\times 2=17$.
(3) if $x+y=z$ and $z=7+a$ then $x+y=7+a$. ☐

Example 1.8. State the property of equality that justifies each of the following.

(1) if $2+11=10+3$ and $10+3=13$ then $2+11=13$.
(2) if $a+8=b+7$ and $b+7=23$ then $a+8=23$.
(3) if $b+c=17$ then $17=b+c$.

Answers:

(1) Transitive.
(2) Transitive.
(3) Symmetric. ☐

Exercise 1.3.

(1) Use the symmetric quality of equality to rewrite $a+101$.
(2) Using the transitive property of equality, what conclusion follows from the statement $a+3=y$ and $y=b+5$?
(3) What property of equality does the following illustrate: $a+b+c=7+2$ so $7+2=a+b+c$?
(4) What property of equality justifies: $a+b+c=d$ and $b+c=9$, so $a+9=d$? [Hint: do not forget the principle of substitution.]
(5) Let a, b and c be any numbers. Prove that If $a=b$ then $a\times c=b\times c$. [Hint: see proof of theorem 1.2.]

1.3.3. What a difference a name makes!

You may be wondering: What difference can a name make? The answer is: A big difference! You have been using that big difference to your advantage for several years.

The value of the sum $\frac{1}{3}+\frac{2}{5}$ is not obvious. But, if we write

$$\frac{1}{3} \text{ using the name } \frac{5}{15}$$

and

$$\frac{2}{5} \text{ using the name } \frac{6}{15},$$

then $\frac{11}{15}$ is obviously the value of the sum.

Example 1.9. Find the sum $\frac{3}{7} + \frac{5}{8}$ and note when one name is substituted for another name.

$$\frac{2}{7} + \frac{5}{8} = \underbrace{\frac{16}{56} + \frac{35}{56}}_{\text{substitutions}} = \frac{51}{56}. \quad \square$$

Exercise 1.4.

(1) If a names the number 3, then what number is $2 + a \times 3$?
(2) If a represents the number 7, then what number is $(5 + a) \times 3$?
(3) If a names the number 4, and b number 9, then what is $a+b+2$?
(4) If a is 10, and b is 7, then what is $a + b$?. (Note: since phrases like "names the number" and "represents the number" are tiring to write and speak, we often just say, for example, "a is 7".)

1.4. Sets

The word **set** means a collection of objects. For example, when we speak of the set of Alice's stamps, we mean her collection of stamps. The objects in a set are called **members** or **elements** of the set. The elements of Alice's set of stamps are the stamps. Often, the members of a set have some quality in common. The members of Alice's set of stamps have in common that they are each a stamp belonging to Alice. Notice that a set has qualities of its own. For example, the set of Alice's stamps has the quality that it contains 100 stamps. And, the members of a set may have a quality that the set itself cannot be said to have. For instance, the set of Alice's stamps is not itself a stamp. And, although each of Alice's stamps might be blue in color, the set of her stamps certainly is not blue. The words "set", "collection", "class", and "aggregate" are used synonymously.

In mathematics, we are often concerned with sets that contain numbers as members. For example, we might speak of the set of even numbers or the set of odd numbers less than 7. Notice that, like Alice's set of stamps, the set of odd numbers less than 7 has the quality that it contains 3 numbers and this is a quality none of its members can possess. While the members of the set of odd numbers less than 7 are each odd, the set of them cannot be said to be "odd".

1.4.1. Set notation

To indicate a set of items, simply place set brackets "{" and "}" around the items in the set. The symbol $\{1, 3, 8\}$ is pronounced "The set whose members are $1, 3, 8$". Examples will make clear how set notation works.

Example 1.10.

1. $\{1, 2, 3, 4, 5\}$ means the set of numbers $1, 2, 3, 4, 5$.
2. $\{3, 6, 9, 12\}$ means the set of numbers $3, 6, 9, 12$.
3. The set of even numbers between 4 and 14 is written $\{6, 8, 10, 12\}$.
4. The set of prime numbers less than 17 is written $\{2, 3, 5, 7, 11, 13\}$.
5. The set of factors of 12: $\{1, 2, 3, 4, 6, 12\}$.
6. The set of prime factors of 12: $\{2, 3\}$.
7. $\{1, 2, 3, \cdots, 100\}$ is the set of numbers from 1 to 100.
8. $\{1, 2, 3, \cdots\}$ is the set of numbers $1, 2, 3, \cdots$. □

1.4.2. Set membership

When we wish to say that a number is a member of a set, we use the sign "$\in$". We pronounce "$a \in \{a, b, c\}$" as "a is an element of the set whose members are a, b, c". The phrases "is an element of", "is a member of", "is contained in" or simply "is in" are synonymous.

Example 1.11.

1. $2 \in \{1, 2, 3, 4, 5\}$ is true, because 2 *is* a member of $\{1, 2, 3, 4, 5\}$.
2. $4 \in \{1, 2, 3, 4, 5\}$ is true.
3. $5 \in \{1, 2, 3, 4, 5\}$ is true, because 5 is in $\{1, 2, 3, 4, 5\}$.
4. $9 \in \{3, 6, 9, 12\}$ is true.
5. $2 \in \{3, 6, 9, 12\}$ is false.
6. $\{6\} \in \{3, 6, 9, 12\}$ is false, but
7. $\{6\} \in \{3, 6, 9, 12, \{6\}\}$ is true.
8. $23 \in \{1, 2, 3, \cdots, 100\}$ is true.
9. $18 \in \{2, 4, 6, 8, \cdots\}$ is true. □

We can assert that 7 is not a member of $\{1, 3, 6, 9, 12, 15\}$ by writing $7 \notin \{1, 3, 6, 9, 12, 15\}$. Sets may be named for convenience. For example, if $\mathbb{A} = \{1, 2, 3, 4, 5, 6\}$, then $2 \in \mathbb{A}$ and $7 \notin \mathbb{A}$ are true statements.

1.4.3. Special sets

Two special sets are the **Empty Set**, denoted by the symbol $\emptyset$ or by $\{\}$ and the **Universal Set**, often denoted by the symbol U. The Empty Set has no members and the Universal set contains everything.

1.4.4. Subsets

When all the members of one set are contained in another set, we say the one set is a **subset** of the other. If $\mathbb{A}$ is a subset of $\mathbb{B}$, we write $\mathbb{A} \subseteq \mathbb{B}$. To deny that $\mathbb{A}$ is a subset of $\mathbb{B}$, write $\mathbb{A} \not\subseteq \mathbb{B}$. A set is a subset of itself, $\mathbb{A} \subseteq \mathbb{A}$. If all the members of $\mathbb{A}$ are in $\mathbb{B}$, but some member of $\mathbb{B}$ is not in $\mathbb{A}$, then $\mathbb{A}$ is called a **proper subset** of $\mathbb{B}$ and we write $\mathbb{A} \subset \mathbb{B}$. The empty set is a subset of every set; that is, for any set $\mathbb{A}, \emptyset \subseteq \mathbb{A}$.

Example 1.12. Let $\mathbb{A} = \{1, 3, 6, 7, 8, 10, 12\}$, and $\mathbb{B} = \{1, 10\}$

(1) $3 \in \mathbb{A}$ is true.
(2) $10 \in \mathbb{B}$ is true.
(3) $\{10\} \subseteq \mathbb{B}$ is true.
(4) $\{3, 7\} \not\subseteq \mathbb{B}$ is true.
(5) $\{3, 7, 10\} \subseteq \mathbb{A}$ is true.
(6) $\mathbb{B} \subseteq \mathbb{A}$ is true.
(7) $\mathbb{A} \not\subseteq \mathbb{B}$ is true.
(8) $\emptyset \subseteq \mathbb{A}$ is true.
(9) $\emptyset \subseteq \mathbb{B}$ is true. □

Exercise 1.5.
Say whether each statement is true or false.

(1) $9 \in \{1, 2, 5, 7, 9, 15, 27\}$.
(2) $9 \in \{2, 4, 8, 16, 32\}$.
(3) $28 \in \{1, 3, 5, 7, \cdots\}$.
(4) $6 \notin \{1, 3, 5, 7, \cdots\}$.
(5) $\{2\} \in \{1, 2, 5, 7, \cdots\}$.
(6) $\{2\} \in \{1, \{2\}, 5, 7, \cdots\}$.
(7) $\{1, 5, 11\} \subseteq \{1, 3, 5, 7, \cdots\}$.
(8) $\emptyset \in \{1, 2, 5, 7, \cdots\}$.
(9) $\emptyset \subseteq \{1, 2, 5, 7, \cdots\}$.
(10) $\emptyset \subseteq \emptyset$.

Insert the symbols $\in$ or $\notin$ to make each statement true.

(11) 4_____$\{1, 2, 4, 6, 8, 10, \cdots\}$.

(12) $30\_\_\_\_\_\{1, 3, 6, 9, 12, \cdots\}$.
(13) $11\_\_\_\_\_\{2, 4, 6, 8, \cdots\}$.
(14) $\{7\}\_\_\_\_\_\{1, 2, 4, 6, \{7\}, 8, 10, \cdots\}$.

Give a verbal description of each set

(15) $\{2, 4, 6, 8, \cdots\}$.
(16) $\{10, 15, 20, 25, \cdots, 105\}$.

1.5. Three fundamental ideas

A given collection of numbers may possess many qualities. Of these qualities, a few tell us a great deal about the character of a collection of numbers. Addition, subtraction, multiplication, and division are some of the operations performed on numbers. Below, the symbol $\star$ stands any operation.

Definition 1.2. Let a, b, c represent any members of a set of numbers that we will denote by the symbol $\mathbb{A}$. Let $\star$ represent on operation on the elements of set $\mathbb{A}$, then

Closed: The set $\mathbb{A}$ is **closed** under the operation $\star$, if the result of $a \star b$ is a member of $\mathbb{A}$.

Commutative: The set $\mathbb{A}$ is **commutative** under the operation $\star$, if $a \star b = b \star a$.

Associative: The set $\mathbb{A}$ is **associative** under the operation $\star$, if $(a \star b) \star c = a \star (b \star c)$.

A set of numbers need not be large in order to have the three characteristics identified above. The table below defines a made-up operation $\oplus$.

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

TABLE 1.1. $(\{0, 1, 2\}, \oplus)$

To evaluate $1 \oplus 2$, find 1 in the left-most column, then follow that row across to the column headed by 2. So, $1 \oplus 2 = 0$.

Just to be sure we know how to use the table, we evaluate $2 \oplus (1 \oplus 0)$.

$$\begin{aligned} 2 \oplus (1 \oplus 0) &= 2 \oplus 1 \\ &= 0 \end{aligned}$$

The set $\{0, 1, 2\}$ under the operation $\oplus$ is closed, commutative, and associative.

Closed: because the result of every computation is an element of $\{0, 1, 2\}$.

Commutative: because $0 \oplus 0 = 0 \oplus 0, 0 \oplus 1 = 1 \oplus 0, 0 \oplus 2 = 2 \oplus 0$, and so on for every pair of elements.

Associative: because $0 \oplus (0 \oplus 1) = (0 \oplus 0) \oplus 1$ and so on for every triplet of elements.

Once you know that a set of numbers is closed, commutative, and associative under an operation, you may take advantage of these qualities when you work with those numbers and that operation.

The natural numbers are $1, 2, 3, \cdots$.

Remark 1.2. It is a fact that the natural numbers are closed, commutative, and associative under the operations addition and multiplication.

Example 1.13. $2 + 3$ is a natural number. So is $12053 + 2378$. Since $a + b$ is a natural number for any natural numbers a, b, we know the natural numbers are closed under addition. □

Example 1.14. $2 + 3 = 3 + 2$ and $23 + 5 = 5 + 23$. Since $a + b = b + a$ for any natural numbers a, b, we know the natural numbers are commutative under addition. □

Example 1.15. $2+(3+5) = (2+3)+5$ and $7+(10+8) = (7+10)+8$. Since $a + (b + c) = (a + b) + c$ for any natural numbers a, b, c, we know the natural numbers are associative under addition. □

Example 1.16. The set of natural numbers is not closed under the operation of division, because $1 \div 2$ is $\frac{1}{2}$ and $\frac{1}{2}$ is not a natural number. □

Example 1.17. Rewrite $2 + (3 + 5) + 7$ as $(3 + 2) + 5 + 7$.
Solution:

$$\begin{aligned} 2 + (3 + 5) + 7 &= (2 + 3) + 5 + 7 && \text{addition is associative} \\ &= (3 + 2) + 5 + 7 && \text{addition is commutative} \end{aligned}$$

Example 1.18. Rewrite $2 \times (3 \times 5) \times 7$ as $3 \times 2 \times (7 \times 5)$.
Solution:

$$\begin{aligned} 2 \times (3 \times 5) \times 7 &= 2 \times 3 \times (5 \times 7) && \text{multiplication is associative.} \\ &= 2 \times 3 \times (7 \times 5) && \text{multiplication is commutative.} \\ &= 3 \times 2 \times (7 \times 5). && \text{multiplication is commutative} \end{aligned}$$

$\square$

Example 1.19. Rewrite $(3+8)+1$ as $1+(3+8)$.
Solution:

$$(3+8)+1 = 1+(3+8), \text{ addition is commutative.} \quad \square$$

This last example requires you to see $(3+8)$ not as several symbols, but as one blob of stuff. Imagine that you see $(3+8)$ as ♦. So instead of $(3+8)+1$ you see ♦ $+ 1$. Now you are more likely to see how commutativity applies: ♦ $+ 1 = 1 +$ ♦.

With experience, you will sense when to see the details and when to see a blob.

Example 1.20. Rewrite $2+3+(5+7)$ as $(7+2)+3+5$.
Solution:

$$\begin{aligned} 2+3+(5+7) &= 2+3+(7+5) && \text{addition is commutative.} \\ &= 2+(3+7)+5 && \text{addition is associative.} \\ &= 2+(7+3)+5 && \text{addition is commutative.} \\ &= (2+7)+3+5 && \text{addition is associative.} \\ &= (7+2)+3+5. && \text{addition is commutative} \end{aligned}$$

$\square$

Exercise 1.6. Use the ideas commutative and associative, as in example (1.20), to work problems 1-4.

(1) Rewrite $(2+7)+13$ as $(2+13)+7$.
(2) Rewrite $(9 \times 11) \times 6$ as $(9 \times 6) \times 11$.
(3) Rewrite $1+4+(7+12)$ as $12+(1+7)+4$.
(4) Rewrite $3+(7+12)+(4 \times 7)$ as $(4 \times 7)+12+(7+3)$.
(5) Which of the following are natural numbers: $2, 1001, \frac{1}{3}, 6, 2.5$?
(6) Is $\frac{12}{3}$ a natural number?

Chapter 2

Integers

The purpose of this chapter is to introduce the integers. We begin with the natural numbers, because the integers are an extension of the natural numbers.

2.1. Natural numbers

Definition 2.1 (The natural numbers)**.** The numbers $1, 2, 3, \cdots$ are called "**natural numbers**". The symbol $\mathbb{N}$ is typically used to represent the set of natural numbers.

Example 2.1. The numbers $3, 731, 1002$ are natural numbers, because they are in the set $\{1, 2, 3, \cdots\}$. But the numbers $\frac{1}{2}, \frac{2}{5}, \frac{9}{8}$ are not natural numbers, because they are not in $\{1, 2, 3, \cdots\}$. $\square$

Someone is bound to ask whether $\dfrac{3}{1}$ and $\dfrac{10}{2}$ are natural numbers. We know that $\dfrac{3}{1} = 3$ and $\dfrac{10}{2} = 5$. Since $\dfrac{3}{1}$ names the number 3 and $\dfrac{10}{2}$ names the number 5, we conclude that $\dfrac{3}{1}$ and $\dfrac{10}{2}$ are natural numbers, although the names $\dfrac{3}{1}$ and $\dfrac{10}{2}$ obscure that fact.

2.1.1. Properties of the natural numbers

The natural numbers are closed, commutative, and associative under the operations of addition and multiplication.

No one knows when any of these numbers were first discovered. We can imagine our ancestors happily counting sheep, goats, and various other items. Eventually, our ancestors learned to add, multiply, subtract, and even divide within the set of natural numbers.

A merchant could by subtraction know how many bushels of oats remained of 100 if 70 bushels were sold. Multiplication could tell the merchant how much money the 70 bushels would bring. All was well. For a while, anyway. Until someone asked how many sheep remain if a shepherd sells all 40 of a herd of 40 sheep.

Oops. No one knew the answer to $40 - 40$. Worse yet, no one knew of any number that could possibly be the answer. The numbers had run out. An unhappy state if ever there were one. Applying the ideas of the previous chapter, we would say that the natural numbers are not closed under subtraction.

The discovery of the number zero provided the answer to "What remains of 40 sheep if 40 are sold?" Numbers were extended to include 0. But, happiness would not have reigned long.

Some troublemaker must have asked What is 5 subtract 6? Oops. Out of numbers, again! Even with 0 included, the set of numbers $\{0, 1, 2, 3, \cdots\}$ is not closed under subtraction.

2.2. Subtraction - one view

There will be another view of subtraction a few pages from now.

You might have first learned to add by *counting on*. And, first learned to subtract by *counting back*.

What is $5 + 3$? Counting on:

$$1, 2, 3, 4, \overset{\curvearrowright}{5}, \overset{\curvearrowright}{6}, \overset{\curvearrowright}{7}, 8, \cdots$$

What is $5 - 3$? Counting back:

$$1, \overset{\curvearrowleft}{2}, \overset{\curvearrowleft}{3}, \overset{\curvearrowleft}{4}, 5, 6, \cdots$$

What is $5 - 6$? Counting back:

$$\overset{\curvearrowleft}{\blacksquare}, \overset{\curvearrowleft}{0}, \overset{\curvearrowleft}{1}, \overset{\curvearrowleft}{2}, \overset{\curvearrowleft}{3}, \overset{\curvearrowleft}{4}, 5, 6, \cdots$$

We do not yet know the number, or even that such a number exists. But if there is such a number, it *should* be at $\blacksquare$. Our goal, in the next few pages, is to discover the number for $\blacksquare$.

2.3. Two mathematical systems

A tiny mathematical system

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Table 2.1. $(\{0,1,2\},\oplus)$

The tiny system consisting of $\{0,1,2\}$ and the made-up operation $\oplus$ defined by Table 2.1 is humble. It would hardly enable the poorest shepherd to count his sheep.

But observe that no matter which of the numbers $0,1,2$ we choose, 0 can be produced:

$$
\begin{aligned}
&0 \oplus 0 = 0,\\
&1 \oplus 2 = 0,\\
&2 \oplus 1 = 0.
\end{aligned}
$$

A huge mathematical system

$+$	0	1	2	$\cdots$	
0	0	1	2	$\cdots$	
1	1	2	3	$\cdots$	no 0 in this row
2	2	3	4	$\cdots$	no 0 in this row
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	no 0 in these rows

Table 2.2. $(\{0,1,2,3,\cdots\},+)$

The huge system shown in Table 2.2 is familiar, because it is the numbers $0,1,2,3,\cdots$ together with ordinary addition. However, notice that there is no number that added to 1 results in 0, no number that added to 2 results in 0, and so on.

Tiny versus huge

The tiny system is algebraically richer than the huge system. Every member of $\{0,1,2\}$ under $\oplus$ can be obtained from the other members by the operation $\oplus$. A similar claim cannot be made for $\{0,1,2,\cdots\}$ under $+$.

2.4. Discovering the integers

Let us take another look at

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$$(\{0,1,2\},\oplus)$$

Notice that for every member of $\{0,1,2\}$ there is another member such that the operation $\oplus$ on the pair of members results in 0.

$$0 \oplus 0 = 0.$$
$$1 \oplus 2 = 0.$$
$$2 \oplus 1 = 0.$$

The "other number" is called the **inverse** of the first number under the operation $\oplus$. In $\{0,1,2\}$ under $\oplus$,

The inverse of 0 is 0, because $0 \oplus 0 = 0$.
The inverse of 1 is 2, because $1 \oplus 2 = 0$.
The inverse of 2 is 1. because $2 \oplus 1 = 0$.

If the huge set of numbers $\{0,1,2,3,\cdots\}$ is extended by including the inverse element under addition of each of $1,2,3,\cdots$, the resulting set is called the integers.

In the integers under addition, every number, including 0, is the sum of two different integers. 0 is the sum of any integer a and the inverse of a under addition. There is a very nice way to name the inverse elements in the integers. For addition

the inverse of 1 is -1,
the inverse of 2 is -2,
the inverse of 3 is -3.
$\vdots$

In general, the inverse under addition of a is written $-a$.

Definition 2.2 (Integers). The **integers** are the numbers $\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$.

(1) 0 is called the “**identity element**” for addition. $a + 0 = a$ and $0 + a = a$, for any integer a.
(2) For each integer a, there exists another integer written $-a$ that is the inverse of a under addition. $a + (-a) = 0$ and $(-a) + a = 0$.
(3) The inverse of a under addition is usually called the **additive inverse** of a. □

At the end of Section 2.2, we knew where the number equal to $5-6$ belonged.

$$\overset{\curvearrowleft}{\blacksquare}, \overset{\curvearrowleft}{0}, \overset{\curvearrowleft}{1}, \overset{\curvearrowleft}{2}, \overset{\curvearrowleft}{3}, \overset{\curvearrowleft}{4}, 5, 6, \cdots$$

We just did not know the number. Now we do. We fill in ■ with the additive inverse of 1.

$$\overset{\curvearrowleft}{-1}, \overset{\curvearrowleft}{0}, \overset{\curvearrowleft}{1}, \overset{\curvearrowleft}{2}, \overset{\curvearrowleft}{3}, \overset{\curvearrowleft}{4}, 5, 6, \cdots$$

In fact, we can continue filling in all the numbers to the left of 0.

$$\cdots, -6, -5, -4, -3, -2, -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \cdots$$

2.4.1. Names of subsets of integers

Names used for the integers and important subsets of integers are

$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$, the set of integers.
$\mathbb{Z}^+ = \{1, 2, 3, \cdots\}$, the set of positive integers.
$\mathbb{Z}^- = \{\cdots, -3, -2, -1\}$, the set of negative integers.

The set of numbers $\{0, 1, 2, 3, \cdots\}$ is called the set of “nonnegative integers” and has no special symbol that names it. Until part way through Chapter 3, the word “number” will be understood to mean an integer.

For the time being, “number” means an integer.

$$\underbrace{\underbrace{\cdots 6, -5, -4, 3, -2, -1}_{\text{negative integers } \mathbb{Z}^-}, \underbrace{0,\ \underbrace{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \cdots}_{\text{positive integers } \mathbb{Z}^+}}_{\text{nonnegative integers}}}_{\text{integers}}$$

FIGURE 1. Important subsets of the integers

Example 2.2. The sum $2 + (-2) = 0$, because (-2) is the additive inverse of 2. We place parenthesis around -2 to emphasize that -2 is a single symbol that names a particular number. □

Example 2.3. For each of the following, write an equation using the number given and its inverse on the left hand side and 0 on the right hand side.

(1) 2
(2) -2
(3) 100
(4) -17
(5) -5
(6) $a, a \in \mathbb{Z}$
(7) $-a, a \in \mathbb{Z}$

Answers

(1) $2 + (-2) = 0$.
(2) $(-2) + 2 = 0$.
(3) $100 + (-100) = 0$.
(4) $(-17) + 17 = 0$.
(5) $(-5) + 5 = 0$.
(6) $a + (-a) = 0$.
(7) $(-a) + a = 0$.

Exercise 2.1. For each of the following, write an equation using the number given and its inverse on the left hand side and 0 on the right hand side.

(1) 7
(2) 4
(3) 2090
(4) -33
(5) -51
(6) 8
(7) $x, x \in \mathbb{Z}$
(8) $-x, x \in \mathbb{Z}$

Questions $9-12$ refer to the following table that defines an operation $\star$ on the set $\{1,2,3,4\}$.

$\star$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

(9) What number plays the role that 0 had in Table 2.1?
(10) What is the inverse element of 3 under $\star$?
(11) Does every element of $\{1,2,3,4\}$ have an inverse element?
(12) Is $\{1,2,3,4\}$ commutative under $\star$?

Questions $13-16$ refer to the following table that defines an operation $\star$ on the set $\{a,b,c,d\}$.

$\star$	a	b	c	d
a	a	b	c	d
b	b	d	a	c
c	c	a	d	b
d	d	c	b	a

(13) What letter plays the role that 0 had in Table 2.1?
(14) What is the inverse element of b under $\star$?
(15) Does every element of $\{a,b,c,d\}$ have an inverse element?
(16) Is $\{a,b,c,d\}$ commutative under $\star$?

2.5. Addition and subtraction with integers

Having discovered the integers, it would be nice to know how they behave under addition and subtraction. If a and b represent positive integers ($a,b\in\mathbb{Z}^+$), there are 8 cases to consider.

case 1: $a+b$
case 2: $a-b$
case 3: $(-a)+b$
case 4: $(-a)-b$
case 5: $a+(-b)$
case 6: $a-(-b)$
case 7: $(-a)+(-b)$
case 8: $(-a)-(-b)$

Cases 1−4: $a + b, a - b, (-a) + b, (-a) - b$

The first four cases are easily handled by counting on and counting back. Just like in second grade. We illustrate cases 3 and 4 with numeric examples.

What is $-3 + 2$? Counting on:

$$\cdots, -4, \overset{\curvearrowright}{-3}, \overset{\curvearrowright}{-2}, -1, 0, 1, 2, 3, \cdots$$

so, $-3 + 2 = -1$. □

What is $-3 - 2$? Counting back:

$$\cdots, -6, -5, \overset{\curvearrowleft}{-4}, \overset{\curvearrowleft}{-3}, -2, -1, 0, 1, 2, 3, \cdots$$

so, $-3 - 2 = -5$. □

There are several more ways to think of subtraction. They are illustrated in the following example.

Example 2.4. Evaluate $3 - 8$ in several different ways.

(1) Imagine the number line. Counting back from 3 to 0 takes 3 steps. That leaves 5 steps to go. So $3 - 8 = -5$.

$$\cdots, -7, -6, -\underbrace{5, -4, -3, -2, -1,}_{\text{5 steps}}\ \mathbf{0}\ \underbrace{1,\ 2,\ 3}_{\text{3 steps}},\ 4,\ 5, \cdots$$

So $3 - 8 = -5$.

(2) Think of -8 as $-5 - 3$. Then substituting $-3 - 5$ for -8, produces

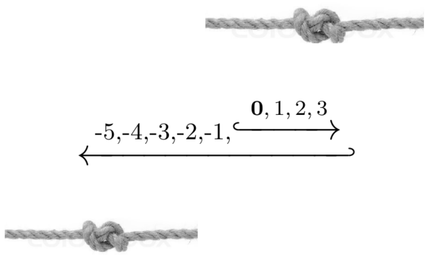
$$\begin{aligned} 3 - 8 &= 3 - 3 - 5 \\ &= -5. \end{aligned}$$

The diagram below goes well with this way of thinking.

$$\cdots, -7, -6, -5, \xleftarrow{-5} \mathbf{0} \xleftarrow{-3} 3,\ 4,\ 5, \cdots$$

□

(3) Tug of war. 3 right, then 8 left. Ends at -5.



Would the tug of war have the same result if the order of events were: 8 left, then 3 right? $\square$

(4) This may be the easiest method.
  (a) Judge whether the result will be positive or negative. For $3 - 8$ result will be **negative**,
  (b) Find the positive difference of the numbers. Here it is $8 - 3 = 5$,
  (c) Write the number from step (b) with a negative sign if indicated at step (a). $\therefore 3 - 8 = -5$. $\square$

The symbol $\therefore$ means “Therefore”.

Exercise 2.2.
Subtract as required.

(1) $6 - 9$
(2) $1 - 7$
(3) $2 - 3$
(4) $5 - 7$
(5) $-3 - 4$
(6) $-2 - 14$
(7) $0 - 6$
(8) $8 - 17$
(9) $-1 - 4$
(10) $12 - 28$

Cases 5 − 8 will require some work. However, once we figure out one of the cases, the remaining three cases will go quickly.

Case 5: $a + (-b)$
Let us explore several specific examples of this case.

Example 2.5. What number is $3 + (-2)$?

It is in no way clear how to count on by −2. Instead, we reason as follows.

$$\begin{aligned} 3 + (-2) &= (1 + 2) + (-2)), && \text{substitution, } 3 = 1 + 2. \\ &= 1 + (2 + (-2)), && \text{associative.} \\ &= 1 + 0, && \text{inverse elements.} \\ &= 1, && \text{identity element.} \end{aligned}$$

Conclusion, $3 + (-2) = 1$. Let's note that $3 - 2 = 1$, too. □

Example 2.6. What number is $7 + (-4)$?

$$\begin{aligned} 7 + (-4) &= (3 + 4) + (-4), && \text{substitution, } 7 = 3 + 4. \\ &= 3 + (4 + (-4)), && \text{associative.} \\ &= 3 + 0, && \text{inverse elements.} \\ &= 3, && \text{identity element.} \end{aligned}$$

Conclusion, $7 + (-4) = 3$. Similar to previous example, note that that $7 - 4 = 3$. □

Could $a + (-b)$ mean $a - b$? Our two examples certainly point in that direction!

Let us see if we can use the same reasoning for *any* pair of positive numbers a and b.

Idea 1.

Let a and b be positive numbers. Then $a + (-b) = a - b$.

Reasoning. Let a and b be positive numbers. Suppose there is a number, call it c, such that $a - b = c$. This means that $a = c + b$. We are going to first substitute $c + b$ for a. At the end, we will substitute $a - b$ for c.

$$\begin{aligned} a+(-b) &= (c+b)+(-b), && \text{substitution, } a=c+b. \\ &= c+(b+(-b)), && \text{associative.} \\ &= c+0, && \text{inverse elements.} \\ &= c, && \text{identity element.} \\ &= a-b, && \text{substitution, } c=a-b. \end{aligned}$$

Therefore.

$$a+(-b)=a-b. \qquad \square \tag{2.1}$$

In other words, adding the additive inverse of a positive number means subtracting the number. And, subtracting a positive number means adding the additive inverse of the number.

Example 2.7. Find the sum $10+(-3)$.
Solution.

$$\begin{aligned} 10+(-3) &= 10-3. && \text{using Idea 1} \\ &= 7. && \square \end{aligned}$$

Not only have we disposed of Case 5, but we have a good start on Case 6. First, let's note that additive inverses come in pairs. In other words $-a$ is the additive inverse of a and a is the additive inverse of $-a$.

Case 6: $a-(-b)$

This is the subtraction of the integer $(-b)$. But, according to equation (2.1), that is accomplished by adding the additive inverse of $-b$. Since the additive inverse of $-b$ is b, $a-(-b)=a+b$. The familiar computation $a+b$ takes the place of the unfamiliar computation $a-(-b)$. This reasoning leads to Idea 2.

Idea 2. Let a and b represent any positive numbers.
Then $a-(-b)=a+b$. $\square$

In other words, subtracting the additive inverse of a positive number means adding the number. And, adding a positive number means subtracting the additive inverse of the number.

Case 7: $(-a) + (-b)$
Case 8: $(-a) - (-b)$

We treat cases 7 and 8 respectively using the following two ideas. The reasoning for these two ideas is left to the reader in the exercises for this section.

Idea 3. Let a and b represent any positive numbers.
Then $(-a) + (-b) = (-a) - b$. ☐

Idea 4. Let a and b represent any positive numbers.
Then $(-a) - (-b) = (-a) + b$. ☐

2.5.1. No one likes eight cases

The eight cases, though instructive, make this topic seem more complicated than it is. Let us see if we can pare them down.

All eight cases fell to the four ideas. Perhaps there is some common idea that is present in all four ideas If so, maybe all we really need is that common idea. The ideas are listed below for reference.

For a and b positive numbers,

(1) $a + (-b) = a - b$.
(2) $a - (-b) = a + b$.
(3) $(-a) + (-b) = (-a) - b$.
(4) $(-a) - (-b) = (-a) + b$.

In each of these four equations, subtraction appears on one side and addition on the other side. If we use the symmetric property of equality, we can make this very visible.

$$a - b = a + (-b). \tag{2.2}$$

$$a - (-b) = a + b. \tag{2.3}$$

$$(-a) - b = (-a) + (-b). \tag{2.4}$$

$$(-a) - (-b) = (-a) + b. \tag{2.5}$$

(2.6)

The idea that is common to all four almost jumps off the page! Subtraction *is* addition of the additive inverse. We commemorate this insight in the following definition.

Definition 2.3 (Subtraction)**.** For any numbers x and y,

$$x - y = x + (-y). \quad \square$$

The "other view" of subtraction, promised earlier.

It is interesting to ask where the four ideas acquired their common feature. Let's return to the reasoning provided for Idea 1. It began like this:

> **Reasoning.** Let a and b be positive numbers. Suppose there is a number, call it c, such that $a - b = c$. This means that $a = c + b$. We are going to first substitute $c + b$ for a. At the end, we will substitute $a - b$ for c.

The key idea is that

$$a - b = c \text{ means that } a = c + b.$$

When you were quite young, you answered questions like "What is $7 - 5$?" by saying "$7 - 5 = 2$, because $2 + 5 = 7$." You already knew that that "if b added to c produces a, then subtracting b from a produces c". That is the idea that defines subtraction.

Example 2.8. Evaluate $12 - 20$.

(1) Result will be **negative**,
(2) $20 - 12 = 8$,
(3) $12 - 20 = -8$.

Example 2.9. Evaluate $3 - (-8)$.

$$\begin{aligned} 3 - (-8) &= 3 + 8, && \text{add the additive inverse of } -8. \\ &= 11. \\ \therefore 3 - (-8) &= 11. \end{aligned}$$

Example 2.10. Evaluate $3 + (-8)$.

$$\begin{aligned} 3 + (-8) &= 3 - 8 \\ &= -5. \\ \therefore 3 + (-8) &= -5. \qquad \square \end{aligned}$$

Example 2.11. Evaluate $10 - 7 \times 2$.

$$\begin{aligned} 10 - 7 \times 2 &= 10 - 14 \\ &= -4. \\ \therefore 10 - 7 \times 2 &= -4. \qquad \square \end{aligned}$$

Example 2.12. Evaluate $((-3) - (-7)) \times 5$.

$$\begin{aligned} ((-3) - (-7)) \times 5 &= ((-3) + 7) \times 5. \\ &= 4 \times 5. \\ &= 20. \\ \therefore ((-3) - (-7)) \times 5 &= 20. \qquad \square \end{aligned}$$

Exercise 2.3.

(1) Provide reasoning for Idea 3, which says for any positive numbers a and b,
$(-a) + (-b) = (-a) - b$.

(2) Provide reasoning for Idea 4: for any positive numbers a and b,
$(-a) - (-b) = (-a) + b$.

[Hint for (1) and (2): see reasoning for Ideas 1 and 2]

Supplementary Exercise 1

Evaluate each expression.

1) $(-2)-((-2)+2)$

2) $3+(-3)-4$

3) $2\times 2-2$

4) $4((-4)-3)$

5) $((-3)-3)\times 4$

6) $1+4\times 2$

7) $(-2)-((-1)-(-2))$

8) $(-4)-((-3)-2)$

9) $4-(1-3)$

10) $(-4)+(-2)+1$

11) $(-4)-(3+3)$

12) $(-3)-((-3)-(-4))$

13) $(1-(-3))\times 2$

14) $4+(-4)-(-3)$

15) $(-4)-2\times 4$

16) $(-1)-(1-(-3))$

17) $((-2)+4)\times 4$

18) $1-2\times 4$

19) $(-2)+2+2$

20) $(-1)+3-(-1)$

21) $(-2)+(-3)+3$

22) $(-4)-4\times 2$

23) $(-3)-3\times 4$

24) $3-2-4$

25) $(-4)+3+2$

26) $(-3)-((-1)-1)$

27) $4-((-3)-1)$

28) $1+2-(-3)$

29) $3\times 2-(-1)$

30) $2-1-(-3)$

2.6. The additive inverse of the additive inverse

It is a fact that the additive inverse of the additive inverse of a number is that number. Any fact so much fun to say deserves to be a theorem.

Theorem 2.1. *For any number a, $-(-a) = a$.*

Proof. Let a represent any number.

$$\begin{aligned} -(-a) &= 0 + (-(-a)), && \text{identity element.}\\ &= (a + (-a)) + (-(-a)), && \text{inverse element.}\\ &= a + ((-a) + (-(-a)), && \text{associative.}\\ &= a + 0, && \text{inverse elements.}\\ &= a, && \text{identity element.} \end{aligned}$$

Therefore for any number a, $-(-a) = a$. □

2.7. Is $-a$ necessarily a negative number?

It is tempting to think that $-a$ is a negative number, because we see the symbol "$-$" in front of a. In nearly all of the proofs we have produced and the problems we have solved, the phrase "Let a represent any number" appears. If a is a positive number, for instance 5, then $-a$ is -5, the additive inverse of 5. In this case, a is negative number. But, a is supposed to represent *any* number. So, a represents negative as well as positive numbers. Suppose, for example, a represents -3. Then $-a$ is the additive inverse of -3 and the additive inverse of -3 is 3. In this case, $-a$ is a positive number. Moral: do not fall into the trap of thinking $-a$ must be a negative number. $-a$ is positive when a is negative and negative when a is positive. By the way, Who is -0? This should be the additive inverse of 0. And that is 0, because $0 + 0 = 0$. 0 is its own additive inverse. We write 0, not -0, for the additive inverse of 0.

This is surely worth a couple of points on the SAT.

Example 2.13.
Simplify the expression on the left hand side of the equation.

(1) $-(-5) = 5$.
(2) $-(-(-5)) = -5$.
(3) $-(-(-(-5))) = 5$.
(4) $-(-(-(-(-5)))) = -5$.

Apply theorem (2.1).

(5) If $a = -2$ then $-a = 2$. □

Exercise 2.4.

Simplify.

1. $-(-7)$
2. $-(-113)$
3. $-(-9))$
4. $15-(-(-(-3)))$

Answer the following.

5. In section (2.5), we discussed cases $a-b$ and $a+b$, for a and b positive numbers. If we let b represent *any* number, do we still need both $a-b$ and $a+b$? Explain.
6. Simplify $14-a$, if $a=-7$.
7. Simplify $134-(-a)$, if $a=-1$.

2.8. Multiplication with integers

Let us briefly review how we have arrived here. The natural numbers allowed one to pose the question "What is 5 subtract 6"? But the natural numbers could not provide the answer. This question led to the discovery of numbers that had previously gone unnoticed. Those numbers are the additive inverses of the natural numbers. We extended the natural numbers to the integers −the set that contains all the natural numbers, all their additive inverses, and 0. We then investigated how the operations of addition and subtraction should behave in the integers. It is now time to investigate multiplication in the integers.

2.8.1. First, some useful notation

There are a variety of ways to write multiplication. In levels above arithmetic, the sign "×" is avoided. A few examples will get you started on the alternative notations for multiplication.

Let a and b be any numbers. Then multiplication may be indicated in the following ways.

Letter times letter

1. ab means $a\times b$, using juxtaposition.
2. $a\cdot b$ means $a\times b$, using dot.
3. $(a)(b)$ means $a\times b$, using parentheses.

Number times letter or expression in parentheses.

1. $3a$ means $3\times a$.
2. $3\cdot a$ means $3\times a$.
3. $(3)(a)$ means $3\times a$.

(4) $3(a)$ means $3 \times a$.
(5) $3(a+b)$ means $3 \times (a+b)$.

Number times number

(1) $3 \cdot 5$ means 3×5.
(2) $(3)(2)$ means 3×2.
(3) $3(2)$ means 3×2.

A variety of factors

(1) $3(x+y)$ means $3 \times (x+y)$.
(2) $7a(2x+3y) - 8x$ means $7 \times a \times (2 \times x + 3 \times y) - 8 \times x$.

This last example makes our new notation very appealing.

Exercise 2.5. Rewrite each of the following using (1) juxtaposition, (2) dot, (3) parenthesis.

(1) Rewrite $7 \times b$.
(2) Rewrite $9 \times x$.
(3) Rewrite $a \times d$.
(4) Rewrite $2 \times a \times b$.
(5) Rewrite $5 \times (x+2)$.
(6) Rewrite $9a \times (3y+4) + 2$.

Rewrite each of the following by using juxtaposition.
Example: $2 \times (11+7) = 2(11+7)$.

(7) $5 \times (a+b)$
(8) $7 \times (a+1)$
(9) $2 \times (3 \times a + 4)$
(10) $6 \times (x+y)$
(11) $9 \times (23 - 11)$

Rewrite the following using only the "×" symbol for multiplication.

(12) $3(2a+5)$
(13) $4 \cdot 3 \cdot 9$
(14) $4abc$
(15) $xy(2y+5x)$
(16) $3x(5x+3y+7)$

2.8.2. Distribution (turning a product into a sum)

Multiplication of integers will raise questions similar to those occasioned by addition of integers. For example, What is the product

$(-5)(-6)$? Researching this and similar questions requires that we understand another fundamental idea. To introduce this idea, let us answer the question What is the product $5(3+4)$? There are two ways to perform this computation.

(1) first way

$$\begin{aligned} 5(3+4) &= 5(7) \\ &= 35. \end{aligned}$$

(2) second way

$$\begin{aligned} 5(3+4) &= 5(3)+5(4) \\ &= 15+20 \\ &= 35. \end{aligned}$$

Another example,

(1) first way

$$\begin{aligned} 2(5+7) &= 2(12) \\ &= 24. \end{aligned}$$

(2) second way

$$\begin{aligned} 2(5+7) &= 2(5)+2(7) \\ &= 10+14 \\ &= 24. \end{aligned}$$

The reader is perhaps thinking the author has lost his wits, because who would compute the second way when the first is easier?

The author would answer by saying "Everybody! Even the reader has been using the second way for years."

The following scheme for computing 5(37) is familiar to you.

$$\begin{array}{r} 37 \\ \times 5 \\ \hline 35 \\ 150 \\ \hline 185 \end{array}$$

Now let's annotate it.

$$\begin{array}{r l} 37 & \\ \times 5 & \\ \hline 35 & \leftarrow 5(7) \\ 150 & \leftarrow 5(30) \\ \hline 185 & \leftarrow 5(7) + 5(30) \end{array}$$

This is exactly the "second way"! Thinking of 37 as $30 + 7$,

$$\begin{aligned} 5(37) &= \mathbf{5}(\mathbf{7} + \mathbf{30}) \\ &= \mathbf{5}(\mathbf{7}) + \mathbf{5}(\mathbf{30}) \\ &= 35 + 150 \\ &= 185. \end{aligned}$$

The official name for what we have been calling the "second way" is "**distribution**".

Distribution. Let a, b and c represent any numbers, then

$$a(b + c) = ab + ac. \quad \square$$

You will use distribution almost constantly in your work in algebra.

Eventually, we will collect the several "fundamental ideas" that we have met, christen them "Axioms", and mention that those few ideas are the basis of all the algebraic procedures you will ever learn. Distribution will be among those ideas.

The verb "distribute" is often replaced by the verb "**expand**". For example, the command "Expand $3(x+6)$" is obeyed by writing $3x+18$. In this example, $3x + 18$ is called the "**expanded form**" of $3(x + 6)$.

Example 2.14. Rewrite each of the following products by distribution.

(1) $2(a + b)$
(2) $5(2 + y)$
(3) $3(4x + 5)$
(4) $10(2a + 3b)$

Answers.

(1) $2a + 2b$
(2) $10 + 5y$
(3) $12x + 15$
(4) $20a + 30b)$

Exercise 2.6. Expand the following.

(1) $a(b+c)$
(2) $3(a+2)$
(3) $4(x+y)$
(4) $2(y+5)$
(5) $7(a+3b)$
(6) $6(2+x)$
(7) $5(2a+3b)$
(8) $11(x+2)$

It is a fact that multiplication distributes over the sum of *any number* of terms. You will not be able to prove this until some years from now. But we can prove it for a sum of three terms, a sum of four terms, and so on.

(9) Prove that $a(b+c+d)=ab+ac+ad$.
(10) Prove that $a(b+c+d+e)=ab+ac+ad+ae$.

[Hint. It will be helpful to see a blob. See example 1.19.]

2.8.3. Sign of the product

Since there are three flavors of integers (negative, zero, positive), there are several cases to consider. If $a, b \in \mathbb{Z}^+$, the cases are:

case 1: multiplication by 0, $a \cdot 0$ and $0 \cdot a$
case 2: $a \cdot b$
case 3: $-a \cdot b$ and $a \cdot (-b)$
case 4: $(-a) \cdot (-b)$

Case 1. Everyone knows that the product of 0 and a number is 0. But, just out of interest, lets see if we can prove that this is so.

Theorem 2.2. *Let a be any number. Then $0 \cdot a = 0$ and $0 \cdot a = 0$.*

Proof. Let a be any number.

$$\begin{aligned}
1 &= 1, && \text{equality is reflexive.}\\
1 &= 1+0, && \text{identity element.}\\
a\cdot 1 &= a(1+0), && \text{theorem 1.2.}\\
a\cdot 1 &= a\cdot 1 + a\cdot 0, && \text{distribution.}\\
a &= a + a\cdot 0, && \text{identity element.}\\
(-a)+a &= (-a)+(a+a\cdot 0), && \text{theorem 1.2.}\\
(-a)+a &= (-a+a)+a\cdot 0, && \text{associative.}\\
0 &= 0 + a\cdot 0, && \text{inverse elements.}\\
0 &= a\cdot 0, && \text{identity element.}\\
a\cdot 0 &= 0, && \text{equality is symmetric.}
\end{aligned}$$

$\square$

Well, we proved we can prove it. But it was tedious. For the time being, the proofs you see here and any you write will be tedious. That is because at this stage of your learning, you will benefit from the labor of seeing and writing each application of a theorem or fundamental idea. Soon, though, you will be so familiar with the theorems and fundamental ideas, that there will not be much benefit in continuing with this degree of detail. When that time is reached, the proof above will look more like this:

Proof. Let a be any number.

No step has been "skipped", but many have been combined.

$$\begin{aligned}
a &= a(1+0).\\
a &= a + a\cdot 0.\\
0 &= a\cdot 0.
\end{aligned}$$

$\square$

Remark 2.1. The proof of theorem (2.2) is different than those you have seen before. Until this proof, we started with one side of an equation, say the LHS (Left Hand Side), and then we rewrote that side of the equation until it appeared exactly the same as the RHS (Right Hand Side). In this book, this kind of proof is called a "cross the road" proof. In the proof of theorem (2.2), we began with a true equation, then wrote a sequence of equivalent equations, until we produced the equation we desired to prove. The two strategies are outlined below.

First strategy	Second strategy
$LHS =$	first equation
$=$	second equation
$=$	third equation
$=$	fourth equation
$\vdots$	$\vdots$
$= RHS$	desired equation

Case 2: $a \cdot b$. This you have known forever.

Case 3. $a \cdot (-b) = -(ab)$ and $(-a) \cdot (b) = -(ab)$.

The first part of case 3 says $a \cdot (-b) = -(ab)$. Showing that this is true amounts to proving the following theorem. The proof of the first part, $a \cdot (-b) = -(ab)$, is provided. The proof of the second part, $(-a) \cdot (b) = -(ab)$, is left as an exercise.

Theorem 2.3. *Let a and b be any numbers. Then*

$$a \cdot (-b) = -(ab) \text{ and } (-a) \cdot (b) = -(ab).$$

Proof.
First part, $a \cdot (-b) = -(ab)$

$$\begin{aligned} b + (-b) &= 0. \\ a(b + (-b)) &= 0. \\ ab + a \cdot (-b) &= 0, && \text{distribution.} \\ -(ab) + ab + a \cdot (-b) &= -(ab). \\ a \cdot (-b) &= -(ab). \end{aligned}$$

□

Exercise 2.7.

(1) Prove the other part of theorem (2.2); that is, if a is any number, then $0 \cdot a = 0$. [Hint: see proof of the first part of theorem (2.2).]
(2) Rewrite the proof of the first part of theorem (2.3) without merging so many steps and provide justification for each step.

(3) Prove the second part of theorem (2.3); that is, for any numbers a and b, $(-a)\cdot(b) = -(ab)$. [Hint: see proof of the first part of theorem (2.3).]

Case 4. $(-a)\cdot(-b)$

Theorem 2.4. *Let a and b be any numbers. Then*

$$(-a)\cdot(-b) = ab.$$

Proof. Let a and b be any numbers.

$$\begin{aligned} b + (-b) &= 0. \\ -a(b + (-b)) &= 0. \\ (-a)b + (-a)(-b) &= 0, && \text{distribution.} \\ -(ab) + (-a)(-b) &= 0, && \text{theorem 2.3.} \\ (-a)(-b) &= ab. \end{aligned}$$

□

Several examples illustrate the application of the last four theorems.

Example 2.15. Simplify each of the following.

(1) $3\cdot(-5)$
(2) $-2\cdot(-3)$
(3) $-5\cdot(-5)$
(4) $-6\cdot x$
(5) $-9(-y)$
(6) $-7(-1)$

Answers

(1) -15
(2) 6
(3) 25
(4) $-6x$
(5) $9y$
(6) 7

The $\star$ symbol marks the sign that causes trouble for beginners. Always check your work when distributing a negative number or distributing over a difference.

Example 2.16. Expand $3(x-4)$.
Solution:

$$\begin{aligned} 3(x-4) &= 3x - (3)(4) && \text{think “3 times } \textit{negative } 4\text{”.} \\ &= 3x \overset{\bigstar}{-} 12. \quad \square \end{aligned}$$

Example 2.17. Expand $-5(a+3)$.
Solution:

$$-5(a+3) = -5a \overset{\bigstar}{-} 15. \qquad \text{Think “\textit{negative} 5 times 3”.} \qquad \square$$

Example 2.18. Expand $-7(2a-6)$.
Solution:

$$-7(2a-6) = -14a \overset{\bigstar}{+} 42. \quad \text{Think “\textit{negative} 7 times \textit{negative} 6”.} \qquad \square$$

Example 2.19. Simplify each of the following.

(1) $5 \cdot (a+3)$
(2) $3 \cdot (x-4)$
(3) $7 \cdot (x-5)$
(4) $-2 \cdot (a+6)$
(5) $-3(y-1)$
(6) $-7(x-y)$
(7) $(a+3)(4)$
(8) $(x-5)(3)$
(9) $(x-2)(-8)$
(10) $(-5)(-a-3)$
(11) $-7(-y-4)$

Answers

(1) $5a+15$
(2) $3x-12$
(3) $7x-35$
(4) $-2a-12$
(5) $-3y+3y$
(6) $-7x+7y$
(7) $4a+12$
(8) $3x-15)$
(9) $-8x+16$
(10) $5a+15$
(11) $7y+28$

Example 2.20. Use distribution to rewrite each sum as a product.

1. $2x + 6$
2. $3y + 9$
3. $15x - 5y$
4. $7x - 21$
5. $2x - 14$
6. $9x - 27$

Answers

1. $2(x + 3)$
2. $3(y + 3)$
3. $5(3x - y)$
4. $7(x - 3)$
5. $2(x - 7)$
6. $9(x - 3)$

Supplementary Exercise 2

Expand each expression.

1) $5(5n+5)$

2) $-4(1+5a)$

3) $-5(1+2k)$

4) $-(x+5)$

5) $4(-5x-5)$

6) $-5(n+4)$

7) $5(1+k)$

8) $2(-3p+5)$

9) $2(5x-5)$

10) $4(5-5n)$

11) $-4(5m+3)$

12) $-3(-4+5r)$

13) $5(x+4)$

14) $4(1+n)$

15) $-3(-3+4b)$

16) $-(5v+4)$

17) $5(1+2x)$

18) $4(x-1)$

19) $-3(4+4a)$

20) $-4(p+4)$

21) $-4(-2+3x)$

22) $-5(-5n+3)$

23) $3(m+2)$

24) $-5(r+3)$

25) $-5(4x+3)$

26) $5(4n+3)$

27) $2(1-2v)$

28) $3(-5x+2)$

29) $3(3n+2)$

30) $2(4a+2)$

31) $2(5k+1)$

32) $5(5+2x)$

33) $2(x+2)$

34) $2(-2+2n)$

35) $3(-5+m)$

36) $-(3p+1)$

37) $-(x+1)$

38) $-(n-3)$

39) $-5(3-2m)$

40) $-2(4r+1)$

2.8.3.1. Questions you may wish to discuss in class

(1) Are the parentheses in the expression $(x+3)+(a+2)$ necessary?
(2) Must one always work a sum from left to right?
(3) Suppose $10+8-3+2$ is rewritten as $10+8+(-3)+2$. Then must the computation be performed left to right?
(4) Must you perform the computation $10+8-3+2$ from left to right?
(5) Suppose $10+8-3+2$ is rewritten as $10+(8-3)+2$. Then must the computation be performed left to right?
(6) Suppose a classmate says "It is OK to rewrite in a different order an expression that includes subtraction, but you have to move a number and the '$-$' sign in front of it together. For example, $9+10-7+2 = 9-7+10+2$. The '-7' moved as a block." Is this correct?
(7) Using distribution, $5(2+3) = 10+15$. Does this violate the rules you learned about order of operations?

The author hopes your class concludes that the answers to section (2.8.3.1) #2 and #4 are "No". In practice, people who do mental arithmetic well, take advantage of the fact that that expressions including only sums and differences need not be evaluated left to right. In the following example, first try to find pairs of numbers that add to ten or five, then add the tens and fives.

Example 2.21. Find the sum: $7+2+9+9+6+7+3+4+3+8+1$.
Solution.

Perhaps the reader will make up some similar sums for amusement. Do think back to this later when combining like terms is discussed.

$$\begin{array}{ll}
\cancel{7}+2+9+9+6+7+\cancel{3}+4+3+8+1 & \text{think one 10} \\
\cancel{7}+\cancel{2}+9+9+6+7+\cancel{3}+4+3+\cancel{8}+1 & \text{think two 10's} \\
\cancel{7}+\cancel{2}+\cancel{9}+9+6+7+\cancel{3}+4+3+\cancel{8}+\cancel{1} & \text{think three 10's} \\
\cancel{7}+\cancel{2}+\cancel{9}+9+\cancel{6}+7+\cancel{3}+\cancel{4}+3+\cancel{8}+\cancel{1} & \text{think four 10's} \\
\cancel{7}+\cancel{2}+\cancel{9}+9+\cancel{6}+\cancel{7}+\cancel{3}+\cancel{4}+\cancel{3}+\cancel{8}+\cancel{1} & \text{think five 10's}
\end{array}$$

Five tens plus 9 is 59. Multiple lines were used to show the sequence. In practice, it would be a single line with multiple cancellations and mental additions.

2.9. Like terms

Yup. Pun.

First some terminology. We call items in a sum (or in a difference) "**terms**". For example.

(1) "$3a+2a+5a+a$". Four terms: $3a, 2a, 5a$ and a.

(2) "$3a + 2a + 5b + 8x$". Four terms: $3a, 2a, 5b$ and $8x$.
(3) "$3a + 2a + 11a$". Three terms: $3a, 2a$ and $11a$.
(4) "$3a - 12a + 11a$". Three terms: $3a, (-12a)$ and $11a$.
(5) "$7a + 2a$'. Two terms: $7a$ and $2a$.
(6) "$7a \times 2a$". *No* terms.

Perhaps you have heard the expression "You cannot add apples and oranges." Someone is sure to point out that one can add 3 apples and 6 oranges to get 9 fruit. Granted. But if we wish the sum to be some number of apples or some number of oranges, then we are at a loss as to what is the sum of 3 apples plus 6 oranges. Similarly, the sum of 3 a's plus 6 b's is neither a number of a's nor a number of b's.

On the other hand,

3 apples plus 7 apples is 10 apples,
3 horses plus 7 horses is 10 horses,
3 planets plus 7 planets is 10 planets,
$3a$ plus $7a$ is $10a$,
$3b$ plus $7b$ is $10b$.

We call the terms "3 apples" and " 10 apples" "**like terms**". The following are "unlike terms",

3 horses and 7 sheep,
3 planets and 7 oven-mitts,
$3a$ and $7b$,
$3x$ and $7y$.

Rule. *You may always add and subtract like terms. You may never add or subtract unlike terms.* Add apples to apples, or some number of a's to some other number of a's, but do not add apples to oranges or a's to b's.

No doubt the reader would be better satisfied with an explanation of the like terms rule that is based on the mathematics we have covered rather some old tale about apples and oranges.

Let's try this: $7a$ and $2a$ may be added, because $7a+2a = (7+2)a = 9a$. We used distribution. This goes for apples, too. $7\,apples + 2\,apples = (7 + 2)\,apples = 9\,apples$.

$7a$ and $2b$ do *n*ot add. Using distribution to obtain "(sum of numbers)× letter" is impossible, because the same letter does not appear in both terms. $7a + 2b = (7 + 2) \times ? = ?$. The following should make this reasoning clear.

$$7a + 2a = (7 + 2)a \tag{2.7a}$$

$$= 9a. \tag{2.7b}$$

but

$$7a + 2b = (7 + 2)\underline{\ ???\ } \tag{2.8a}$$

$$= \ ??? \tag{2.8b}$$

Equation (2.7b) is equation (2.7a) simplified. The key equation, (2.7a), can only be obtained by distribution.

Equation (2.8b) would be equation (2.8a) simplified. But, the key equation, (2.8a), cannot be obtained.

Now that you are convinced that like terms add and subtract, but unlike terms do not, let us consider a few examples.

Example 2.22. The following show the LHS of an equation simplified to the RHS of an equation by combining like terms.

(1) $8a + 3a - 2a = 9a$.
(2) $6a - 11a + 3a = -2a$.
(3) $5x + 4x = 9x$. □

When several kinds of like terms appear in an expression, each kind is simplified independently of the other kinds. The expression $6a + 8b + 9a + 7a + 2b$ contains two kinds of like terms. One kind is in a. The other kind is in b. The expression $6a + 8b + 9a + 7a + 2b$ simplifies to $22a + 10b$.

We consider a variety of examples of simplification.

Example 2.23. Only one kind of term.

(1) $3a + 2a$
(2) $6a - 5a + 3a - 10a$
(3) $(2x + 5x) + (9x - x)$

Solutions.

(1) $5a$
(2) $-6a$. No harder than (1), merely more computations.
(3) $(2x+5x)+(9x-x) = 2x+5x+9x-x = 15x$. The parentheses are unnecessary.

Example 2.24. Two kinds of terms.

(1) $3a + 2b$
(2) $9a - 3a + 7b - 12b$
(3) $(2x + 3) + (9x - 1)$

Solutions.

(1) $3a + 2b$ is already simplified.
(2) $6a - 5b$
(3) $11x + 2$. 3 and -1 are like terms, because each contains the same letter. Namely, no letter.

Example 2.25. Several kinds of terms.

(1) $5a + 2b + 3c$
(2) $12a - 3b + 5c + 10b - a + 5c$
(3) $(4x + 3y + 8z - 7) + (3x + 2z + 9y)$

Solutions

(1) $5a + 2b + 3c$ is already simplified.
(2) $11a + 7b + 10c$
(3) $(4x+3y+8z-7)+(3x+2z+9y) = 4x+3y+8z-7+3x+2z+9) = 7x + 12y + 10z - 7$. □

There is a good way of keeping track of terms as you combine them. *Look* at the expression $3a + 9a + 5b + 4b + 2c + 10c$. The first term is in a. Scan across combining terms in a. Keep track of each term you use by *lightly and neatly* striking through it.

If you have ever done a "word search" in history class, you have already done something similar to, but harder than, ferreting out these like terms.

$$\begin{aligned}
&3a + 5b + 9a + 2c + 4b + 10 \\
&\qquad = \cancel{3a} + 5b + \cancel{9a} + 2c + 4b + 10c && \text{\} think } 12a \\
&\qquad = \cancel{3a} + 5b + \cancel{9a} + \cancel{2c} + 4b + \cancel{10c} && \text{\} think } 12a + 12c \\
&\qquad = \cancel{3a} + \cancel{5b} + \cancel{9a} + \cancel{2c} + \cancel{4b} + \cancel{10c} && \text{\} think } 12a + 12c + 9b \\
&\qquad = 12a + 9b + 12c && \text{\} done}
\end{aligned}$$

The several lines above show the sequence of computations. In practice, the work would probably look like this:

$$\cancel{3a} + \cancel{5b} + \cancel{9a} + \cancel{2c} + \cancel{4b} + \cancel{10c} = 12a + 9b + 12c.$$

Example 2.26. Simplify each of the following.

(1) $3a + 2a + 8b + 9b$
(2) $6a - 5a + 3b - b$
(3) $12a + 3a - 3b - 5b$
(4) $15b - 2a + 10b + 3a$
(5) $3x + 2x + 25y + x + 9x + 16y$
(6) $3a + 5b + 2a + 9c - a + b - 2c$
(7) $(3x + 5y) + (2x + 8y)$
(8) $(x + 5y + 3z) + (2x + 5z)$
(9) $3x + 2 + (9x + 5)$
(10) $2x + 3y$
(11) $2x + 8 + 9y + 7$
(12) $x + 7 + 2y - x$

Answers

(1) $5a + 17b$
(2) $a + 2b$
(3) $15a - 8b$
(4) $a + 25b$
(5) $15x + 41y$
(6) $4a + 6b + 7c$
(7) $5x + 13y$
(8) $3x + 5y + 8z$
(9) $12x + 7$
(10) $2x + 3y$, already simplified
(11) $2x + 9y + 15$, 8 and 7 are like terms. Each contains the same letter. Namely, no letter.
(12) $2y + 7$ □

Exercise 2.8. Simplify the following expressions. If an expression is already simplified, say so.

(1) $4a - 2a + 5$
(2) $3a - 2b + 9b + 12a$
(3) $(6x + 5y) + (2x - 3y + 2)$
(4) $9a + 2b - 12c + 7 - 2a + 8c - 3b + 2$
(5) $(2c + 7b - 5) + (5a + c - 8)$
(6) $3x + 7y + 2x + 8z + 2y - 9$
(7) $-5x + 2z + 7y - 8 + 3y + 8z - 5x$
(8) $2a + 2x - 8c - 8 + 3x - 5x$
(9) $2a + 2x + 12c - 7$
(10) $5x + 5y + 5z + 5$

2.9.1. More like terms

A single term may be, and in fact often will be, the product of several letters. For example, $3abc$ is a single term. So is $3ab$. Now, are these two like terms? The answer is “no”, because like terms must have exactly the same letters. While $3abc$ and $3ab$ are not like terms, $3abc$ and $5abc$ are like terms.

$3abc$	is	one term
$3abc + 9ac$	contains	two unlike terms.
$3abc + 9abc$	contains	two like terms.
$3abc + 9ab - 6bc$	contains	three unlike terms.
$3abc + 9ab - 6abc + 2ab$	contains	two kinds of like terms.

Example 2.27. Simplify each of the following.

(1) $3ab + 2ab + 8b + 9b$
(2) $6ac - 5a + 3ac - 6a$
(3) $12a + 3b - 3ab$
(4) $15ab - 2c + 10ab + 2c + 5$
(5) $3xy + 2xyz + 25y + x + 9y + 16xy$
(6) $3abd + 5abc + 2ab + 9bc - 2a$
(7) $2xyz + 3xy - xyz + 9xy - 2$

Answers

(1) already simplified
(2) $9ac - 11a$
(3) already simplified
(4) $25ab + 5$
(5) $x + 19xy + 2xyz + 34y$
(6) already simplified
(7) $xyz + 12xy - 2$ □

There is a very nice notation for writing the product of several identical factors. For example, the product $3 \cdot 3$ is written 3^2 and $5 \cdot 5 \cdot 5$ is written 5^3. Similarly, the product of seven factors of x is written x^7. The following are all different terms and cannot be combined by addition or subtraction: $3x^2$, $3x^3$, $3x^4$. But, $3x^2$ and $5x^2$ are like terms, so $3x^2 + 5x^2 = 8x^2$. It is always true that like terms contain exactly the same letters with exactly the same exponents.

In 3^6, 3 is called the base and 6 is called the exponent.

Example 2.28. Simplify each of the following.

(1) $5a^3 + 2a^3$

(2) $5a^3 + 2a^4$
(3) $2a^4 + 5a^3 + 10a^4 + 2a$
(4) $2a^4b^3 + 5a^4b^3$
(5) $7a^4b^3 + 2a^4b^2$
(6) $(3x^2 + 7x + 3) + (9x^2 + 3x + 11)$

Answers

(1) $8a^3$
(2) Already simplified
(3) $12a^4 + 5a^3 + 2a$
(4) $7a^4b^3$
(5) Already simplified
(6) $12x^2 + 10x + 14$ □

Exercise 2.9. Simplify the following expressions. If an expression is already simplified, say so.

(1) $2a^5 + 6a^5$
(2) $7x^3y + 9x^3y$
(3) $2x^3y + 2x^2y$
(4) $4a^3b^3c^2 + 2a^3b^3c^2$
(5) $4a^3b^3c^2 + 2a^3b^3c^3$

2.9.2. Simplifying expressions that involve distribution

Many expressions will involve like terms and distribution. Such expressions are simplified by first distributing and then combining like terms.

Example 2.29. Simplify each of the following.

(1) $3(x + 5) + 6x$
(2) $2(a + b) + 5a + 9b$
(3) $3(a + 2b) + 8b$
(4) $4(a - 3) + 6$
(5) $5(-x - 4) + 3(x - 2)$
(6) $3(x^2 + y) + 3x^2 + 5y$
(7) $2(a + 3b) + 5(a + b)$
(8) $3(x^2 + x + 5) + 7(x^2 + x + 1)$
(9) $a(a + 6) + a^2 + 3a + 5$
(10) $2a(a + 5) + 3a^2 + 8a - 2$
(11) $2a(b + c) + 5(ab + ac)$
(12) $3(2b + 5c) - 6b - 15c$

Solutions.

(1) $3(x+5)+6x = 3x+15+6x = 9x+15$.
(2) $2(a+b)+5a+9b = 2a+2b+5a+9b = 7a+11b$.
(3) $3(a+2b)+8b = 3a+6b+8b = 3a+14b$.
(4) $4(a-3)+6 = 4a-12+6 = 4a-6$.
(5) $5(x+2y+z)+2x+y+6z = 5x+10xy+5z+2x+y+6z = 7x+y+11z+10xy$.
(6) $5(-x-4)+3(x-2) = -5x-20+3x-6 = -2x-26$.
(7) $2(a+3b)+5(a+b) = 2a+6b+5a+5b = 7a+11b$.
(8) $3(x^2+x+5)+7(x^2+x+1) = 3x^2+3x+15+7x^2+7x+7 = 10x^2+10x+22$.
(9) $a(a+6)+a^2+3a+5 = a^2+6a+a^2+3a+5 = 2a^2+9a+5$.
(10) $2a(a+5)+3a^2+8a-2 = 2a^2+10a+3a^2+8a-2 = 5a^2+18a-2$.
(11) $2a(b+c)+5(ab+ac) = 2ab+2ac+5ab+5ac = 7ab+tac$.
(12) $3(2b+5c)-6b-15c = 6b+15c-6b-15c = 0$. □

Exercise 2.10. Simplify the following expressions. If an expression is already simplified, say so.

(1) $3(a+5)+7a-12$
(2) $5(a+b)+3a+9$
(3) $7(2a+b)+4a-2b+1$
(4) $2(x+3)+5(x-1)$
(5) $5(-x+2)+3(x-1)$
(6) $6(3-2y)+2(y+7)$
(7) $11(-2x-1)+2(3x+5)$
(8) $2(3a+2b-5)+5(2a+b-1)$
(9) $3(a^2+a+7)+5(a^2+a+6)$
(10) $b(a+b+2)+a(2a+3b-1)$
(11) $4(a^3b+a^2b+b)+3a^3+5a^2b+3b-7$ □

Supplementary Exercise 3

Simplify each expression.

1) $10 - 5(8n - 6) - 2n$

2) $-10 - 2(-10k - 7)$

3) $16 - 4(p + 7) - 2p$

4) $-8(10 + 6x) - 6x$

5) $7n - 5(2n + 6)$

6) $-2(10m + 4) + 5$

7) $-9r - 10(7r + 5)$

8) $-9 - 8(x - 4)$

9) $-2(n - 9) - 4n$

10) $-(-5b + 4) - 5b$

11) $7v - 8(-2 + 4v)$

12) $-6(2x + 4) - 1$

13) $-n - 10(10n - 8)$

14) $-3a - 6(1 + a)$

15) $-9(4k - 5) - 5k$

16) $8 - 6(1 - 5p)$

17) $-4(4x - 6) + 5$

18) $-8n - (7 - 6n)$

19) $-10m - 2(1 + 3m)$

20) $-6(r + 7) - 5r$

21) $-2(7 + 6x) - 4$

22) $8n - 9(5n + 6)$

23) $-7(1 + 6b) - 7$

24) $-7(v - 8) - 9$

25) $-4 - 10(x + 9)$

26) $-10(-9 + 5n) - 4n$

27) $9 - 7(-5a + 5)$

28) $-5(8 + 4k) + 1$

29) $-8x - 10(5x - 4)$

30) $-4(1 - 3x) + 10x$

31) $-7(1 + n) - 6$

32) $-3(-3k - 5) - 3k$

33) $-10(-5 + 3p) + 9p$

34) $-8(4x - 5) + 1$

35) $-(n - 1) - 10$

36) $-5m - 4(m + 3)$

37) $-(r - 6) - 3r$

38) $10 - 8(10 + 7x)$

39) $-6(6n + 6) + 2$

40) $-(9b + 6) - 7b$

41) $-10(5 + 9v) - 8(5v - 6)$

42) $-10(7n - 8) - 3(4n - 7)$

43) $-7(-3 + 5x) - 9(1 + 9x)$

44) $-8(8k - 4) - 8(10 - 7k)$

45) $-9(4a - 6) - (2 - 7a)$

46) $-10(p + 4) - 2(p - 6)$

47) $-8(x + 9) - 4(1 - 10x)$

48) $-9(m - 1) - 6(-3 + 5m)$

49) $-8(4 + 2n) - 9(-7n + 5)$

50) $-8(5r + 6) - 4(5r + 5)$

51) $-7(-3 + 8x) - (2x + 4)$

52) $-7(1 - 4n) - 9(10 + 4n)$

53) $-3(b - 5) - 7(3b + 4)$

54) $-7(8v - 7) - 4(-1 + 4v)$

55) $-8(-6 - 5x) - 2(2x + 3)$

56) $-10(n - 7) - 9(6 - 5n)$

57) $-6(3 - 8a) - 4(1 + a)$

58) $-2(-5v - 9) - 8(v + 5)$

59) $-6(5x + 4) - 10(10 - 6x)$

60) $-4(x + 10) - 7(7x - 6)$

2.9.3. Expressions that cause trouble

We have avoided these two bad-actors for as long as we could, but their time has arrived. They are expressions like $10 - 2(x+3)$ and its even worse cousin, $10-(x+3)$. Needless to say, a whole army of these can be produced merely by substituting different numbers for 10, 2, and 3.

Suppose we wish to rewrite without parentheses the expression

$$10 - 2(x+3). \tag{2.9}$$

The presence of the leading 10 is complicating matters. If the 10 were absent, the expression would be

$$-2(x+3).$$

This would cause no trouble. We know

$$-2(x+3) = -2x - 6.$$

So, now we are essentially done. We need only realize that equation (2.9) is equivalent to

$$10 - 2(x+3) = 10 - 2x - 6. \tag{2.10}$$

Replacing 10 in expression (2.9) with any number other than 0, produces no end of fiends like expression (2.9). But, we can take care of them all by generalizing the work that led to equation (2.10). We do so.

Let A, a, b and c be any numbers. Then,

$$-a(b+c) = -ab - ac. \qquad \text{Distribution} \tag{2.11a}$$

$$A - a(b+c) = A - ab - ac. \qquad \text{Theorem 1.2} \tag{2.11b}$$

And this shows that

$$A - a(b+c) = A - ab - ac, \tag{2.12}$$

which we state as a theorem.

Theorem 2.5. *For any numbers A, a, b and c, $A-a(b+c) = A-ab-ac$.*

Example 2.30. What about the worse cousin, $10 - (x+3)$?

Well, theorem 2.5 is true for all numbers, so it is true when $a = 1$. That is,

$$A - 1(b+c)$$

which, by theorem 2.5, equals $A - 1b - 1c = A - b - c$. ☐

When you work with expressions like these, be on the lookout for mistakes with the signs. It almost seems that the mission of these expressions is to fiendishly tempt beginners into making sign mistakes. Always pause a moment to check work when these expressions are involved.

The following two examples should be carefully compared to each other.

Example 2.31. Simplify $16 - 3(x + 4)$.

$$\begin{aligned} A - a(b + c) &= A - ab - ac, \qquad && A = 16, a = 3, b = 1, \mathbf{c} = \mathbf{4}. \\ 16 - 3(x + 4) &= 16 - 3x - 12. && \\ &= 4 - 3x. && \end{aligned}$$

Example 2.32. Simplify $16 - 3(x - 4)$.

$$\begin{aligned} A - a(b + c) &= A - ab - ac. && \\ 16 - 3(x + 4) &= 16 - 3x - (3)(-4), \qquad && A = 16, a = 3, b = 1, \mathbf{c} = (-\mathbf{4}). \\ &= 16 - 3x - (-12) && \\ &= 16 - 3x + 12 && \\ &= 28 - 3x. && \end{aligned}$$

There is a helpful way to think when you simplify these kinds of expressions. In the following examples, the first column is what you write when applying theorem (2.5). The second column is what you might think based on theorem (2.5).

Example 2.33. Simplify $20 - 2(x + 6)$

$$\begin{aligned} 20 - 2(x + 6) &= 20 - 2x - 12 \quad \} \textit{ think: } (-2)(x) \text{ is } -2x, (-2)(6) \text{ is } -12.. \\ &= 8 - 2x. \end{aligned}$$

Example 2.34. Simplify $20 - 2(x - 6)$.

$$\begin{aligned} 20 - 2(x - 6) &= 20 - 2x + 12 \quad \} \textit{ think: } (-2)(x) \text{ is } -2x, (-2)(-6) \text{ is } 12. \\ &= 32 - 2x. \end{aligned}$$

And for the worse cousin.

Example 2.35. Simplify $8 - (x + 2)$.

$$\begin{aligned} 8 - (x + 2) &= 8 - x - 2 \quad \} \textit{ think: } (-1)(x) \text{ is} - x, (-1)(2) \text{ is } -2. \\ &= 6 - x. \end{aligned}$$

Example 2.36. Simplify $8 - (x - 2)$

$$\begin{aligned} 8 - (x - 2) &= 8 - x + 2 \quad \} \textit{ think: } (-1)(x) \text{ is} - x, (-1)(-2) \text{ is } 2. \\ &= 10 - x. \end{aligned}$$

Example 2.37. Simplify each of the following.

(1) $15 - 2(x + 5)$
(2) $20 - 3(x - 2)$
(3) $9 - 2(3x + 1)$
(4) $6 - 3(-x - 2)$
(5) $-5 - 4(x - 2)$

Solutions

(1) $15 - 2(x + 5) = 15 - 2x - 10 = 5 - 2x$.
(2) $20 - 3(x - 2) = 20 - 3x + 2 = 22 - 3x$.
(3) $9 - 2(3x + 1) = 9 - 6x - 3 = 6 - 6x$.
(4) $6 - 3(-x - 2) = 6 + 3x + 6 = 12 + 3x$.
(5) $-5 - 4(x - 2) = -5 - 4x + 8 = 13 - 4x$. □

Supplementary Exercise 4

Simplify each expression.

1) $-(-9n-4)+10n$

2) $2a-8(-6-5a)$

3) $-6(8k+8)-7$

4) $-3(5+8x)+6$

5) $6x-6(x-3)$

6) $-8(1+7n)+10n$

7) $2-6(-4+7m)$

8) $-p-6(7p-9)$

9) $7x-(3+6x)$

10) $-7(-n+6)-2n$

11) $-(1+2m)+4m$

12) $-2(2r-2)-10$

13) $3x-9(10x-3)$

14) $-7(7-3n)-6n$

15) $-4(9b-3)+7$

16) $-10(1-9v)+5$

17) $10x-4(x+8)$

18) $3-(5-4x)$

19) $-2(2+9a)-5$

20) $8k-10(10k+9)$

21) $-8(7+8p)-1$

22) $-9-5(-3x+8)$

23) $-3(-8n+8)-9n$

24) $3m-10(-4m+7)$

25) $-5-8(-r+7)$

26) $-5(3+7x)+8$

27) $4n-4(n-2)$

28) $9b-6(b+2)$

29) $4-6(5v-2)$

30) $-4(x-2)-4x$

31) $9n-(5-2n)$

32) $9-9(a-3)$

33) $-8-6(3-3k)$

34) $-4(10-3x)+5x$

35) $4x-(7x-4)$

36) $-4-9(5n+9)$

37) $-6(10m+9)+9$

38) $3p-6(1-7p)$

39) $-8x-10(x+9)$

40) $-7(-8n+8)+5$

The following examples involve expressions that are more complicated, but require no new ideas or techniques.

Example 2.38. Simplify $8 - 3(x+3) + 2(2x+9)$.
Solution.

$$\begin{aligned} 8 - 3(x+3) + 2(2x+9) &= 8 \overbrace{-3x - 9}^{\text{1st dist}} \underbrace{+4x + 18}_{\text{2nd dist}} \\ &= x + 17. \end{aligned}$$

Example 2.39. Simplify $7(a-2) - 5(2a+3)$.
Solution.

$$\begin{aligned} 7(a-2) - 5(2a+3) &= \overbrace{7a - 14}^{\text{1st dist}} \underbrace{-10a - 15}_{\text{2nd dist}} \\ &= -3a - 29. \end{aligned}$$

Example 2.40. Simplify $9 - 3(a-5) - 8(a+1)$.
Solution.

$$\begin{aligned} 9 - 3(a-5) - 8(a+1) &= 9 \overbrace{-3a + 15}^{\text{1st dist}} \underbrace{-8a - 8}_{\text{2nd dist}} \\ &= -11a + 16. \end{aligned}$$

Example 2.41. Simplify $14 - (a+6) - 5(a-3) + 4(a-2)$.
Solution.

$$\begin{aligned} 14 - (a+6) - 5(a-3) + 4(a-2) &= 14 \overbrace{-a - 6}^{\text{1st dist}} \underbrace{-5a + 15}_{\text{2nd dist}} \overbrace{+4a - 8}^{\text{3rd dist}} \\ &= -2a + 15. \end{aligned}$$

Have the simplifications in these last examples been "harder" than those of previous examples? These examples involved no new ideas, strategies, or techniques. Perhaps they are no harder than previous "simplify" problems. They are, however, *more complicated* than the simplifications in earlier examples. You do the same old stuff, just more of it on these last four problems.

Supplementary Exercise 5

Simplify each expression.

1) $-8(7m-4)-4(3+4m)$

2) $-3(r+5)-5(r-6)$

3) $-(-6x-3)-7(1+7x)$

4) $-8(n+5)-5(5n+7)$

5) $-4(b-7)-5(6+7b)$

6) $-(2v-7)-4(1+7v)$

7) $-6(4x+3)-4(-3x+6)$

8) $-8(x-8)-3(1+6x)$

9) $-4(8a-3)-6(a-4)$

10) $-8(-6k+2)-8(1-8k)$

11) $-2(7p-8)-2(3p+5)$

12) $-8(1-5x)-2(8+5x)$

13) $-5(3-5n)-3(1+3n)$

14) $-8(-7+5m)-5(m-1)$

15) $-7(3r-2)-8(3-5r)$

16) $-8(1-2x)-8(7x-5)$

17) $-(4n+1)-7(8n-5)$

18) $-7(-6+b)-8(3b+3)$

19) $-4(1-3v)-6(v-6)$

20) $-8(3x+2)-4(1+5x)$

21) $-5(n-8)-6(n-8)$

22) $-3(-5+a)-5(5a+1)$

23) $-8(1-6k)-5(4k+1)$

24) $-7(2x+5)-(x+4)$

25) $-5(1+8x)-3(1-x)$

26) $-5(n+3)-2(7n+6)$

27) $-4(8m+1)-(-2+7m)$

28) $-(1-7p)-(-6-8p)$

29) $-7(-3+3x)-2(7x+7)$

30) $-7(n+2)-(2n+7)$

31) $-4(1+2b)-2(b+6)$

32) $-7(7-5r)-4(1+r)$

33) $-3(5x-8)-8(4x+6)$

34) $-7(n+5)-8(5n+6)$

35) $-3(b-7)-7(1-4b)$

36) $-7(6v-2)-(v-8)$

37) $-8(3x-2)-6(-5+5x)$

38) $-5(3x+8)-6(8x-4)$

39) $-2(a+1)-(1-a)$

40) $-7(2k-6)-5(2-5k)$

Chapter 3

Rational numbers

The purpose of this chapter is to introduce the rational numbers. We begin with the integers, because the rational numbers are an extension of the integers.

3.1. Integers

The discovery of the integers gave our ancestors mathematical riches beyond those of the natural numbers, while retaining the wealth of the natural numbers. Over a period of more than 2000 years, mathematically insightful humans discovered ever deeper qualities of the integers. Recently, 1994, Andrew Wiles proved a theorem known as "Fermat's Last Theorem". The theorem resisted proof for over 300 years. In proving Fermat's Last Theorem, Wiles discovered a surprising connection between the integers and another area of mathematics. The study of the integers is called "Number Theory".

3.1.1. Division with integers

In many cases, the result of the division of an integer by an integer is an integer. For example, $15 \div 3 = 5$. In other cases, dividing an integer by an integer does not result in an integer. For example, $15 \div 2$ is not an integer. Based on our past experience, we naturally wonder if the integers can be extended in such a way that every division results in a number.

The number 1 plays the same role in multiplication as the number 0 does in addition. Let a represent any integer. Just as $a + 0 = a$ so too does $a \cdot 1 = a$. We called 0 the identity element for addition. We call 1 the identity element for multiplication.

Every integer, except 1, can be written as the product of two distinct integers. For example, $3 = 3 \cdot 1$. But no pair of distinct integers exists whose product is 1.

In a previous section, we discovered that 0, the identity element for addition, is the sum of a number and its additive inverse. Perhaps each number has a multiplicative inverse so that the product of the number and its multiplicative inverse is 1, the identity element for multiplication.

You have for some time known that

$$2 \cdot \frac{1}{2} = 1$$

$$3 \cdot \frac{1}{3} = 1$$

$$4 \cdot \frac{1}{4} = 1.$$

And, in general, for any integer a other than 0,

$$a \cdot \frac{1}{a} = 1.$$

The number $\frac{1}{a}$ is the multiplicative inverse of a, because the product of a and $\frac{1}{a}$ is 1.

When the integers are extended by including the multiplicative inverse of every integer, except 0, the result is the set of rational numbers. We defined the integers by listing the members $\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$.

It is a little inconvenient to list the rational numbers, but we do so for the positive rational numbers in table (3.1). Notice that the positive integers are included in table (3.1). They appear in the first column.

$$\begin{array}{cccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{1}{5} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{1}{5} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{1}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

TABLE 3.1. the positive rational numbers

The following definition of the rational numbers is usually given in elementary textbooks.

Definition 3.1 (The Rational Numbers)**.** A number that can be written in the form $\frac{a}{b}$ where a and b are integers and b is not 0 is called a **rational number**. The set of all such numbers is called the set of rational numbers and is often denoted by the symbol $\mathbb{Q}$. □

In this book. we will use the terms "rational number" and "**fraction**" interchangeably.

Remark 3.1. The integers are contained in the set of rational numbers. Every integer is a rational number, but some rational numbers are not integers. In other words, the set of integers is a proper subset of the set of rational numbers, $\mathbb{Z} \subset \mathbb{Q}$.

Definition 3.2 (Irrational Number)**.** A number that is not a rational number is called an **irrational number**. □

This will be important later. For now it will not be used.

Example 3.1.

(1) $\frac{2}{3}$ is a rational number, because 2 and 3 are integers.

(2) $\frac{119}{107}$ is a rational number, because 119 and 107 are integers.

(3) 5 is a rational number, because it can be written $\frac{5}{1}$ and 5 and 1 are integers.

(4) $\frac{-2}{7}$ is a rational number, because -2 and 7 are integers.

(5) $\frac{9}{-11}$ is a rational number, because 9 and -11 are integers.

Numbers like $\sqrt{2}$ will be discussed later in this book.

(6) $\frac{\sqrt{2}}{5}$ is an irrational number, because $\sqrt{2}$ is not an integer.

(7) $\frac{0}{2}$ is a rational number, because 0 and 2 are integers.

(8) $\frac{1}{0}$ is not a rational number, because the denominator 0 violates definition 3.1.

(9) $\frac{0}{0}$ is not a rational number, because the denominator is 0.

As for the last two items, we will soon discuss the fact that the mix of symbols $\frac{a}{0}$, where a is any number, is meaningless, or, as is usually said, "undefined".

Someone is bound to ask whether

$$\frac{\frac{2}{5}}{\frac{7}{9}}$$

is a rational number. The answer to this question will have to wait for a few pages until we discuss the ideas and techniques needed to provide an answer to it.

3.2. Division

Of course

$$6 \cdot 2 = 12. \tag{3.1}$$

Suppose we multiply both sides by $\frac{1}{2}$,

$$6 \cdot 2 \cdot \frac{1}{2} = 12 \cdot \frac{1}{2}$$

Since 2 and $\frac{1}{2}$ are multiplicative inverses, and 1 is the identity element for multiplication,

$$6 \cdot 1 = 12 \cdot \frac{1}{2}$$

$$6 = 12 \cdot \frac{1}{2}.$$

Noting that $6 = 12 \div 2$

$$12 \div 2 = 12 \cdot \frac{1}{2} \tag{3.2}$$

Equation (3.2) quite clearly suggests that dividing by a number and multiplying by the multiplicative inverse of that number have the same meaning, because they each name the same number. Whenever we can write an equation like equation (3.1), we can obtain an equation like equation (3.2). And, equation (3.1) simply equates the number 12 with the product of its factors, 2 and 6.

You probably used the idea of equation (3.1) when you first learned about division. Suppose when you were first learning about division, you were asked

> "What is
>
> $$15 \div 5?"$$
>
> you may thought
>
> $$5 \times \blacksquare = 15.$$
>
> and answered
>
> "3".

You knew that multiplication and division are related operations.

Let us mention two more multiplication facts.

$$2 \cdot 3 = 6$$
$$2 \cdot 4 = 8.$$

Knowing these two multiplication facts, you could perform the following two divisions

$$6 \div 2 = 3$$
$$8 \div 2 = 4.$$

But what about

$$7 \div 2 = \blacksquare?$$

If the equations are placed in the following order

$$2 \cdot 3 = 6 \tag{3.3}$$

$$2 \cdot \blacksquare = 7 \tag{3.4}$$

$$2 \cdot 4 = 8, \tag{3.5}$$

then it is obvious there is no integer equal to $7 \div 2$. Before the discovery of the rational numbers, this was as far as any human could go, because there is no integer that, substituted for $\blacksquare$, would may equation (3.4) true. But, there is a rational number that makes equation (3.4) true. We can find that number like this.

Suppose there exists a number, call it a, for which

$$7 \div 2 = a. \tag{3.6}$$

This would mean that

$$2 \cdot a = 7.$$

Multiply both sides by $\frac{1}{2}$,

$$\frac{1}{2} \cdot 2 \cdot a = \frac{1}{2} \cdot 7.$$

Since 2 and $\frac{1}{2}$ are multiplicative inverses, and 1 is the identity element for multiplication,

$$1 \cdot a = \frac{1}{2} \cdot 7$$

$$a = \frac{1}{2} \cdot 7. \tag{3.7}$$

Equation (3.6) says $a = 7 \div 2$ and we substitute that value of a into equation (3.7) to obtain

$$7 \div 2 = \frac{1}{2} \cdot 7. \tag{3.8}$$

Equation (3.8) suggests that dividing by a number and multiplying by the multiplicative inverse of that number have the same meaning. This is similar to what we found when we considered $12 \div 2$, but in this example 2 is not a factor of 7.

If you believe that the essential feature of division is expressed in equations (3.2) and (3.8), then our next definition will strike you as being exactly right.

Definition 3.3 (Division)**.** Provided $b \neq 0$, the expressions

$$a \div b, \quad \frac{a}{b}, \quad a \cdot \frac{1}{b}$$

have identical meaning; that is, the expressions name the same number. If $b = 0$, then none of these expressions is meaningful. ☐

Example 3.2.
Find each quotient using the same reasoning as was used to rewrite equation (3.6) as equations (3.7).

(1) $14 \div 7$
(2) $39 \div 7$

Solutions.

(1)

$$14 \div 7 = x$$

means that

$$7 \cdot x = 14.$$

Multiply both sides by $\frac{1}{7}$,

$$\frac{1}{7} \cdot 7 \cdot x = \frac{1}{7} \cdot 14.$$

Since 7 and $\frac{1}{7}$ are multiplicative inverses, and 1 is the identity element for multiplication,

$$\begin{aligned} 1 \cdot x &= \frac{1}{7} \cdot 14 \\ x &= \frac{1}{7} \cdot 14 \\ x &= 2. \\ \therefore \quad 14 \div 7 &= 2. \quad \square \end{aligned}$$

(2)

$$39 \div 7 = x$$

means that

$$7 \cdot x = 39.$$

Multiply both sides by $\frac{1}{7}$,

$$\frac{1}{7} \cdot 7 \cdot x = \frac{1}{7} \cdot 39.$$

Since 7 and $\frac{1}{7}$ are multiplicative inverses, and 1 is the identity element for multiplication,

$$1 \cdot x = \frac{1}{7} \cdot 39$$

$$x = \frac{1}{7} \cdot 39$$

$$x = \frac{39}{7}$$

$$\therefore \quad 39 \div 7 = \frac{39}{7}. \qquad \square$$

Example 3.3. Find the quotient using the definition of division.
Solution.

$$231 \div 53 = 231 \cdot \frac{1}{53}.$$

$$\therefore \quad 231 \div 53 = \frac{231}{53}. \qquad \square$$

Exercise 3.1.
Find each quotient by imitating example (3.2).

(1) $27 \div 3$
(2) $31 \div 6$

Use the definition of division to find the following quotients.

(3) $9 \div 3$
(4) $17 \div 8$
(5) $52 \div 13$
(6) $5 \div 12$

3.2.1. Division by zero is undefined

There are several ways to see this is true. We mention two of them.

3.2.1.1. Contradiction

Division by zero is meaningless. To see why this is so, suppose it is not so. That is, suppose, for the sake of argument, that the expression $\frac{a}{0}$ does equal a number. We will call that number c. Then,

$$\frac{a}{0} = c.$$

By theorem (1.2),

$$\frac{a}{0} \cdot 0 = c \cdot 0.$$

Then

$$\frac{a}{\cancel{0}} \cdot \cancel{0} = c \cdot 0.$$

By theorem (2.2),

$$a = 0.$$

We have just shown that if we suppose $\frac{a}{0}$ is meaningful, every number must equal 0! This is about as false as false can be. So, our supposition that $\frac{a}{0}$ is meaningful must have been wrong.

3.2.1.2. No multiplicative inverse

According to our definition, division by 0 is multiplication by the multiplicative inverse of 0. But any number times 0 is 0, not 1. This means 0 has no multiplicative inverse. So, division by 0 is meaningless.

3.3. Summary

3.3.1. Numbers

The natural numbers were extended to the integers, and the integers were extended to the rational numbers. Until the rational numbers are extended to what are known as the real numbers, the word "number" in this book will denote a rational number. If for some reason we wish to limit a discussion to a subset of the rational numbers, that will be made clear at the time.

3.3.2. Definitions

In mathematics, a definition tells what a thing *is*, by telling us the essential qualities such a thing must possess. It often requires considerable insight, intuition, and effort to discover the correct definition of a mathematical object. For example, the area of mathematics called "calculus" depends on a single definition. The mathematical quest for that definition was handed down through generations of mathematicians. Generations −because the correct definition took over two hundred years to discover!

3.3.3. Axioms

We have used several ideas that in this book have been called "fundamental" - or sometimes "basic" - ideas. Mathematicians realized that only a few of these basic ideas were needed to provide the foundation of algebra. These few basic ideas were singled out and called "**axioms**". All the rules of algebra follow from the axioms. Since the axioms are the starting place, no proof is given for them. This book began with a brief discussion of obvious ideas. The axioms are supposed to be obvious ideas.

3.3.4. Axioms of the rational numbers

Addition and multiplication are commutative. For every number x and every number y,

$$x \cdot y = y \cdot x \qquad\qquad x + y = y + x$$

Addition and multiplication are associative. For every x, every y, and every z,

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \qquad\qquad (x + y) + z = x + (y + z)$$

Identity elements. For every x,

$$x \cdot 1 = x \qquad\qquad x + 0 = x$$

Inverse elements. For every x, there exists a y for which,

$$x \cdot y = 1, \ \text{ provided } x \neq 0 \qquad\qquad x + y = 0$$

Multiplication is distributive over addition. For every x, every y, and every z,

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

3.3.5. Theorems

A theorem is a statement of a mathematical fact that has been proved. The proof of a theorem uses only axioms, definitions, and previously proved theorems. Proof guarantees that the theorem is as true as the axioms and definitions used in its proof.

3.3.6. Division and subtraction

Notice that the axioms are all about addition and multiplication. There is no mention of subtraction and division. As we realized in Chapter 2, subtraction is *adding* the multiplicative inverse of a number and division is *multiplying* by the multiplicative inverse of a number.

Remark 3.2. We very often use the phrase "Let a represent any number." When we say this, what qualities do we suppose a to posses? Answer: only the qualities that every number is known to possess. And those common qualities are exactly the qualities stated in the axioms. Of course a number has certain qualities that make it the special number it is. For example, 2 is the only even prime number −that's quite a distinction. It is certainly not a quality that 2 has in common with all other numbers.

When we say "a represents *any* number", we ignore any attributes that might make a number special.

Remark 3.3. Theorems in Chapter 2 are valid for rational numbers. The word "number" that appears in the theorems of Chapter 2 should now be understood to refer to a rational number.

3.4. Negative and Positive Rational Numbers

The sign of a sum, difference, or product of rational numbers is determined exactly as it would be for integers. The sign of the quotient of rational numbers is determined just as it would be for the product. This should be no surprise, since division is defined as multiplication by the multiplicative inverse.

3.4.1. Writing positive and negative fractions

The following are three equivalent forms of one fraction.

$$-\frac{3}{5} = \frac{-3}{5} = \frac{3}{-5}.$$

While each of the above forms is mathematically correct, some are better style than others. As you gain experience, you will always write good style. Don't lose any sleep over it now.

Example 3.4. Good and bad style

Bad style	Good style
$-\frac{7}{3}\cdot\frac{1}{-4}$	$\frac{-7}{3}\cdot\frac{-1}{4}$
$-\frac{5}{3}\cdot-\frac{1}{4}$	$\frac{5}{3}\left(-\frac{1}{4}\right)$
$\frac{11}{5}--\frac{2}{7}$	$\frac{11}{5}-\left(-\frac{2}{7}\right)$
$\frac{11}{5}--\frac{2}{7}$	$\frac{11}{5}-\frac{-2}{7}$
$\frac{9}{13}+-\frac{15}{16}$	$\frac{9}{13}+\left(-\frac{15}{16}\right)$
$\frac{9}{13}+-\frac{15}{16}$	$\frac{9}{13}+\frac{-15}{16}$

3.4.2. Product and Quotient of Positive and negative fractions

Example 3.5. Find the product of $\frac{3}{7}$ and $\frac{-5}{2}$.

Solution. The product is computed just as in earlier grades: $\frac{\text{product of numerators}}{\text{product of denominators}}$. Be careful to get the signs correct.

$$\begin{aligned} \frac{3}{7}\cdot\frac{-5}{2} &= \frac{3(-5)}{7\cdot 2} \\ &= \frac{-15}{14}. \quad \square \end{aligned}$$

Example 3.6. Find the product of $\frac{-5}{8}$ and $\frac{3}{-7}$.

Solution. $\frac{3}{-7} = \frac{-3}{7}$. So,

$$\begin{aligned} \frac{-5}{8} \cdot \frac{3}{-7} &= \frac{-5}{8} \cdot \frac{-3}{7} \\ &= \frac{(-5)(-3)}{8 \cdot 7} \\ &= \frac{15}{56}. \quad \square \end{aligned}$$

Example 3.7. Each of the following shows the LHS simplified to the RHS.

(1) $\frac{2}{3} \cdot \frac{-5}{7} = \frac{-10}{21}$.

(2) $\frac{2}{5}\left(-\frac{3}{7}\right) = \frac{-6}{35}$.

(3) $\frac{-6}{5} \cdot \frac{-5}{11} = \frac{30}{55}$.

(4) $\left(-\frac{2}{3}\right)\left(\frac{-1}{5}\right) = \frac{2}{15}$.

(5) $\left(-\frac{7}{3}\right)\left(\frac{1}{-4}\right) = \frac{7}{12}$.

Example 3.8. Find the product $\left(\frac{8}{15}\right)\left(\frac{-21}{64}\right)$.

Solution.

$$\begin{aligned} \left(\frac{8}{15}\right)\left(\frac{-21}{64}\right) &= \left(\frac{\cancel{8}^{1}}{\cancel{15}_{3}}\right)\left(\frac{\cancel{-21}^{-7}}{\cancel{64}_{8}}\right) \\ &= \frac{-7}{24}. \end{aligned}$$

All mathematics you learned in previous years still applies. Cancel common factors before you multiply.

Example 3.9. Find the product $\left(\frac{3}{-6}\right)\left(\frac{-15}{-12}\right)$.
Solution.

$$\begin{aligned}\left(\frac{3}{-6}\right)\left(\frac{-15}{-12}\right) &= \left(\frac{-3}{6}\right)\left(\frac{-(-15)}{12}\right)\\ &= \left(\frac{-3}{6}\right)\left(\frac{15}{12}\right)\\ &= \left(\frac{\cancel{-3}^{-1}}{\cancel{6}_2}\right)\left(\frac{\cancel{15}^5}{\cancel{12}_4}\right)\\ &= \frac{-5}{8}.\end{aligned}$$

Example 3.10. Find the quotient $\frac{-2}{5} \div (-23)$.
Solution.

$$\begin{aligned}\frac{-2}{5} \div (-23) &= \frac{-2}{5} \cdot \frac{-1}{23}\\ &= \frac{2}{115}.\end{aligned}$$

Example 3.11. Find the quotient $\frac{-2}{5} \div (5) \div (-7)$.
Solution.

$$\begin{aligned}\frac{-2}{5} \div (5) \div (-7) &= \frac{-2}{5} \cdot \frac{1}{5} \cdot \frac{-1}{7}\\ &= \frac{2}{175}.\end{aligned}$$

Example 3.12. Find the product $\frac{-a}{3} \cdot \frac{-2}{b} \cdot \frac{7}{5}$.
Solution.

$$\begin{aligned} \frac{-a}{3} \cdot \frac{-2}{b} \cdot \frac{7}{5} &= \frac{(-a)(-2)(7)}{(3)(b)(5)} \\ &= \frac{14a}{15b}. \end{aligned}$$

Exercise 3.2. Find the product or quotient.

(1) $\frac{-3}{4} \cdot \frac{5}{7}$

(2) $\frac{-12}{5} \cdot \frac{-3}{11}$

(3) $\left(\frac{-7}{11}\right)\left(\frac{-2}{4}\right)$

(4) $\left(\frac{3}{4}\right)\left(\frac{-24}{27}\right)$

(5) $-\frac{6}{9} \cdot \frac{-12}{2}$

(6) $-4 \cdot \frac{-7}{3}$

(7) $-5 \cdot \frac{-11}{5} \cdot \frac{-13}{33}$

(8) $\frac{3}{8}\left(\frac{-12}{21}\right)$

(9) $9 \cdot \frac{2}{-3} \cdot \frac{-1}{16}$

(10) $\frac{2}{-9} \cdot \frac{-3}{5}$

(11) $\frac{2}{11} \cdot \frac{-3}{-2}$

(12) $\dfrac{4}{5} \div (-3)$

(13) $\dfrac{-3}{8} \div (-2)$

(14) $-\dfrac{5}{9} \div (-10)$

(15) $\dfrac{-12}{5} \div (6)$

(16) $\dfrac{1}{-5} \div (-4)$

(17) $\dfrac{-a}{2} \cdot \dfrac{-b}{3}$

(18) $\dfrac{-3a}{2} \cdot \dfrac{-6}{5}$

(19) $\dfrac{3x}{5} \cdot \dfrac{-4}{y}$

(20) $\dfrac{-4x}{7} \cdot \dfrac{-14}{2}$

(21) $\dfrac{3a}{2} \cdot \dfrac{-7}{3a}$

(22) $\dfrac{5a}{12} \div (-5b)$

(23) $-4 \cdot \dfrac{-a}{5} \div 2a$

(24) $-a \div b \div c$

(25) $-7a \div b \div c$

3.4.3. Sum and Difference of Positive and Negative Fractions

We will call fractions that have the same denominator “like fractions”. As you have known for some time, like fractions may be combined by addition and subtraction. This sounds quite similar to the rule discussed section (2.9) which says that “like terms” may be added and subtracted. The similarity is real. The fraction $\frac{a}{b}$ can always be rewritten as $a \cdot \frac{1}{b}$. Provided $b \neq 0$,

$$\frac{a}{b} = a \cdot \frac{1}{b}.$$

If we think of $\dfrac{1}{b}$ as indicating a “basic unit”, then $a \cdot \dfrac{1}{b}$ denotes a quantity a of such units. For example,

(1) $\dfrac{2}{3}$ denotes 2 thirds, just as “2 apples” denotes two apples.

(2) $\dfrac{8}{5} = 8 \cdot \dfrac{1}{5}$ which indicates 8 of the basic unit the fifth.

(3) $\dfrac{7}{5}$ denotes 7 fifths.

In section (2.9) we used distribution to provide a mathematically compelling reason for the common wisdom that “you can add apples to apples but not apples to sheep”. We can use distribution in the context of fractions, too. Fifths may be added to fifths, because

$$\begin{aligned}\frac{3}{5} + \frac{4}{5} &= 3 \cdot \frac{1}{5} + 4 \cdot \frac{1}{5} \\ &= (3+4)\frac{1}{5} \\ &= (7)\frac{1}{5} \\ &= \frac{7}{5}. \qquad \square\end{aligned}$$

Fifths may not be added to sevenths, because

$$\begin{aligned}\frac{3}{5} + \frac{4}{7} &= 3 \cdot \frac{1}{5} + 4 \cdot \frac{1}{7} \\ &= (3+4) \cdot ??? \qquad \square\end{aligned}$$

Of course, if $\frac{3}{5}$ and $\frac{4}{7}$ are written with a common denominator, the basic units will match so that the addition can be performed.

$$\begin{aligned} \frac{3}{5}+\frac{4}{7} &= \frac{21}{35}+\frac{20}{35} \\ &= 21\cdot\frac{1}{35}+20\cdot\frac{1}{35} \\ &= (21+20)\frac{1}{35} \\ &= (41)\frac{1}{35} \\ &= \frac{41}{35}. \qquad \square \end{aligned}$$

Will we, given any two fractions, always be able to write them with a common denominator (common basic unit) so that they may be added or subtracted? The answer is "Yes". A few pages from now, we will prove that this is so.

3.4.4. Proofs of facts we already knew

Some of the motivation for obtaining proofs of facts we already know is to bolster our confidence in the system of axioms, definitions, and theorems by noting that the system produces the results it *should* produce.

As long as we are in the category "proofs of stuff I've known for years", we may as well do a few more. The reader will be asked to prove the next two theorems as exercises.

Theorem 3.1. *For any number a other than 0,* $\dfrac{a}{a}=1$. $\square$

Theorem 3.2. *For any number a,* $a=\dfrac{a}{1}$. $\square$

Theorems (3.1) and (3.2) are used in the proofs of the next several theorems.

Theorem 3.3. *For any numbers a and b with neither 0,* $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$.

Proof.

$$\begin{aligned}
\text{RHS} &= \frac{1}{a} \cdot \frac{1}{b} \\
&= \frac{ab}{ab} \cdot \frac{1}{a} \cdot \frac{1}{b} && \text{Theorem (3.1)} \\
&= \frac{1}{ab} \cdot a \cdot \frac{1}{a} \cdot b \cdot \frac{1}{b} && \text{definition division} \\
&= \frac{1}{ab} \cdot 1 \cdot 1 && \text{definition division} \\
&= \frac{1}{ab} \\
&= \text{RHS}
\end{aligned}$$

□

Theorem 3.4. *For any numbers a, c and $b \neq 0, d \neq 0$,* $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

Proof.

$$\begin{aligned}
\text{LHS} &= \frac{a}{b} \cdot \frac{c}{d} \\
&= a \cdot \frac{1}{b} \cdot c \cdot \frac{1}{d} && \text{definition division} \\
&= ac \cdot \frac{1}{b} \cdot \frac{1}{d} && \text{multiplication commutative} \\
&= ac \cdot \frac{1}{bd} && \text{Theorem (3.3)} \\
&= \frac{ac}{ab} && \text{definition division} \\
&= \text{RHS}
\end{aligned}$$

□

The next theorem answers the question "Will we, given any two fractions, always be able to write them with a common denominator so that they may be added or subtracted?

Theorem 3.5. *For any numbers a, b, c, d with $b \neq 0$ and $d \neq 0$,*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

Proof.

$$\begin{aligned}\frac{a}{b} + \frac{c}{d} &= \frac{a}{b} \cdot 1 + \frac{c}{d} \cdot 1 \\ &= \frac{a}{b} \cdot \frac{d}{d} + \frac{c}{d} \cdot \frac{b}{b} \\ &= \frac{ad}{bd} + \frac{bc}{bd} \\ &= \frac{ad + bc}{bd}.\end{aligned}$$

□

The next theorem is similar to Theorem (3.5). Its proof is left as an exercise.

Theorem 3.6. *For any numbers a, b, c, d with $b \neq 0$,*

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}.$$

Exercise 3.3.

(1) Prove Theorem (3.1): $\dfrac{a}{a} = 1, a \neq 0$.

(2) Prove Theorem (3.2): $a = \dfrac{a}{1}$.

(3) Provide justifications for the steps in the proof of theorem (3.5).

(4) Prove Theorem (3.6): $\dfrac{a}{b} + \dfrac{c}{b} = \dfrac{a + c}{b}, \quad b \neq 0$.

The next theorem may not be in the category of "stuff I've known forever". It is about the multiplicative inverse of the rational number $\frac{a}{b}$.

Theorem 3.7. *For any numbers a, b where $a \neq 0$ and $b \neq 0$,*

$$\frac{1}{\frac{a}{b}} = \frac{b}{a}.$$

Proof. The right hand side of this equation is just the left hand side multiplied by the number 1 in a fancy form. That is easier to see if you think of $\frac{b}{a}$ as a blob.

$$\begin{aligned}
\frac{1}{\dfrac{a}{b}} &= \frac{1}{\dfrac{a}{b}} \cdot \frac{\dfrac{b}{a}}{\dfrac{b}{a}} \\
&= \frac{\dfrac{b}{a}}{\dfrac{a}{b} \cdot \dfrac{b}{a}} \\
&= \frac{\dfrac{b}{a}}{\dfrac{ab}{ab}} \\
&= \frac{\dfrac{b}{a}}{1} \\
&= \frac{b}{a}.
\end{aligned}$$

$\square$

Remark 3.4. Theorem (3.7) says that the multiplicative inverse of $\dfrac{a}{b}$ is $\dfrac{b}{a}$. This means that, when neither a, b, nor d are 0,

$$\frac{c}{d} \div \frac{a}{b} = \frac{c}{d} \cdot \frac{b}{a}.$$

The number $\frac{b}{a}$ is sometimes called the **reciprocal** of the number $\frac{a}{b}$.

3.4.5. A leftover question

We now consider a question from the beginning of section (3.1.1). Is

$$\frac{\dfrac{2}{5}}{\dfrac{7}{9}} \tag{3.9}$$

a rational number?

Our definition of a rational number says that the expression (3.9) is rational, *if* it can be written as an integer over an integer. Now, expression (3.9) is just a fraction divided by a fraction.

This means

$$\begin{aligned}\frac{\frac{2}{5}}{\frac{7}{9}} &= \frac{2}{5} \div \frac{7}{9}\\ &= \frac{2}{5}\cdot\frac{9}{7}\\ &= \frac{18}{35}.\end{aligned}$$

which is a rational number. ☐

Example 3.13. Simplify $\dfrac{\frac{9}{16}}{\frac{27}{8}}$

Solution.

$$\begin{aligned}\frac{\frac{9}{16}}{\frac{27}{8}} &= \frac{9}{16} \div \frac{27}{8}\\ &= \frac{9}{16}\cdot\frac{8}{27}\\ &= \frac{1}{6}.\end{aligned}$$

Example 3.14. Simplify $\dfrac{2a}{3} + \dfrac{a}{2}$

Solution. Just as in previous years, the first task is to get a common denominator.

$$\begin{aligned}\frac{2a}{3} + \frac{a}{2} &= \frac{2a}{3}\left(\frac{2}{2}\right) + \frac{a}{2}\left(\frac{3}{3}\right)\\ &= \frac{4a}{6} + \frac{3a}{6}\\ &= \frac{7a}{6}.\end{aligned}$$

Example 3.15. Simplify $\dfrac{2a+5}{3}+\dfrac{a+1}{5}$.
Solution.

$$\begin{aligned}
\frac{2a+5}{3}+\frac{a+1}{5} &= \frac{2a+5}{3}\left(\frac{5}{5}\right)+\frac{a+1}{5}\left(\frac{3}{3}\right)\\
&= \frac{5(2a+5)}{15}+\frac{3(a+1)}{15}\\
&= \frac{5(2a+5)+3(a+1)}{15}\\
&= \frac{10a+25+3a+3}{15}\\
&= \frac{13a+28}{15}.
\end{aligned}$$

Example 3.16. Simplify $\dfrac{7a+5}{4}+\dfrac{3a-6}{2}$.
Solution.

$$\begin{aligned}
\frac{7a+5}{4}+\frac{3a-6}{2} &= \frac{7a+5}{4}+\frac{3a-6}{2}\left(\frac{2}{2}\right)\\
&= \frac{7a+5}{4}+\frac{2(3a-6)}{4}\\
&= \frac{7a+5+2(3a-6)}{4}\\
&= \frac{7a+5+6a-12}{4}\\
&= \frac{13a-7}{4}.
\end{aligned}$$

Example 3.17. Simplify $\dfrac{2a+3b}{5}+\dfrac{6a-7}{10}$.
Solution.

$$\begin{aligned}\frac{2a+3b}{5}+\frac{6a-7}{10} &= \frac{2a+3b}{5}\left(\frac{2}{2}\right)+\frac{6a-7}{10}\\ &= \frac{4a+6b}{10}+\frac{6a-7}{10}\\ &= \frac{4a+6b+6a-7}{10}\\ &= \frac{10a+6b-7}{10}.\end{aligned}$$

Example 3.18. Simplify $\dfrac{a+4}{3}-\dfrac{5a-2}{5}$.

Solution.

$$\begin{aligned}\frac{a+4}{3}-\frac{5a-2}{5} &= \frac{a+4}{3}\left(\frac{5}{5}\right)-\frac{5a-2}{5}\left(\frac{3}{3}\right)\\ &= \frac{5(a+4)}{15}-\frac{3(5a-2)}{15},\end{aligned}$$

the next step will help you to avoid a popular mistake,

$$\begin{aligned}&= \frac{5(a+4)}{15}+\frac{-3(5a-2)}{15}\\ &= \frac{5a+20-15a+6}{15}\\ &= \frac{-10a+26}{15}. \qquad \square\end{aligned}$$

There are still no new ideas or techniques, even in example (3.18). But example (3.18) involves so much that you have only just learned, that you should not be surprised if you make mistakes. Remember, you are pushing against the boundaries of your knowledge and skill. With time and practice, your hand will work expressions like example (3.18) while you hold a conversation on a different topic.

Example 3.19. Simplify $\frac{5}{3}\left(\frac{a}{2}+\frac{2}{3}\right)-\frac{a}{9}$.

Solution.

$$\begin{aligned}\frac{5}{3}\left(\frac{a}{2}+\frac{2}{3}\right)-\frac{a}{9} &= \frac{5a}{6}+\frac{10}{9}-\frac{a}{9}\\ &= \frac{15a}{18}+\frac{20}{18}-\frac{2a}{18}\\ &= \frac{15a+20-2a}{18}\\ &= \frac{13a+20}{18}. \qquad \square\end{aligned}$$

The next example is of an expression in which fractions involving a letter are written in what the author considers the "second best" style. Here are two styles of writing the same fraction.

When in the problem sets the second best style is used, do your work, including copying the problem, in the best style.

Best style: $\frac{3a}{8}$.

Second best style: $\frac{3}{8}\,a$.

Either style is fine in print, but the second best style when written by hand can make it hard for the reader to distinguish

$$\frac{3a}{8} \quad \text{from} \quad \frac{3}{8a}.$$

Exercise 3.4. Simplify. It the expression is already in simplest for, say so.

(1) $\dfrac{\frac{5}{7}}{\frac{15}{28}}$

(2) $\dfrac{\frac{16}{13}}{\frac{32}{26}}$

(3) $\dfrac{\frac{3\mathrm{a}}{5b}}{\frac{9\mathrm{a}}{35}}, a \neq 0, b \neq 0$

(4) $\dfrac{8x}{11} + \dfrac{3x}{44}$

(5) $\dfrac{12a}{13} + \dfrac{3}{26}$

(6) $\dfrac{3a+1}{4} + \dfrac{2a+3}{2}$

(7) $\dfrac{a+1}{4} + \dfrac{5a-2}{3}$

(8) $\dfrac{6a+5}{3} + \dfrac{2a-9}{5}$

(9) $\dfrac{7x-2}{5} + \dfrac{x-3}{2}$

(10) $\dfrac{6x-5}{11} + \dfrac{9x-6}{22}$

(11) $\dfrac{a+2}{3} - \dfrac{3a-2}{5}$

(12) $\dfrac{4a+1}{10} - \dfrac{a-1}{5}$

(13) $\dfrac{2a-5}{2} - \dfrac{3a-2}{4}$

(14) $8 - \frac{2y-1}{6} - \frac{5y-9}{3}$

(15) $1 - \frac{-3x+7}{5} - \frac{x+2}{10}$

(16) $\frac{-1}{4} - \frac{-a-2}{5} - \frac{a-6}{2}$

(17) $\frac{a-2b}{7} - \frac{2a-3b}{3}$

(18) $\frac{-x-1}{2} - \frac{-(x+1)}{3}$

(19) $\frac{2}{3} - \frac{a+3}{a}$

(20) $\frac{2}{3b} - \frac{a-2}{2a}$

(21) $\frac{2}{5}a - \frac{3a}{10}$

(22) $\frac{3}{7}a - \frac{2a-5}{2}$

(23) $\frac{3}{x} - \frac{2}{5}$

(24) $\frac{x-y}{2} - \frac{2(3x-y)}{-2}$

(25) $\frac{\frac{\text{a+3}}{2}}{\frac{2}{5}} - \frac{a+1}{5}$

Supplementary Exercise 6

---.

Simplify each expression.

1) $-3(-2x+2)$

2) $-3(2+2n)$

3) $2(m-3)$

4) $3(r+2)$

5) $-2n+\frac{1}{2}n$

6) $x-\frac{7}{3}+\frac{3}{2}x+\frac{1}{2}$

7) $-\frac{7}{2}v+\frac{3}{2}v$

8) $-b+\frac{5}{2}-\frac{1}{2}b$

9) $x+\frac{1}{2}-x$

10) $-\frac{10}{3}n-\frac{1}{2}+n+\frac{1}{3}$

11) $-\frac{4}{3}a-\frac{5}{3}a$

12) $\frac{1}{2}k-\frac{1}{2}+k+\frac{3}{2}$

13) $\frac{8}{3}\left(\frac{3}{2}p-\frac{5}{2}\right)$

14) $\frac{1}{3}\left(-\frac{1}{3}x+1\right)$

15) $\frac{4}{3}\left(n+\frac{4}{3}\right)$

16) $\frac{1}{3}\left(\frac{2}{3}m+\frac{5}{2}\right)$

17) $-\left(\frac{1}{2}r+2\right)$

18) $-\left(\frac{3}{2}x+\frac{1}{2}\right)$

19) $-\frac{11}{3}\left(\frac{4}{3}n-\frac{3}{2}\right)$

20) $-\left(\frac{4}{3}b-\frac{8}{3}\right)$

21) $\frac{2}{3}\left(-2v+\frac{5}{2}\right)-\frac{1}{3}v$

22) $-\frac{5}{3}\left(-\frac{7}{2}x+\frac{3}{2}\right)+\frac{4}{3}x$

23) $\frac{4}{3}\left(-\frac{1}{3}n-\frac{7}{2}\right)+\frac{1}{2}$

24) $\frac{1}{3}-\frac{5}{3}\left(\frac{1}{2}a-1\right)$

25) $-\frac{3}{2}\left(v-\frac{7}{3}\right)+\frac{1}{3}$

26) $-\frac{5}{3}-\frac{11}{3}\left(x+\frac{3}{2}\right)$

27) $-\frac{3}{2}\left(\frac{1}{2}a-\frac{2}{3}\right)-\frac{3}{2}$

28) $\frac{2}{3}-3\left(\frac{1}{3}n-\frac{8}{3}\right)$

29) $\frac{3}{2}\left(\frac{4}{3}k-\frac{4}{3}\right)+\frac{5}{2}$

30) $-3\left(p+\frac{2}{3}\right)+\frac{5}{3}$

31) $-\frac{2}{3}\left(-\frac{11}{3}x-1\right)-x$

32) $3\left(k+\frac{2}{3}\right)-\frac{2}{3}$

33) $-2\left(\frac{2}{3}m+\frac{5}{3}\right)+\frac{2}{3}m$

34) $\frac{4}{3}r+\frac{1}{2}\left(\frac{5}{2}r+\frac{1}{3}\right)$

35) $-2x-\frac{1}{2}\left(2x-\frac{5}{3}\right)$

36) $\left(-\frac{3}{2}n+1\right)+n$

Supplementary Exercise 7

Simplify each expression.

1) $2p + \frac{1}{3}\left(-\frac{3}{2}p - \frac{11}{3}\right)$

2) $-\frac{3}{2} - \frac{11}{3}\left(\frac{2}{3}n - 1\right)$

3) $-\frac{7}{3}\left(x - \frac{5}{3}\right) + \frac{1}{2}x$

4) $\frac{4}{3}\left(\frac{1}{3}x + 1\right) + x$

5) $-\frac{5}{3} - \frac{2}{3}\left(\frac{4}{3}r - 1\right)$

6) $-\frac{11}{3}\left(x + \frac{10}{3}\right) + \frac{5}{3}x$

7) $\frac{3}{2}\left(n - \frac{3}{2}\right) + \frac{1}{3}n$

8) $-1 - 2\left(\frac{1}{3}b + 2\right)$

9) $v + \frac{5}{3}\left(\frac{5}{2}v - \frac{7}{2}\right)$

10) $-\frac{3}{2} + \frac{2}{3}\left(-\frac{1}{3}x + \frac{7}{3}\right)$

11) $\frac{7}{3}\left(\frac{5}{3}x - \frac{7}{3}\right) - \frac{5}{2}x$

12) $-\frac{5}{2}a - \frac{7}{3}\left(2a - \frac{1}{2}\right)$

13) $-\frac{1}{3}k - \frac{3}{2}\left(\frac{1}{2}k + 1\right)$

14) $\frac{8}{3}p + 2\left(\frac{1}{3}p - \frac{7}{2}\right)$

15) $-\frac{7}{2}x + \frac{1}{3}\left(-2x - \frac{4}{3}\right)$

16) $-\frac{3}{2}\left(-2x - \frac{11}{3}\right) + \frac{1}{3}\left(-\frac{5}{2}x - \frac{1}{2}\right)$

17) $-\frac{4}{3}\left(\frac{1}{3}m + \frac{5}{3}\right) - \frac{4}{3}\left(m + \frac{4}{3}\right)$

18) $\frac{1}{2}\left(r - \frac{1}{2}\right) - \frac{5}{2}\left(r + \frac{1}{2}\right)$

19) $\frac{5}{3}\left(\frac{3}{2}x + \frac{5}{2}\right) + \frac{1}{2}\left(\frac{1}{3}x + \frac{1}{2}\right)$

20) $\frac{7}{3}\left(n - \frac{1}{2}\right) + \frac{2}{3}\left(-\frac{7}{3}n + \frac{4}{3}\right)$

21) $-\frac{4}{3}\left(b + \frac{8}{3}\right) - \frac{8}{3}\left(\frac{1}{2}b - 2\right)$

22) $\frac{3}{2}\left(v + \frac{1}{2}\right) - \frac{11}{3}\left(\frac{3}{2}v + \frac{1}{2}\right)$

23) $\frac{3}{2}\left(-\frac{3}{2}n + 2\right) - 2\left(\frac{1}{3}n + 1\right)$

24) $2\left(-\frac{4}{3}x + \frac{4}{3}\right) - \left(-\frac{2}{3}x + \frac{4}{3}\right)$

25) $-\frac{4}{3}\left(a + \frac{3}{2}\right) - \frac{8}{3}\left(2a + \frac{3}{2}\right)$

26) $-2\left(\frac{4}{3}k - 2\right) + \frac{4}{3}\left(-\frac{3}{2}k + \frac{1}{3}\right)$

27) $2\left(x + \frac{5}{3}\right) + \frac{2}{3}\left(2x + \frac{2}{3}\right)$

28) $\frac{1}{2}\left(\frac{5}{2}x - \frac{3}{2}\right) + \frac{4}{3}\left(\frac{1}{2}x + 1\right)$

29) $-\frac{10}{3}\left(x - \frac{10}{3}\right) + \frac{1}{3}\left(x + \frac{5}{3}\right)$

30) $-\frac{7}{2}\left(m - \frac{3}{2}\right) + \frac{3}{2}\left(m + \frac{3}{2}\right)$

Chapter 4

Equations

4.1. The idea of an equation

An **equation** is a statement. "A=B" states that the expression on the left hand side (LHS) and the expression on the right hand side (RHS) are names for the same object.

A statement can be either true or false. The statement "A=B" is true, if "A" and "B" (the LHS and the RHS) do name the same object. Otherwise, the statement is false.

The equation

$$8 = 8 \tag{4.1}$$

is so obviously true that is seems silly to even mention that it is true.

The statement

$$6 + 2 = 8 \tag{4.2}$$

is just as obviously true as equation (4.1) once we simplify the LHS,

$$\begin{aligned} LHS &= 6 + 2 \\ &= 8 \\ &= RHS. \qquad \square \end{aligned}$$

The statement

$$6 + 2 = 5 + 3 \tag{4.3}$$

is almost as obviously true as equation (4.2) once we simplify both the LHS and the RHS,

$$\begin{aligned} LHS &= 6 + 2 \\ &= 8. \end{aligned}$$

$$\begin{aligned} RHS &= 5+3 \\ &= 8. \end{aligned}$$

$$\therefore\ LHS = RHS. \qquad \Box$$

On the other hand, it is not quite so obvious that

$$\frac{1}{2}\left(\frac{1}{2}-\frac{1}{3}\right)-\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2}\right)=\frac{1}{2}-\frac{1}{3} \tag{4.4}$$

is true. But it is true, because

Can you say which of the three properties of equality is the backbone of this demonstration that equation (4.4) is true?

$$\begin{aligned} LHS &= \frac{1}{2}\left(\frac{1}{2}-\frac{1}{3}\right)-\frac{1}{2}\left(\frac{1}{3}-\frac{1}{2}\right) \\ &= \frac{1}{12}-\frac{1}{3} \\ &= \frac{1}{6}. \end{aligned}$$

$$\begin{aligned} RHS &= \frac{1}{2}-\frac{1}{3} \\ &= \frac{1}{6}. \end{aligned}$$

$$\therefore LHS = RHS. \qquad \Box$$

Sometimes we wish to show that a statement is false. For example, show that “$7 \cdot 12 = 123 - 38$” is false.

$$\begin{aligned} LHS &= 7 \cdot 12 \\ &= 84. \end{aligned}$$

$$\begin{aligned} RHS &= 123 - 38 \\ &= 85. \end{aligned}$$

But $LHS \neq RHS$, so “$7 \cdot 12 = 123 - 38$” is false. $\Box$

The reader can verify that when 2 is substituted for x, the following equation is true.

$$3x + 12 = 18. \tag{4.5}$$

A variety of words and expressions are employed to say that a given number when substituted for the unknown makes an equation true. Speaking of equation (4.5), you may say

(1) "2 is the solution of equation (4.5)."
(2) "2 solves equation (4.5)."
(3) "2 satisfies equation (4.5)."
(4) "The number that satisfies equation (4.5) is 2."

Exercise 4.1. For each equation and proposed value, show that the value is (or is not) a solution of the equation.

(1) $5x + 8 = 23, \quad x = 3.$

(2) $7x - 9 = 26, \quad x = 5.$

(3) $5a - 4 = 20, \quad a = 6.$

(4) $\dfrac{2y}{5} + \dfrac{3}{2} = 3, \quad y = \dfrac{15}{4}.$ □

4.2. Solving equations

We have a surefire way to test whether or not a stated value of the unknown satisfies an equation. But, how do we find the number to test? We need is a method that will *make obvious* the number that satisfies an equation. Several ideas will be helpful as we search for such a method. They are introduced below.

Definition 4.1. (Equivalent equations)
Equations are said to be **equivalent** when they have exactly the same solution. For example, the equations $x + 8 = 12$ and $18 - x = 14$ and $12 \div 3 = x$ are equivalent equations because the number 4 is the solution to all of them.

Remark 4.1. Equivalent equations may be said to be different forms of the same equation. For example, $x + 5 = 2$ and $x = -3$ are forms of the same equation.

Remark 4.2. It is a fact, though it will be many years before you will be able to prove it, that an equation obtained from a prior equation by correct application of any combination of axioms, definitions, and theorems is equivalent to the prior equation.

We indicate that equations are equivalent by the symbols "$\Longleftrightarrow$" , "$\equiv$', or "iff". Each of the following is true:

1. "$x + 3 = 0 \Longleftrightarrow x + 9 = 6$"
2. "$2x = 8 \equiv 7 + x = 11$"
3. "$x + 8 = 20$ iff $x + 24 = 36$."

Equivalence and equality are two different relations. Expressions are said to be "equal", for example $3 + 7 = 20 \div 2$. Equations are said to be "equivalent", for example $2x + 1 = 7 \Longleftrightarrow 6x + 3 = 21$. Like equality, equivalence is reflexive, symmetric, and transitive.

We should note that contrary to what you may read and hear, the letter x in the equation $x + 4 = 9$ is not a *variable*. It is called a *letter* and it represents an *unknown* quantity, not a varying quantity.

With vocabulary and notation taken care of, let's return to our search for a method to find the solution of $13x - 117 = 104$.

The author hopes that the value of x that makes $13x - 117 = 104$ true is not obvious to you. If it is, perhaps you will humor the author and pretend that it is not. Our strategy is to begin with the equation $13x - 117 = 104$ and produce a chain of equivalent equations

$$\begin{aligned} 13x - 117 = 104 &\Longleftrightarrow \\ &\vdots \\ &\Longleftrightarrow \\ &\vdots \\ &\Longleftrightarrow x = 17. \end{aligned}$$

Until the last equation $x = 17$ is produced. The solution of the last equation of the chain,

$$x = 17,$$

is, as they say, "obvious to even the most casual observer". The value of x that makes $x = 17$ true is 17. No doubt about it!

The good news is that you have already acquired the ideas and techniques needed to carry out the strategy we have outlined. Here goes.

Example 4.1. Solve $13x - 117 = 104$ for the unknown.
Solution.
Add 117 to both sides,

$$13x - 117 = 104 \iff 13x = 104 + 117 \tag{4.6}$$

$$\iff 13x = 221, \tag{4.7}$$

multiply both sides by $\frac{1}{13}$,

$$\iff x = \frac{221}{13} \tag{4.8}$$

$$\iff x = 17. \tag{4.9}$$

$\therefore$ the value of x that satisfies equation (4.6) is 17. □

Since each equation of the chain connecting the first equation (4.6) to the last equation (4.9) was obtained by application of only axioms, definitions, and theorems,

$$13x - 117 = 104 \iff x = 17.$$

But this means that the solution of the last equation, where it is obvious, must be the solution to the first equation where it was non-obvious.

Remark 4.3 (Checking answers)**.**
Great, but how do we know that each link of the chain is sound? Maybe a mistake was made and has gone undetected? If an error had occurred, it would mean that the value $x = 17$ is not necessarily the solution. Now that's a pesky thought, isn't it? But, that is why checking answers is advised. And we know how to check. Just substitute 17 for x in the equation $13x - 117 = 104$, then verify that in fact $LHS = RHS$. We do so.

Of course $x = 17$ could be the solution if we made several errors that canceled each other out.

$$\begin{aligned}
\text{LHS} &= 13(17) - 117 \\
&= 221 - 117 \\
&= 104. \\
\text{RHS} &= 104. \\
\text{LHS} &= \text{RHS}. \qquad \square
\end{aligned}$$

As you progress to higher levels, many of the problems you work will be quite tedious to check by substitution. In those cases, most people check by searching their work for an error or by repeating the solution. That is in mathematics class. In practice, the amount of checking depends on what is at stake. You can bet that an engineering equation to determine the required diameter of steel cable that holds up a suspension bridge is checked *hundreds* or more times, regardless of the inconvenience.

4.3. Linear equations

You will best appreciate what a linear equation is when you learn about other equations that are not linear. In spite of that, we say that a "**linear equation**" is an equation in which the unknown is of degree 1. An exponent of 1 is usually not written, because $x^1 = x$. The degree of the equation is determined by the greatest exponent to which the unknown is raised. Examples of equations of various degrees appear in Table (4.1).

Equation	Degree	Common Name
$3x + 7 = 0$	1	linear
$5x^2 + 3x - 17 = 0$	2	quadratic
$5x^3 + 3x^2 - 8x + 1 = 0$	3	cubic
$x^4 + 4x^3 - 9x^2 + x - 7 = 0$	4	quartic
$7x^5 - 3x^4 + 7x^3 - 2x^2 - 5x + 17 = 0$	5	quintic

TABLE 4.1. Table (Equations of various Degrees)

No matter how complicated a linear equation may be, it can always be rewritten in the form

$$ax + b = 0,$$

where x is the unknown. This is the backgroud for the following definition.

Definition 4.2. An equation is **linear** if it can be written in the form $ax + b = 0$ where x is the unknown and a and b are particular integers with $a \neq 0$. □

4.3.1. Addition and subtraction only

These equations may be solved merely by adding or subtracting from both sides of the equation.

Example 4.2. Solve for y, if $y + 21 = 2$.
Solution.

$$y + 21 = 2 \iff y = -19.$$

Example 4.3. Solve for x, if $3 + x = -20$.
Solution.

$$3 + x = -20 \iff x = -23.$$

Example 4.4. Solve for x, if $5x = 12 + 4x$.
Solution.

$$5x = 12 + 4x \iff x = 12.$$

Example 4.5. Solve for x, if $x + \frac{2}{5} = 3$.
Solution.

$$\begin{aligned} x + \frac{2}{5} = 3 &\iff x = 3 - \frac{2}{5} \\ &\iff x = \frac{13}{5}. \end{aligned}$$

Example 4.6. Solve for a, if $a + \frac{3}{7} = -1$.
Solution.

$$\begin{aligned} a + \frac{3}{7} = -1 &\iff a = -1 - \frac{3}{7} \\ &\iff a = \frac{-10}{7}. \end{aligned}$$

4.3.2. Multiplication and division only

The word "**coefficient**" means the number that multiplies a letter.

Example 4.7. State the coefficient.

(1) In the expression $3x + 5$, 3 is the coefficient of x.

(2) In the expression $9a - 2$, 9 is the coefficient of a.

(3) In the expression $-7 - 105y$, -105 is the coefficient of y.

(4) In the expression $-7 - \frac{2}{3}y$, $\frac{-2}{3}$ is the coefficient of y.

There is more to say about the idea of a coefficient, but we have said enough for now.

Equations that involve only multiplication and division may be solved merely by multiplying both sides of the equation by the multiplicative inverse of the coefficient of the unknown.

Example 4.8. Solve for x, if $5x = 35$.
Solution.

Since the coefficient of x is 5, multiply both sides by $\frac{1}{5}$.

$$\begin{aligned} 5x = 35 &\iff \frac{1}{5}(5x) = \frac{1}{5}(35) \\ &\iff x = \frac{35}{5} \\ &\iff x = 7. \end{aligned}$$

Example 4.9. Solve for x, if $12x = 13$.
Solution.

Multiply both sides by $\frac{1}{12}$.

$$\begin{aligned} 12x = 13 &\iff \frac{1}{\cancel{12}_1}(\cancel{12}^1 x) = \frac{1}{12}(13) \\ &\iff x = \frac{13}{12}. \end{aligned}$$

Example 4.10. Solve for x, if $\frac{x}{3} = \frac{2}{5}$.
Solution.

The coefficient of x is $\frac{1}{3}$, so multiply both sides by 3.

$$\begin{aligned}\frac{x}{3} = \frac{2}{5} &\Longleftrightarrow 3 \cdot \frac{x}{3} = 3 \cdot \frac{2}{5} \\ &\Longleftrightarrow x = \frac{6}{5}.\end{aligned}$$

Example 4.11. Solve for x, if $\frac{3x}{5} = 45$.
Solution.

Multiply both sides by $\frac{5}{3}$.

$$\begin{aligned}\frac{3x}{5} = 45 &\Longleftrightarrow \frac{5}{3}\left(\frac{3x}{5}\right) = \frac{5}{3}(45) \\ &\Longleftrightarrow x = 75.\end{aligned}$$

Example 4.12. Solve for x, if $\frac{x}{2012} = \frac{2}{2013}$.
Solution.

Multiply both sides by 2012.

$$\begin{aligned}\frac{x}{2012} = \frac{2}{2013} &\Longleftrightarrow (\cancel{2012}) \cdot \frac{x}{\cancel{2012}} = (2012) \cdot \frac{2}{2013} \\ &\Longleftrightarrow x = \frac{2 \cdot 2012}{2013} \\ &\Longleftrightarrow x = \frac{4024}{2013}.\end{aligned}$$

Exercise 4.2. Solve.

(1) $5x = 30$.

(2) $3x = 2$.

(3) $\frac{x}{5} = 30$.

(4) $\frac{x}{3} = 45$.

(5) $\dfrac{x}{2} = \dfrac{3}{7}$.

(6) $\left(\dfrac{2}{3}\right) x = \dfrac{1}{5}$.

(7) $\dfrac{5x}{6} = 35$.

(8) $\dfrac{x}{113} = \dfrac{112}{113}$.

(9) $\dfrac{7x}{15} = \dfrac{4}{11}$.

(10) $\left(\dfrac{6}{13}\right) x = \dfrac{3}{26}$.

(11) $\left(\dfrac{23}{1137}\right) x = \dfrac{1}{1137}$.

(12) $\dfrac{10x}{1331} = \dfrac{3}{1332}$.

Example 4.13. Solve for x, if $0.3\,x = 12$.

Solution. Rewriting 0.3 as $\dfrac{3}{10}$ makes the arithmetic easier.

$$\begin{aligned} 0.3\,x = 12 &\Longleftrightarrow \frac{3x}{10} = 12 \\ &\Longleftrightarrow x = \frac{10 \cdot \cancel{12}^{4}}{\cancel{3}} \\ &\Longleftrightarrow x = 40. \end{aligned}$$

Example 4.14. Solve for x, if $0.35\,x = 11$.

Solution 1. If you are not required to answer with a decimal, then rewriting 0.35 as $\frac{35}{100}$ makes the arithmetic easier.

$$\begin{aligned}0.35\,x = 11 &\iff \frac{35x}{100} = 11\\ &\iff x = \frac{11\cdot \cancel{100}^{20}}{\cancel{35}_{7}}\\ &\iff x = \frac{220}{7}.\end{aligned}$$

Solution 2. If you are required to answer with a decimal, then you would continue Solution 1 by writing $\frac{220}{7}$ as a decimal.

$$\begin{aligned}&\iff x = \frac{220}{7}\\ &\iff x = 31.4286.\end{aligned}$$

If a decimal answer is required, it might seem easier to keep 0.35 in decimal form. Then,

$$\begin{aligned}0.35\,x = 11 &\iff \frac{35x}{100} = 11\\ &\iff x = \frac{11}{0.35}\\ &\iff x = 31.4286.\end{aligned}$$

This only seems easier, because to finish, you end up computing $1100 \div 35$. This is no easier than $220 \div 7$.

Usually clearing decimals makes the arithmetic easier. Several examples of clearing decimals follow.

Example 4.15.

(1) $0.3\,x = 0.8 \iff (10)(0.3)x = (10)(0.8) \iff 3x = 8.$
(2) $0.27\,x = 0.5 \iff (100)(0.27)x = (100)(0.5) \iff 27x = 50.$
(3) $36.4\,x = 0.71 \iff (100)(36.4)x = (100)(0.71)$
$\iff 3640x = 71.$

Exercise 4.3. Rewrite without decimals.

(1) $0.25x = 0.72.$
(2) $0.801x = 0.4.$
(3) $1.31x = 0.602.$
(4) $11.91x = 1.3.$
(5) $0.001x = 0.5.$

4.3.3. Several operations

Solving linear equations in one unknown uses nearly all the algebra you have learned so far. Even so, it is a process that will become routine for you. The usual strategy is

(1) simplify both sides of the equation including clearing fractions,
(2) add or subtract to get all terms with the unknown on one side of the equation, numbers on the other side,
(3) if the coefficient of the unknown is not already 1, then multiply the equation by the reciprocal of the coefficient of the unknown.

Example 4.16. Solve $15x + 7 = 9x - 8$.
Solution.

$$\begin{aligned} 15x + 7 = 9x - 8 &\iff 15x = 9x - 15, && \text{subtract } 7 \\ &\iff 6x = -15, && \text{subtract } 9x \\ &\iff x = \frac{-15}{6}, && \text{multiply by } \frac{1}{6} \\ &\iff x = \frac{-5}{3}. \end{aligned}$$

We have been conscientious in stating that each equation in the chain of equations is equivalent ($\iff$) to the preceding one. In practice, the equivalence is not usually stated, it being understood that a chain of equivalent equations is intended. We will follow that practice.

Example 4.17. Solve $3x - 17 = 5x + 3$.
Solution.

$$\begin{aligned} 3x - 17 &= 5x + 3 \\ 3x &= 5x + 20 \\ -2x &= 20 \qquad (4.10) \\ x &= -10. \qquad \square \qquad (4.11) \end{aligned}$$

The reader should think the annotation for each line of the solution. For instance, when passing from equation (4.10) to equation (4.11), think “multiply by negative one-half” or “divide by negative two”.

Supplementary Exercise 8

Solve each equation for the unknown.

1) $-1 = 2n - 3n$

2) $2b + 1 - 2 = 5$

3) $-4 = r - 3 - 1$

4) $6 = x + x$

5) $-2 = -1 - 2n + 3$

6) $2 = b + 3 + 1$

7) $-v - 2v = 3$

8) $2x + 3 + 1 = 0$

9) $5 = -2x - 3x$

10) $2 = 2a - 1 - 3$

11) $8 = -2k - 2k$

12) $-2p + p = -2$

13) $2 = 3 + 2x - 1$

14) $-2 = -3n + n$

15) $-1 = -m + 2 - 2m$

16) $3r + r = 0$

17) $-7 = x - 2 - 2$

18) $3n + 3 + 3n = 3$

19) $-2 = b - 2b$

20) $5 = 1 + v + 1$

21) $-3 = x - 2x$

22) $-3 = 2n - 1 + 2$

23) $a + 3 + 2 = 2$

24) $2k - k = 0$

25) $x + 2 + x = 4$

26) $-x + 3x = -4$

27) $-2n + n = 2$

28) $-m + 2m = -2$

29) $1 = 3p - 2 - 2p$

30) $-3 = -2x + 3x$

Equations often involve distribution.

Example 4.18. Solve $10 + 3(2x + 4) = x - 13$.

Same strategy.
Simplify
Add (Subtract)
Multiply (Divide).

Solution.

$$\begin{aligned} 10 + 3(2x + 4) &= x - 13 \\ 10 + 6x + 8 &= x - 13, && \text{simplify} \\ 6x + 18 &= x - 13, && \text{simplify} \\ 6x &= x - 31, && \text{add or subtract} \\ 5x &= -31, && \text{add or subtract} \\ x &= \frac{-31}{5}, && \text{mutiply or divide.} \quad \square \end{aligned}$$

In all the previous examples we followed the routine introduced at the start of section (4.3.3): (1) simplify, (2) add (subtract), (3) multiply (divide). The next example is more complicated, but not harder, because it just more of what we have already been doing.

Example 4.19. Solve $6 - 3(x - 7) = -(5x + 13) + 20$.
Solution.

$$\begin{aligned} 6 - 3(x - 7) &= -(5x + 13) + 20 \\ 6 - 3x + 21 &= -5x - 13 + 20 \\ -3x + 27 &= -5x + 7 \\ -3x &= -5x - 20 \\ 2x &= -20 \\ x &= -10. \qquad \square \end{aligned}$$

Supplementary Exercise 8A

Solve each equation.

1) $79 = 5(1 - 4n) - 6$

2) $5(-3a - 3) + 3a = -63$

3) $-114 = -6(5k + 4)$

4) $-6(4p - 1) = -90$

5) $-74 = -2(6x + 1)$

6) $-5(1 + 6n) - 6n = 67$

7) $-70 = 5(-5 + 3m)$

8) $-6(5 + 3r) = -102$

9) $4(-5 + 4x) = -100$

10) $2(5n + 3) + 3n = 71$

11) $96 = 6(-2b + 4)$

12) $63 = 3(-4 + 5r)$

13) $-64 = -4(4x + 4)$

14) $-3(1 + 6n) = 69$

15) $6 + 6(3 - 3a) = -84$

16) $5(6 + 4v) = 130$

17) $2(x + 2) + 4(-3x + 6) = 38$

18) $2(-4x - 6) - 2(3 + 6x) = -58$

19) $-6(n - 1) + 4(n - 6) = -16$

20) $37 = -(k - 6) - 6(1 + 6k)$

21) $-4(3p - 6) - 5(1 - p) = 26$

22) $-42 = 4(x + 4) - 6(5x + 1)$

23) $6(1 - 2n) + 5(4 - 3n) = -55$

24) $27 = -(m + 1) - 3(6m - 3)$

25) $-2(3r + 3) + 3(5 - 2r) = -51$

26) $28 = -4(1 + 3x) + 4(x - 2)$

27) $4 = -3(2n + 4) + 4(n + 3)$

28) $-29 = 5(b - 1) + 6(-4 + 5b)$

29) $30 = 2(-6v - 3) - 2(2v + 6)$

30) $-36 = -6(5x + 2) + 4(6x - 6)$

31) $-6(4 - 6n) - 6(n + 4) = 12$

32) $-(6 + 5a) - 5(a + 2) = 4$

Supplementary Exercise 8B

Solve each equation.

1) $68 = -4(1 - 6k)$

2) $-5(2 + 5p) = -110$

3) $-2 + 5(2x - 3) = -67$

4) $-77 = 4(-4n + 2) - 5$

5) $-140 = -5(1 - 5m) + 2m$

6) $6(1 - 3r) = 78$

7) $6(1 + 5x) = -114$

8) $-6(6 - 2n) + 3 = -81$

9) $-3(4 + 5b) = 78$

10) $3(-6v + 1) = 93$

11) $144 = 4(-6x + 5) + 4$

12) $-72 = -4(2 + 4n)$

13) $-79 = 4a + 5(1 + 2a)$

14) $6(4 - 5v) = -66$

15) $-4(-1 + 4x) - 1 = -93$

16) $-3(4x - 5) - 3x = 75$

17) $-6(n + 4) + 4(n + 5) = -2$

18) $-14 = 4(1 - k) + 6(-3 + 5k)$

19) $-2 = -4(3 + 4p) - 2(-5p - 2)$

20) $24 = 4(5 - 4x) - 6(2 - 4x)$

21) $17 = 6(n + 3) - (n + 6)$

22) $36 = 6(5 + 6m) - 6(m + 4)$

23) $6(-3r - 2) - (-4 + r) = 30$

24) $-37 = 3(1 + 5x) + 5(2x + 2)$

25) $-27 = -2(-5 + 2n) - 3(-1 + 2n)$

26) $-4 = -5(-5b + 2) + 2(5b + 3)$

27) $59 = 3(1 + 4v) - 4(1 - 2v)$

28) $-10 = -2(2x - 2) - 3(1 - 5x)$

29) $-(5n + 3) + 6(2n - 3) = -42$

30) $4(4a - 5) + 3(5 - 2a) = 35$

31) $-49 = -(-3b + 4) - 6(b + 5)$

32) $5(2 + 4p) - (p - 4) = 14$

Example 4.20. Solve $\frac{5x}{6} + 14 = 4x + 3$.
Solution.
Take advantage of the fact that an equation may be multiplied by any number. Use this fact to clear fractions. In this example, multiply both sides by 6.

$$\frac{5x}{6} + 14 = 4x + 3.$$

$$\cancel{6}\left(\frac{5x}{\cancel{6}}\right). = 6(4x - 11) \qquad \text{Clear fractions.} \tag{4.12}$$

$$5x = 24x - 66. \qquad \text{Simplify.}$$

$$-19x = -66. \qquad \text{Subtract.}$$

$$x = \frac{66}{19}. \qquad \text{Multiply.}$$

(4.13)

When clearing fractions, a distribution is usually introduced. So, equation (4.12) is typical.

Example 4.21. Solve $\frac{7x}{3} = 11x - 4$.
Solution.

$$\frac{7x}{3} = 11x - 4.$$

$$\cancel{3}\left(\frac{7x}{\cancel{3}}\right) = 3(11x - 4).$$

$$7x = 33x - 12.$$

$$-26x = -12.$$

$$x = \frac{6}{13}.$$

Example 4.22. Solve $-\frac{11}{12} = -\frac{7}{2}n + \frac{1}{3} + \frac{2}{3}n$.
Solution.

$$\begin{aligned}
-\frac{11}{12} &= -\frac{7}{2}n + \frac{1}{3} + \frac{2}{3}n \\
12\left(-\frac{11}{12}\right) &= 12\left(-\frac{7}{2}n + \frac{1}{3} + \frac{2}{3}n\right) \\
-11 &= \cancel{12}^6\left(-\frac{7}{\cancel{2}_1}n\right) + \cancel{12}^4\left(\frac{1}{\cancel{3}_1}\right) + \cancel{12}^4\left(\frac{2}{\cancel{3}_1}n\right) \tag{4.14} \\
-11 &= -42n + 4 + 8n \\
-15 &= -34n \\
\frac{-15}{-34} &= n \\
\therefore n &= \frac{15}{34}.
\end{aligned}$$

Example 4.23. Solve $3 - \frac{2}{3}x + \frac{7}{12} = -\frac{3}{4}x + \frac{2}{3} + \frac{5}{6}x$.
Solution.

$$\begin{aligned}
3 - \frac{2}{3}x + \frac{7}{12} &= -\frac{3}{4}x + \frac{2}{3} + \frac{5}{6}x \\
12\left(3 - \frac{2}{3}x + \frac{7}{12}\right) &= 12\left(-\frac{3}{4}x + \frac{2}{3} + \frac{5}{6}x\right) \tag{4.15} \\
36 - 8x + 7 &= -9x + 8 + 10x \\
-8x + 43 &= x + 8 \\
-9x &= -35 \\
x &= \frac{35}{9}.
\end{aligned}$$

In example (4.23) we did the distribution and arithmetic indicated in equation (4.15) mentally. Less clutter makes the work easier to follow than at equation (4.14) where less was done mentally. You may find that you make *fewer* mistakes by doing more algebra and arithmetic mentally.

Well, how much work *should* I show? −Enough for a good student at your level to follow it.

Supplementary Exercise 9

Solve each equation.

1) $-\frac{3}{2} = r - \frac{11}{3} + \frac{5}{3}$

2) $-\frac{1}{3}x - \frac{4}{3}x = -\frac{25}{9}$

3) $-\frac{23}{6} = \frac{5}{3}b + \frac{3}{2} - 2$

4) $\frac{61}{9} = -\frac{7}{2}n + \frac{3}{2} + \frac{1}{3}n$

5) $\frac{1}{3}v - \frac{4}{3}v = \frac{7}{2}$

6) $\frac{1}{3} = \frac{4}{3}x + 1 - 2$

7) $\frac{3}{2}x + \frac{7}{2}x = \frac{15}{2}$

8) $0 = \frac{3}{2}a + 1 - 1$

9) $\frac{23}{3} = -\frac{7}{3}k - \frac{3}{2}k$

10) $\frac{17}{12} = \frac{3}{2}p + \frac{4}{3}p$

11) $\frac{11}{3}x + 1 - \frac{1}{2} = \frac{29}{3}$

12) $-\frac{2}{3}n + n = \frac{1}{2}$

13) $-\frac{41}{12} + x - \frac{11}{3}x = -\frac{7}{2}x - \frac{4}{3}$

14) $-\frac{16}{3} - \frac{1}{2}n = -n - \frac{7}{2}n$

15) $\frac{7}{3} - m = -\frac{4}{3}m + \frac{3}{2}$

16) $-5 - \frac{1}{2}r = \frac{3}{2}r - \frac{1}{3}$

17) $-\frac{3}{2}x + 1 = \frac{5}{3} - x$

18) $\frac{5}{9} + n + \frac{3}{2} + \frac{7}{3}n = n + \frac{1}{2}$

19) $-\frac{5}{3}b - 3b = -\frac{32}{9} + \frac{2}{3}b$

20) $v + \frac{1}{2} = \frac{1}{4} + 1\frac{1}{2}v$

21) $-\frac{1}{2}x - x - \frac{3}{2} = -\frac{5}{2}x + 1 - 2$

22) $-\frac{8}{3}n + 1 - 3 = -\frac{5}{6} - \frac{3}{2}n$

23) $\frac{3}{2}b + 1 = 2b + \frac{1}{2}$

24) $\frac{2}{3}b + \frac{3}{2} = \frac{19}{6} - b$

Supplementary Exercise 9A

Solve each equation.

1) $0 = -2n - \frac{5}{3}n$

2) $r - \frac{7}{2}r = -\frac{35}{6}$

3) $2x + 2x = -4$

4) $\frac{3}{2}m + \frac{5}{3} - 1\frac{1}{2} = -\frac{13}{3}$

5) $\frac{1}{3}n - \frac{4}{3} - \frac{1}{2}n = -\frac{23}{18}$

6) $3 = 3b - \frac{3}{2}b$

7) $-\frac{41}{9} = -\frac{1}{3}r + 1 + 2r$

8) $-\frac{3}{2}x + \frac{5}{2}x = \frac{8}{3}$

9) $-\frac{7}{6} = \frac{1}{3}n - \frac{1}{3} + \frac{4}{3}n$

10) $\frac{1}{2}a + \frac{5}{2}a = -4$

11) $-v - \frac{8}{3}v = \frac{22}{3}$

12) $x - \frac{1}{2} + \frac{1}{2}x = -3$

13) $\frac{13}{9} - 2x = -\frac{5}{3}x + 1$

14) $-\frac{7}{3}n + 1 + 2n = \frac{21}{4} + 2\frac{1}{2}n$

15) $-\frac{35}{18} + k - \frac{2}{3}k = \frac{1}{2}k - 2$

16) $\frac{2}{3}p - \frac{8}{9} = p + \frac{1}{3}p$

17) $5 - \frac{4}{3}x = -x - \frac{10}{3}x$

18) $\frac{1}{2}n + 1 = -\frac{1}{6} + n$

19) $\frac{5}{18} + \frac{4}{3}r = r + \frac{1}{2}$

20) $\frac{5}{3}m - \frac{5}{2} - 2m = \frac{7}{9} + \frac{4}{3}m + \frac{3}{2} - 2$

21) $-\frac{4}{3}x + 2x = \frac{25}{9} + \frac{4}{3}x - \frac{11}{3} - \frac{4}{3}x$

22) $\frac{5}{3}b + \frac{3}{2} = -2b - \frac{13}{6}$

23) $\frac{2}{3}n + \frac{4}{3} = \frac{56}{9} + 2n$

24) $-\frac{3}{2}v + \frac{31}{12} = \frac{1}{3}v + \frac{3}{2} - \frac{5}{3}$

4.3.4. Round up the usual suspects

There are a couple of troublemakers that sometimes show up when solving equations. Let's get acquainted with them before we continue solving linear equations.

The following two expressions tempt students into making invalid computations:

$$a\left(\frac{b}{a}+d\right), \tag{4.16}$$

$$\frac{ab+cd}{a}. \tag{4.17}$$

The invalid computations are these.

$$a\left(\frac{b}{a}+d\right)=\cancel{a}\left(\frac{b}{\cancel{a}}+d\right). \tag{4.18}$$

$$\frac{ab+cd}{a}=\frac{\cancel{a}b+cd}{\cancel{a}}. \tag{4.19}$$

To show that the computation (4.18) is invalid, it is enough to find one example where the computation results in a false statement.

$$\begin{aligned}\cancel{3}^1\left(\frac{7}{\cancel{3}_1}+5\right)&=7+5\\&=12.\quad \textbf{(FALSE)}\end{aligned}$$

Because,

$$\begin{aligned}3\left(\frac{7}{3}+5\right)&=\cancel{3}\left(\frac{7+15}{\cancel{3}}\right)\\&=22.\end{aligned}$$

To show that the computation (4.19) is invalid, consider

$$\begin{aligned}\frac{(2)(3)+(5)(6)}{2}&=\frac{(\cancel{2}^1)(3)+(5)(6)}{\cancel{2}_1}\\&=3+30\\&=33.\quad \textbf{(FALSE)}\end{aligned}$$

Because,

$$\begin{aligned}\frac{(2)(3)+(5)(6)}{2} &= \frac{6+30}{2}\\ &= 15.\end{aligned}$$

Exercise 4.4.

(1) Show that the following computation is invalid.

$$\frac{1}{a}(a+b) = \frac{1}{\cancel{a}}(\cancel{a}+b).$$

(2) Show that the following computation is valid. [Hint: use distribution in numerator.]

$$\frac{1}{a}(ab+ad) = \frac{1}{\cancel{a}}(\cancel{a}b+\cancel{a}d).$$

Simplify each of the following. If an expression is already simplified, say so.

(3) $\frac{1}{2}(4a+1)$

(4) $\frac{1}{3}(3a+3)$

(5) $10 - \frac{2}{5}(3a+5)$

(6) $11 - \frac{7}{3}(6a+3)$

(7) $\frac{7+3x}{7}$

(8) $\frac{6x+5}{3}$

(9) $\frac{4+2a}{2}$

(10) $\frac{4+4a}{2}$

(11) $\frac{2+4a}{2}$

(12) $\frac{11+11b}{11}$

(13) $\frac{2+2b}{2}$

(14) $\frac{2+32b}{3}$

(15) $\frac{27+9b}{3}$

(16) $\frac{81-9b}{3}$

4.3.5. Linear equations with subtle features

Let us solve the following two linear equations side-by-side so that we might compare the solutions:

$$\frac{2}{3}x+5=x-2 \tag{4.20}$$

$$\frac{2}{3}(x+5)=x-2. \tag{4.21}$$

Solutions.

$$\begin{array}{lcc}
 & \frac{2}{3}x+5=x-2 & \frac{2}{3}(x+5)=x-2 \\
(4.22) & 3\left(\frac{2}{3}x+5\right)=3(x-2) & \cancel{3}\left(\frac{2}{\cancel{3}}\right)(x+5)=3(x-2) \\
 & 2x+15=3x-6 & 2x+10=3x-6 \\
 & -x=-21 & -x=-16 \\
 & x=21. & x=16.
\end{array}$$

Notice the difference at equations (4.22). In the equation in the right hand column,

$$3\left(\frac{2}{3}\right)$$

is a product of two fractions and cancellation is possible.

But, in the equation in the left hand column

$$3\left(\frac{2}{3}x+5\right)$$

is a product of a fraction and the sum $\frac{2}{3}x+5$. As such, cancellation is not possible.

Now, $3\left(\frac{2}{3}x+5\right)$ may be rewritten

$$3\left(\frac{2x+15}{3}\right).$$

Although the fraction in the parentheses has a complicated numerator, the expression *is* never-the-less a product of fractions, so the cancellation

$$\cancel{3}\left(\frac{2x+15}{\cancel{3}}\right)$$

is valid.

Supplementary Exercise 10

Solve each equation.

1) $-\frac{637}{18} = 2k - \frac{7}{2}\left(-\frac{10}{3}k + 1\right)$

2) $-\frac{310}{9} = -\frac{10}{3}\left(-3x + \frac{4}{3}\right)$

3) $-\frac{11}{3}\left(2a - \frac{8}{3}\right) = \frac{286}{9}$

4) $-\frac{7}{2}\left(\frac{8}{3}n - 1\right) = \frac{217}{6}$

5) $-\frac{127}{4} = -3\left(-\frac{7}{2}p - \frac{5}{3}\right)$

6) $\frac{4}{3} - \frac{7}{2}\left(-\frac{10}{3}x + \frac{8}{3}\right) = -\frac{94}{3}$

7) $\frac{1837}{54} = -\frac{11}{3}\left(\frac{8}{3}x + \frac{1}{2}\right)$

8) $-\frac{7}{2}\left(-\frac{10}{3}m + 1\right) + \frac{1}{2} = -\frac{263}{6}$

9) $-\frac{1}{2}\left(x - \frac{5}{3}\right) = -\frac{1}{3}\left(\frac{4}{3}x + \frac{1}{2}\right)$

10) $\frac{7}{3}\left(n + \frac{3}{2}\right) = -\frac{10}{3} + \frac{4}{3}\left(\frac{2}{3}n + \frac{5}{3}\right)$

11) $\frac{4}{3} + \frac{1}{2}\left(\frac{3}{2}x + 1\right) = -\frac{10}{3}\left(x - \frac{2}{3}\right)$

12) $2b + \frac{1}{3}\left(\frac{1}{2}b + \frac{5}{2}\right) = 2\left(\frac{4}{3}b + 1\right) - \frac{1}{2}$

13) $-\frac{5}{2}\left(-\frac{3}{2}n + \frac{1}{2}\right) = -\frac{7}{2}\left(n + \frac{4}{3}\right) + \frac{8}{3}$

14) $-\frac{3}{2}\left(\frac{1}{2}v + 1\right) = -\frac{1}{2}\left(v + \frac{7}{3}\right)$

15) $2b + \frac{1}{2} - 2b = \frac{3}{2}\left(\frac{1}{2}b - \frac{5}{3}\right) - \frac{7}{2}\left(\frac{3}{2}b - \frac{5}{2}\right)$

16) $-\frac{4}{3}k - \frac{5}{2}\left(-\frac{2}{3}k - 2\right) = \frac{4}{3}\left(k + \frac{3}{2}\right)$

Supplementary Exercise 10A

---.

Solve each equation.

1) $-\frac{205}{6} = 3\left(\frac{5}{2}k - 1\right) + k$

2) $\frac{5}{3}x + \frac{7}{2}\left(-\frac{10}{3}x + 1\right) = \frac{67}{2}$

3) $3\left(-\frac{10}{3}x - \frac{7}{2}\right) = -\frac{71}{2}$

4) $-\frac{7}{2}\left(-\frac{7}{2}p + 1\right) - \frac{4}{3} = -\frac{1145}{24}$

5) $-\frac{11}{3}\left(-\frac{10}{3}m + 1\right) - \frac{1}{2}m = -\frac{233}{6}$

6) $-\frac{286}{9} = -\frac{11}{3}\left(-\frac{5}{2}x - \frac{1}{2}\right)$

7) $\frac{141}{4} = -\frac{5}{2}a + 2\left(-\frac{7}{2}a + 1\right)$

8) $-\frac{7}{2}\left(\frac{8}{3}n + \frac{3}{2}\right) = -\frac{133}{4}$

9) $\frac{4}{3}\left(-\frac{8}{3}b + \frac{5}{2}\right) = -\frac{5}{3}\left(b + \frac{1}{2}\right) + \frac{3}{2}$

10) $2\left(-\frac{10}{3}r + \frac{1}{3}\right) = -r - \left(r - \frac{10}{3}\right)$

11) $-\frac{10}{3}b - b = \frac{8}{3}\left(-\frac{1}{2}b + \frac{1}{2}\right) + 2\left(\frac{4}{3}b + 2\right)$

12) $-\left(\frac{8}{3}x + 1\right) + \frac{5}{2}x = 2\left(\frac{1}{2}x + \frac{7}{3}\right)$

13) $\frac{1}{2}n - n = \frac{1}{2}\left(\frac{5}{2}n + 2\right) - \frac{3}{2}\left(\frac{3}{2}n - \frac{10}{3}\right)$

14) $\frac{7}{3}\left(-\frac{5}{3}b - \frac{1}{2}\right) = -\frac{11}{3}\left(\frac{4}{3}b + 1\right)$

15) $-\left(-\frac{5}{2}x + \frac{3}{2}\right) = -\frac{11}{3}\left(x + \frac{3}{2}\right)$

16) $\frac{5}{2}x - \left(-\frac{3}{2}x + 1\right) = -2x - \frac{11}{3}\left(x + \frac{2}{3}\right)$

4.4. Rational expressions revisited

4.4.1. Factoring out $-1, -2, \cdots$

Students are sometimes surprised at what can be accomplished by recognizing -1 as a factor. Initially one might think that the expression

$$\frac{a-1}{1-a}$$

cannot be simplified. But, it does simplify.

Example 4.24. Simplify $\frac{a-1}{1-a}$, for $a \neq 1$.
Solution.

$$\begin{aligned}\frac{a-1}{1-a} &= \frac{-1(1-a)}{1-a}\\ &= -1\left(\frac{1-a}{1-a}\right)\\ &= -1\left(\frac{\cancel{1-a}^{1}}{\cancel{1-a}_{1}}\right)\\ &= (-1)(1)\\ &= -1. \qquad \square\end{aligned}$$

Example 4.25. Simplify $\frac{2b-2}{1-b}, b \neq 1$.
Solution.

$$\begin{aligned}\frac{2b-2}{1-b} &= \frac{-2(1-b)}{b-1}\\ &= -2\left(\frac{1-b}{1-b}\right)\\ &= -2\left(\frac{\cancel{1-b}^{1}}{\cancel{1-b}_{1}}\right)\\ &= (-2)(1)\\ &= -2. \qquad \square\end{aligned}$$

Exercise 4.5. Simplify.

(1) $\dfrac{x-1}{1-x}, x \neq 1$

(2) $\dfrac{1-b}{b-1}, b \neq 1$

(3) $\dfrac{5-5a}{5a-5}, a \neq 1$

(4) $\dfrac{8-8x}{2x-2}, x \neq 1$

(5) $\dfrac{7-7y}{3-3y}, y \neq 1$

(6) $-\dfrac{12-12x}{3-3x}, x \neq 1$

(7) $21-\dfrac{24-24x}{2-2x}, x \neq 1$

4.5. A closer look at linear equations

Theorem 4.1. *Every linear equation in one unknown has a solution in the rational numbers and this solution is unique.* □

Theorem (4.1) makes two statements in a compact sentence. The two statements are:

(1) Every linear equation in one unknown has a solution in the rational numbers.
(2) The solution mentioned in (1) is unique. There are no other solutions.

Exercise 4.6.

(1) The equation $\frac{3}{7}x+2=0$ as written does not appear to be a linear equation. Why? [Hint: you need definition (4.2)].

(2) Rewrite the equation $\frac{2}{5}x+\frac{1}{2}=3x-6$ in the form required by definition (4.2) and conclude that the equation is linear.
‘

(3) Solve the equation $3(2x-7)+5=2(3x-11)+12$.
   (a) What do you think the strange result means?
   (b) Does the result obtained contradict theorem (4.1)?

(4) Solve the equation $3(2x - 7) - 9 = 2(3x - 11) - 8$.
   (a) What do you think this strange result means?
   (b) Does the result obtained contradict theorem (4.1)?

(5) Make up several equations in which the unknown, call it x, occurs with an exponent of 1 and show that each can be rewritten in the form $ax + b = 0$.

(6) * Theorem (4.1) says that every linear equation in one unknown has a solution in the rational numbers. Prove this.

(7) ** Theorem (4.1) also says that the solution of every linear equation in one unknown is unique. Prove this.

Chapter 5

Applications

Chapter 6

Linear Function

Appendices

Appendix A

Answers to Exercises

Exercise 1.1

(1) $0, 1, 2, \cdots, 1000$.
(2) $7, 8, 9, \cdots, 93$.
(3) $5, 6, 7, \cdots$.
(4) Yes. Exhibit a 1-1 correspondence.
(5) Yes. Exhibit a 1-1 correspondence.
(6) Yes. Exhibit a 1-1 correspondence.

Exercise 1.2

(1) $a = 1 \times a$.

(2) $\dfrac{a}{a} = 1$.

(3) $0 + b = 0$.

(4) $\dfrac{a}{c} + \dfrac{b}{c} = \dfrac{a+b}{c}$

(5) $a \times b = b \times a$.

(6) $2 \times y$, where $y = 1, 2, 3, \cdots$.

Exercise 1.3

(1) $101 + a$.
(2) $a + 3 = b + 5$.
(3) Symmetric.
(4) The Principle of Substitution
(5) **Proof.** Let a, b and c be any numbers. Suppose that $a = b$.

$a \times c = a \times c$	equality is reflexive
$a \times c = b \times c$	substitution, we supposed $a = b$.

Therefore, if $a = b$ then $a \times c = b \times c$. □

Exercise 1.4

(1) 11
(2) 36
(3) 15
(4) 17

Exercise 1.5

(1) True.
(2) False.
(3) False.
(4) True.
(5) False.
(6) True.
(7) True.
(8) False.
(9) True.
(10) True.
(11) $\in$.
(12) $\in$.
(13) $\notin$.
(14) True.
(15) The set of even numbers.
(16) The set of multiples of 5 that are no less than 10.

Exercise 1.6

(1)

$$\begin{aligned} (2+7)+13 &= 2+(7+13) && \text{associative} \\ &= 2+(13+7) && \text{commutative} \\ &= (2+13)+7 && \text{commutative} \end{aligned}$$

(2)

$$\begin{aligned} (9\times 11)\times 6 &= 9\times(11\times 6) && \text{associative} \\ &= 9\times(6\times 11) && \text{commutative} \\ &= (9\times 6)\times 11 && \text{associative} \end{aligned}$$

(3)

$$\begin{aligned} 1+4+(7+12) &= 1+(7+12)+4 && \text{commutative} \\ &= (1+7)+12+4 && \text{associative} \\ &= 12+(1+7)+4 && \text{commutative} \end{aligned}$$

(4)

$$\begin{aligned} 3+(7+12)+(4\times 7) &= (3+7)+12+(4\times 7) && \text{associative} \\ &= (7+3)+12+(4\times 7) && \text{commutative} \\ &= (7+3)+(4\times 7)+12 && \text{commutative} \\ &= (4\times 7)+(7+3)+12 && \text{commutative} \\ &= (4\times 7)+12+(7+3) && \text{commutative} \end{aligned}$$

(5) $2, 1001, 6$
(6) Yes. Because $\frac{12}{3} = 4$.

Exercise 2.1

(1) $7+(-7)=0$
(2) $4+(-4)=0$
(3) $2090+(-2090)=0$
(4) $-33+33=0$

(5) $-51 + 51 = 0$
(6) $-8 + 8 = 0$
(7) $x + (-x) = 0$
(8) $-x + (-(-x)) = 0$

Exercise 2.2

(1) -3
(2) -6
(3) -1
(4) -2
(5) -7
(6) -16
(7) -6
(8) -9
(9) -5
(10) -16

Exercise 2.3

(1) Let a and b be positive numbers. Since the integers are closed under subtraction, there is a number, call it c, such that $-a - b = c$. This means that $-a = c + b$.

$$\begin{aligned} -a + (-b) &= (c + b) + (-b) && \text{substitution, } -a = c + b \\ &= c + (b + (-b)) && \text{associative} \\ &= c + 0 && \text{inverse elements} \\ &= c && \text{identity element} \\ &= -a - b && \text{substitution, } c = a - b \end{aligned}$$

Therefore.

$$-a + (-b) = -a - b. \quad \square \tag{A.1}$$

(2) This is the subtraction of the integer $(-b)$. But, according to equation (A.1), that is accomplished by adding the additive inverse of $-b$. Since the additive inverse of $-b$ is b, $-a-(-b) = -a + b$.

Exercise 2.4

Simplify.

(1) 7
(2) 113
(3) 9
(4) $15+3=18$
(5) No. We no longer need the separate cases $a+(-b)$ and $a+b$. When b is positive, case $a+b$. When b is negative, case $a+(-b)$.
(6) 21
(7) 133

Exercise 2.5

(1) $7b$
(2) $9x$
(3) ad
(4) $2ab$
(5) $5(x+2)$
(6) $9a((3y+4)+2)$
(7) $5(a+b)$
(8) $7(a+1)$
(9) $2(3\times a+4)$
(10) $6(x+y)$
(11) $9(23-11)$
(12) $3\times(2\times a+5)$
(13) $4\times 3\times 9$
(14) $4\times a\times b\times c$
(15) $x\times y\times(2\times y+5\times x)$
(16) $3\times x\times(5\times x+3\times y+7)$

Exercise 2.6

(1) $ab+ac$
(2) $3a+6$
(3) $4x+4y$
(4) $2y+10$
(5) $7a+21b$
(6) $12+6x$
(7) $10a+15b$

(8) $11x + 22$

(9) **Proof.**

$$\begin{aligned} a(b+c+d) &= ab + a(c+d) \\ &= ab + ac + ad \end{aligned}$$

$\therefore a(b+c+d) = ab + ac + ad$ □

(10) **Proof.**

$$a(b+c+d+e) = a(b+c+d) + ae$$

Using the result of problem 9

$$= ab + ac + ad + ae$$

$\therefore a(b+c+d+e) = ab + ac + ad + ae$ □

Exercise 2.7

(1) **Proof.** Let a be any number.

$$\begin{aligned} 1 &= 1 && \text{equality is reflexive} \\ 1 + 0 &= 1 && \text{identity element} \\ a(1+0) &= a \cdot 1 && \text{theorem 1.2} \\ a \cdot 1 + a \cdot 0 &= a \cdot 1 && \text{distribution} \\ a + a \cdot 0 &= a && \text{identity element} \\ (-a) + (a + a \cdot 0) &= (-a) + a && \text{theorem 1.2} \\ (-a + a) + a \cdot 0 &= (-a) + a && \text{associative} \\ 0 + a \cdot 0 &= 0 && \text{inverse elements} \\ a \cdot 0 &= 0 && \text{identity element} \end{aligned}$$

$\therefore a \cdot 0 = 0$

□

(2) **Proof.** We prove the first part of case 3; that is, $a \cdot (-b) = -(ab)$.

$$\begin{aligned} b + (-b) &= 0 && \text{inverse elements} \\ a(b + (-b)) &= a \cdot 0 && \text{theorem 1.2} \\ a(b + (-b)) &= 0 && \text{theorem 2.2} \\ ab + a \cdot (-b) &= 0 && \text{Distribution} \\ -(ab) + ab + a \cdot (-b) &= -(ab) + 0 && \text{theorem 1.2} \\ -(ab) + ab + a \cdot (-b) &= -(ab) && \text{identity element} \\ 0 + a \cdot (-b) &= -(ab) && \text{inverse element} \\ a \cdot (-b) &= -(ab) && \text{identity element} \end{aligned}$$

$\therefore a \cdot (-b) = -(ab)$

$\square$

(3) **Proof.** Second part, $(-a) \cdot (b) = -(ab)$

$$\begin{aligned} a + (-a) &= 0 \\ b(a + (-a)) &= 0 \\ ba + b \cdot (-a) &= 0 && \text{Distribution} \\ -(ba) + ba + b \cdot (-a) &= -(ba) \\ b \cdot (-a) &= -(ba) \\ (-a) \cdot (b) &= -(ab) \end{aligned}$$

$\square$

Exercise 2.8

Simplify the following expressions. If an expression is already simplified, say so.

(1) $2a + 5$
(2) $15a + 7b$
(3) $8x + 2y + 2$
(4) $7a - b - 4c + 9$
(5) $5a + 7b + 3c - 13$
(6) $5x + 9y + 8z - 9$
(7) $-10x + 10y + 10z - 8$
(8) $2a - 8c - 8$
(9) Simplified.
(10) Simplified.

Exercise 2.9

1. $7a^5$
2. $16x^3y$
3. Already simplified.
4. $6a^3b^3c^2$
5. Already simplified.

Exercise 2.10

1. $10a-7$
2. $8a+5b+9$
3. $18a+5b+1$
4. $7x+1$
5. $-2x+7$
6. $32+2y$
7. $-16x-1$
8. $16a+9b-15$
9. $8a^2+8a+51$
10. $2a^2+4ab-a+2b+b^2$
11. $4a^3b+9a^2b+7b+3a^3-7$

Exercise 3.1

Find each quotient by imitating example (3.2).

(1)

$$27 \div 3 = x$$

means that

$$3 \cdot x = 27.$$

Multiply both sides $\frac{1}{3}$,

$$\frac{1}{3} \cdot 3 \cdot x = \frac{1}{3} \cdot 27.$$

Since 3 and $\frac{1}{3}$ are multiplicative inverses, and 1 is the identity element for multiplication,

$$\begin{aligned} 1 \cdot x &= \frac{1}{3} \cdot 27 \\ x &= \frac{1}{3} \cdot 27 \\ x &= 9 \\ \therefore \quad 27 \div 3 &= 9. \quad \square \end{aligned}$$

(2)

$$31 \div 6 = x$$

means that

$$\begin{aligned} 6 \cdot x &= 31 \\ \frac{1}{6} \cdot 6 \cdot x &= \frac{1}{6} \cdot 31 \\ 1 \cdot x &= \frac{1}{6} \cdot 31 \\ x &= \frac{31}{6}. \end{aligned}$$

(3) $9 \div 3 = 9 \cdot \dfrac{1}{3} = 3.$

(4) $17 \div 8 = 17 \cdot \dfrac{1}{8} = \dfrac{17}{8}.$

(5) $52 \div 13 = 52 \cdot \dfrac{1}{13} = \dfrac{52}{13} = 4.$

(6) $5 \div 12 = 5 \cdot \dfrac{1}{12} = \dfrac{5}{12}.$

Exercise 3.2

(1) $\dfrac{-15}{28}$

(2) $\frac{36}{55}$

(3) $\frac{7}{22}$

(4) $\frac{-2}{3}$

(5) 4

(6) $\frac{28}{3}$

(7) $\frac{-13}{3}$

(8) $\frac{-3}{7}$

(9) $\frac{3}{8}$

(10) $\frac{2}{15}$

(11) $\frac{3}{11}$

(12) $\frac{-4}{15}$

(13) $\frac{3}{16}$

(14) $\frac{1}{18}$

(15) $\frac{-2}{5}$

(16) $\frac{1}{20}$

(17) $\frac{ab}{6}$

(18) $\frac{9a}{5}$

(19) $\frac{-12x}{5y}$

(20) $4x$

(21) $\frac{-7}{2}$

(22) $\frac{-a}{12b}$

(23) $\frac{2}{5}$

(24) $\frac{-a}{bc}$

(25) $\frac{-7a}{bc}$

Exercise 3.3

(1) **Proof.** Let a be any number other than 0.

$$\begin{aligned} \text{LHS} &= \frac{a}{a} \\ &= a \cdot \frac{1}{a}, && \text{definition division} \\ &= 1 && \text{inverse element} \\ &= \text{RHS} \end{aligned}$$

$$\therefore \frac{a}{a} = 1, a \neq 0$$

□

(2) **Proof.** Let a be any number.

$$\begin{aligned} \text{LHS} &= \frac{a}{1} \\ &= a \cdot \frac{1}{1}, && \text{definition division} \\ &= a \cdot 1, && \text{Theorem (3.1) with } a = 1 \\ &= a, && \text{idemtity element} \\ &= \text{RHS} \end{aligned}$$

$$\therefore \frac{a}{1} = a$$

□

(3) **Proof.**

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= \frac{a}{b} \cdot 1 + \frac{c}{d} \cdot 1 && \text{identity element} \\ &= \frac{a}{b} \cdot \frac{d}{d} + \frac{c}{d} \cdot \frac{b}{b} && \text{Theorem (3.1)} \\ &= \frac{ad}{bd} + \frac{cb}{bd} && \text{Theorem (3.4)} \\ &= ad \cdot \frac{1}{bd} + cb \cdot \frac{1}{bd} && \text{Theorem (3.4)} \\ &= \frac{1}{bd}(ad + bc) && \text{distribution} \\ &= \frac{ad + bc}{bd} && \text{definition division} \end{aligned}$$

□

(4) **Proof.**

$$\begin{aligned} \frac{a}{b}+\frac{c}{b} &= a\cdot\frac{1}{b}+c\cdot\frac{1}{b} && \text{definition division} \\ \frac{a}{b}+\frac{c}{b} &= \frac{1}{b}(a+c) && \text{distribution} \\ &= \frac{a+c}{b} && \text{definition division} \end{aligned}$$

□

Exercise 3.4

(1) $\frac{4}{3}$

(2) $\frac{13}{33}$

(3) 1

(4) $\frac{35x}{44}$

(5) $\frac{3+24a}{26}$

(6) $\frac{7a+7}{4}$

(7) $\frac{23a-5}{12}$

(8) $\frac{36a-2}{15}$

(9) $\frac{19x-19}{10}$

(10) $\frac{21x-16}{22}$

(11) $\frac{16-4a}{15}$

(12) $\dfrac{2a+3}{10}$

(13) $\dfrac{a-8}{4}$

(14) $\dfrac{67-12y}{6}$

(15) $\dfrac{5x-6}{10}$

(16) $\dfrac{63-6a}{20}$

(17) $\dfrac{-11a+15b}{21}$

(18) $\dfrac{-x-1}{6}$

(19) $\dfrac{-a-9}{3a}$

(20) $\dfrac{4a+6b-3ab}{6ab}$

(21) $\dfrac{a}{10}$

(22) $\dfrac{35-8a}{14}$

(23) $\dfrac{15-2x}{5x}$

(24) $\dfrac{7x-3y}{2}$

(25) $\dfrac{21a+71}{20}$

Exercise 4.1

(1) $5(3) + 8 = 23, \quad \therefore x = 3$ is a solution.

(2) $7(5) - 9 = 26, \quad \therefore x = 5$ is a solution.

(3) $5(6) - 4 = 26,\ 26 \neq 20, \quad \therefore a = 6$ is not a solution.

(4) $\dfrac{2\left(\dfrac{15}{4}\right)}{5} + \dfrac{3}{2} = 3, \quad \therefore x = \dfrac{15}{4}$ is a solution.

Exercise 4.2

(1) $x = 6$

(2) $x = \dfrac{2}{3}$

(3) $x = 150$

(4) $x = 135$

(5) $x = \dfrac{6}{7}$

(6) $x = \dfrac{3}{10}$

(7) $x = 42$

(8) $x = 112$

(9) $x = \dfrac{60}{77}$

(10) $x = \dfrac{1}{4}$

(11) $x = \dfrac{1}{23}$

(12) $x = \dfrac{1331}{4440}$

Exercise 4.3

(1) $25x = 72$
(2) $801x = 400$
(3) $131x = 60.2$
(4) $1191x = 130$
(5) $x = 500$

Exercise 4.4

(1) Let $a = 2, b = 3$, then

$$\begin{aligned}\frac{1}{a}(a+b) &= \frac{1}{2}(2+3)\\ &= \frac{5}{2}\end{aligned}$$

but,

$$\begin{aligned}\frac{1}{\cancel{a}}(\cancel{a}+b) &= \frac{1}{\cancel{2}}(\cancel{2}+3)\\ &= 1(1+3)\\ &= 4\end{aligned}$$

(2) Let a, b be any numbers provided that $a \neq 0$.

$$\begin{aligned}\text{LHS} &= \frac{1}{a}(ab+ad)\\ &= \frac{1}{a}(a)(b+d)\\ &= b+d.\end{aligned}$$

$$\text{RHS} = \frac{1}{\cancel{a}}(\cancel{a}b+\cancel{a}d)$$

$$\text{RHS} = a+b$$

Since, LHS = RHS, the equation is true.

(3) $\dfrac{4a+1}{2}$

(4) $a+1$

(5) $\dfrac{40-6a}{5}$

(6) $4-14a$

(7) Simplified.

(8) Simplified.

(9) $2+a$

(10) $2+2a$

(11) $1+2a$

(12) $1+b$

(13) $1+b$

(14) Simplified.

(15) $9+3b$

(16) $27-3b$

Exercise 4.5

Exercise A.1. Simplify.

(1) -1
(2) -1
(3) -1
(4) -4
(5) Simplified.
(6) -4

(7) 9

Exercise 4.6

(1) The equation $\frac{3}{7}x + 2 = 0$ as written does not appear to be a linear equation, because the coefficient of x, $\frac{3}{7}$ is not an integer.

(2)

$$\begin{aligned} \frac{2}{5}x + \frac{1}{2} &= 3x - 6 \\ 10\left(\frac{2}{5}x + \frac{1}{2}\right) &= 10(3x - 6) \\ 4x + 5 &= 30x - 60 \\ -24x + 65 &= 0 \end{aligned}$$

which satisfies the requirements of definition (4.2) and is therefore a linear equation.

(3)

$$\begin{aligned} 3(2x - 7) + 5 = 2(3x - 11) + 12 &\iff 6x - 16 = 6x - 10 \\ &\iff 0 = 6 \end{aligned}$$

Apparently the equation $3(2x - 7) + 5 = 2(3x - 11) + 12$ is equivalent to the statement “$0 = 6$”. Since the statement “$0 = 6$” is always false, the equation $3(2x - 7) + 5 = 2(3x - 11) + 12$ is likewise always false. We conclude that there is no value of x that makes $3(2x - 7) + 5 = 2(3x - 11) + 12$ true. We say that $3(2x - 7) + 5 = 2(3x - 11) + 12$ has no solution.

Theorem (4.1) says that every linear equation has a solution. This result appears to contradict that theorem.

Fear not. The principles of mathematics have note gone “Bonkers”, because the equation “$3(2x - 7) + 5 = 2(3x - 11) + 12$” is *not* a linear equation. It is equivalent to “$0 = 6$” which is not of the form $ax + b = 0$, where $a \neq 0$.

(4)

$$\begin{aligned}3(2x-7)-9=2(3x-11)-8 &\Longleftrightarrow 6x-30=6x-30\\ &\Longleftrightarrow 0=0\end{aligned}$$

Apparently the equation $3(2x-7)-9=2(3x-11)-8$ is equivalent to the statement "$0=0$" which is always true. This means that every number when substituted for x makes $3(2x-7)-9=2(3x-11)-8$ true.

This appears to contradict the portion of theorem (4.1) which says the solution of a linear equation is *unique*. But the equation "$3(2x-7)-9=2(3x-11)-8$" is not a linear equation, because it is equivalent to "$0=0$". And, "$0=0$" is not of the form $ax+b=0$, where $a \neq 0$.

(5) Many possible answers.

(6) Proof of Theorem (4.1).

Proof. Suppose an equation is a linear equation. Then, by definition, it can be written in the form $ax+b=0$, where a, b integers with $a \neq 0$.

Since every integer has an inverse under addition, there exists a number $-b$ that we add to both sides.

$$ax+b=0 \Longleftrightarrow ax=-b.$$

Since every integer, by virtue of being a rational number, has an inverse under multiplication, there exists a number $\frac{1}{a}$ (remember, the equation is linear, so $a \neq 0$) by which we multiply both sides.

$$\Longleftrightarrow x=\frac{-b}{a}.$$

The number $\frac{-b}{a}$ exists, because the rational numbers are closed under division. Therefore, every linear equation has a solution in the rational numbers.

□

(7) Proof of Theorem (4.1).

Proof. Suppose a, b integers, where $a \neq 0$. Then the $ax + b = 0$ is a linear equation. Further suppose that there are two different solutions to this equation. Call them x_1 and x_2 where $x_1 \neq x_2$. This implies the following two following equations.

$$ax_1 + b = 0 \tag{A.2}$$

$$ax_2 + b = 0. \tag{A.3}$$

Since equality is transitive,

$$ax_1 + b = ax_2 + b$$

Subtract b from both sides and divide by a which is not 0. Then,

$$x_1 = x_2.$$

This contradicts the supposition that $x_1 \neq x_2$. Therefore, every linear equation has a unique solution in the rational numbers. □

Appendix B

Answers to Supplementary Exercises

Answers to Supplementary Exercise 1

1) −2
2) −4
3) 2
4) −28
5) −24
6) 9
7) −3
8) 1
9) 6
10) −5
11) −10
12) −4
13) 8
14) 3
15) −12
16) −5
17) 8
18) −7
19) 2
20) 3
21) −2
22) −12
23) −15
24) −3
25) 1
26) −1
27) 8
28) 6
29) 7
30) 4

Answers to Supplementary Exercise 2

1) $25n + 25$
2) $-4 - 20a$
3) $-5 - 10k$
4) $-x - 5$
5) $-20x - 20$
6) $-5n - 20$
7) $5 + 5k$
8) $-6p + 10$
9) $10x - 10$
10) $20 - 20n$
11) $-20m - 12$
12) $12 - 15r$
13) $5x + 20$
14) $4 + 4n$
15) $9 - 12b$
16) $-5v - 4$
17) $5 + 10x$
18) $4x - 4$
19) $-12 - 12a$
20) $-4p - 16$
21) $8 - 12x$
22) $25n - 15$
23) $3m + 6$
24) $-5r - 15$
25) $-20x - 15$
26) $20n + 15$
27) $2 - 4v$
28) $-15x + 6$
29) $9n + 6$
30) $8a + 4$
31) $10k + 2$
32) $25 + 10x$
33) $2x + 4$
34) $-4 + 4n$
35) $-15 + 3m$
36) $-3p - 1$
37) $-x - 1$
38) $-n + 3$
39) $-15 + 10m$
40) $-8r - 2$

Answers to Supplementary Exercise 3

1) $40 - 42n$
2) $4 + 20k$
3) $-6p - 28$
4) $-80 - 54x$
5) $-3n - 30$
6) $-20m - 3$
7) $-79r - 50$
8) $23 - 8x$
9) $-6n + 18$
10) -4
11) $-25v + 16$
12) $-12x - 25$
13) $-101n + 80$
14) $-9a - 6$
15) $-41k + 45$
16) $2 + 30p$
17) $-16x + 29$
18) $-2n - 7$
19) $-16m - 2$
20) $-11r - 42$
21) $-18 - 12x$
22) $-37n - 54$
23) $-14 - 42b$
24) $-7v + 47$
25) $-94 - 10x$
26) $90 - 54n$
27) $-26 + 35a$
28) $-39 - 20k$
29) $-58x + 40$
30) $-4 + 22x$
31) $-13 - 7n$
32) $6k + 15$
33) $50 - 21p$
34) $-32x + 41$
35) $-n - 9$
36) $-9m - 12$
37) $-4r + 6$
38) $-70 - 56x$
39) $-36n - 34$
40) $-16b - 6$
41) $-2 - 130v$
42) $-82n + 101$
43) $12 - 116x$
44) $-8k - 48$
45) $-29a + 52$
46) $-12p - 28$
47) $32x - 76$
48) $-39m + 27$
49) $-77 + 47n$
50) $-60r - 68$
51) $17 - 58x$
52) $-97 - 8n$
53) $-24b - 13$
54) $-72v + 53$
55) $42 + 36x$
56) $35n + 16$
57) $-22 + 44a$
58) $2v - 22$
59) $30x - 124$
60) $-53x + 2$

Answers to Supplementary Exercise 4

1) $19n + 4$
2) $42a + 48$
3) $-48k - 55$
4) $-9 - 24x$
5) 18
6) $-8 - 46n$
7) $26 - 42m$
8) $-43p + 54$
9) $x - 3$
10) $5n - 42$
11) $2m - 1$
12) $-4r - 6$
13) $-87x + 27$
14) $-49 + 15n$
15) $-36b + 19$
16) $-5 + 90v$
17) $6x - 32$
18) $4x - 2$
19) $-9 - 18a$
20) $-92k - 90$
21) $-57 - 64p$
22) $-49 + 15x$
23) $15n - 24$
24) $43m - 70$
25) $-61 + 8r$
26) $-7 - 35x$
27) 8
28) $3b - 12$
29) $16 - 30v$
30) $-8x + 8$
31) $11n - 5$
32) $36 - 9a$
33) $-26 + 18k$
34) $-40 + 17x$
35) $-3x + 4$
36) $-85 - 45n$
37) $-60m - 45$
38) $45p - 6$
39) $-18x - 90$
40) $56n - 51$

Answers to Supplementary Exercise 5

1) $-72m + 20$
2) $-8r + 15$
3) $-43x - 4$
4) $-33n - 75$
5) $-39b - 2$
6) $-30v + 3$
7) $-12x - 42$
8) $-26x + 61$
9) $-38a + 36$
10) $112k - 24$
11) $-20p + 6$
12) $-24 + 30x$
13) $-18 + 16n$
14) $61 - 45m$
15) $19r - 10$
16) $32 - 40x$
17) $-60n + 34$
18) $18 - 31b$
19) $32 + 6v$
20) $-44x - 20$
21) $-11n + 88$
22) $10 - 28a$
23) $-13 + 28k$
24) $-15x - 39$
25) $-8 - 37x$
26) $-19n - 27$
27) $-39m - 2$
28) $5 + 15p$
29) $7 - 35x$
30) $-9n - 21$
31) $-16 - 10b$
32) $-53 + 31r$
33) $-47x - 24$
34) $-47n - 83$
35) $25b + 14$
36) $-43v + 22$
37) $-54x + 46$
38) $-63x - 16$
39) $-a - 3$
40) $11k + 32$

Answers to Supplementary Exercise 6

1) $6x - 6$
2) $-6 - 6n$
3) $2m - 6$
4) $3r + 6$
5) $-\frac{3}{2}n$
6) $\frac{30x - 22}{12}$
7) $-2v$
8) $\frac{5 - 3b}{2}$
9) $\frac{1}{2}$
10) $\frac{-14n - 1}{6}$
11) $-3a$
12) $\frac{3k + 2}{2}$
13) $\frac{12p - 20}{3}$
14) $\frac{3 - x}{9}$
15) $\frac{12n + 16}{9}$
16) $\frac{4m + 15}{18}$
17) $\frac{-4 - r}{2}$
18) $\frac{-3x - 1}{2}$
19) $\frac{-88n + 99}{18}$
20) $\frac{8 - 4b}{3}$
21) $\frac{5 - 5v}{3}$
22) $\frac{43x - 15}{2}$
23) $\frac{-8n - 75}{18}$
24) $\frac{12 - 5a}{6}$
25) $\frac{23 - 9v}{6}$
26) $\frac{-22x - 43}{6}$
27) $\frac{-3a - 2}{4}$
28) $\frac{26 - 3n}{3}$
29) $\frac{4k + 1}{2}$
30) $\frac{-9p - 1}{3}$
31) $\frac{13x + 6}{9}$
32) $\frac{9k + 4}{3}$
33) $\frac{-2m - 10}{3}$
34) $\frac{31x + 2}{12}$
35) $\frac{5 - 18x}{6}$
36) $\frac{2 - n}{2}$

Answers to Supplementary Exercise 7

1) $\frac{27p-22}{18}$

2) $\frac{39-44n}{18}$

3) $\frac{70-33}{18}$

4) $\frac{13x-12}{9}$

5) $\frac{-8r-9}{9}$

6) $\frac{-18x-110}{9}$

7) $\frac{22n-27}{12}$

8) $\frac{-2b-15}{3}$

9) $\frac{31v-35}{6}$

10) $\frac{1-4x}{9}$

11) $\frac{25x-98}{18}$

12) $\frac{7-43a}{6}$

13) $\frac{-13k-18}{12}$

14) $\frac{10p-21}{3}$

15) $\frac{-75x-8}{18}$

16) $\frac{13x+32}{6}$

17) $\frac{-16m-36}{9}$

18) $\frac{-4r-3}{2}$

19) $\frac{32x+53}{12}$

20) $\frac{14n-5}{18}$

21) $\frac{16-24b}{9}$

22) $\frac{-48v-13}{12}$

23) $\frac{12-35n}{12}$

24) $\frac{4-6x}{3}$

25) $\frac{-20a-18}{3}$

26) $\frac{40-42k}{9}$

27) $\frac{30x+34}{9}$

28) $\frac{23x+7}{12}$

29) $\frac{35-9x}{3}$

30) $\frac{15-4m}{2}$

Answers to Supplementary Exercise 8

1) {1}
2) {3}
3) {0}
4) {3}
5) {2}
6) {−2}
7) {−1}
8) {−2}
9) {−1}
10) {3}
11) {−2}
12) {2}
13) {0}
14) {1}
15) {1}
16) {0}
17) {−3}
18) {0}
19) {2}
20) {3}
21) {3}
22) {−2}
23) {−3}
24) {0}
25) {1}
26) {−2}
27) {−2}
28) {−2}
29) {3}
30) {−3}

Answers to Supplementary Exercise 8A

1) $\{-4\}$
2) $\{4\}$
3) $\{3\}$
4) $\{4\}$
5) $\{6\}$
6) $\{-2\}$
7) $\{-3\}$
8) $\{4\}$
9) $\{-5\}$
10) $\{5\}$
11) $\{-6\}$
12) $\{5\}$
13) $\{3\}$
14) $\{-4\}$
15) $\{6\}$
16) $\{5\}$
17) $\{-1\}$
18) $\{2\}$
19) $\{-1\}$
20) $\{-1\}$
21) $\{-1\}$
22) $\{2\}$
23) $\{3\}$
24) $\{-1\}$
25) $\{5\}$
26) $\{-5\}$
27) $\{-2\}$
28) $\{0\}$
29) $\{-3\}$
30) $\{0\}$
31) $\{2\}$
32) $\{-2\}$

Answers to Supplementary Exercise 8B

1) $\{3\}$
2) $\{4\}$
3) $\{-5\}$
4) $\{5\}$
5) $\{-5\}$
6) $\{-4\}$
7) $\{-4\}$
8) $\{-4\}$
9) $\{-6\}$
10) $\{-5\}$
11) $\{-5\}$
12) $\{4\}$
13) $\{-6\}$
14) $\{3\}$
15) $\{6\}$
16) $\{-4\}$
17) $\{-1\}$
18) $\{0\}$
19) $\{-1\}$
20) $\{2\}$
21) $\{1\}$
22) $\{1\}$
23) $\{-2\}$
24) $\{-2\}$
25) $\{4\}$
26) $\{0\}$
27) $\{3\}$
28) $\{-1\}$
29) $\{-3\}$
30) $\{4\}$
31) $\{5\}$
32) $\{0\}$

Answers to Supplementary Exercise 9

1) $\left\{\frac{1}{2}\right\}$

2) $\left\{\frac{5}{3}\right\}$

3) $\{-2\}$

4) $\left\{-\frac{5}{3}\right\}$

5) $\left\{-\frac{7}{2}\right\}$

6) $\{1\}$

7) $\left\{\frac{3}{2}\right\}$

8) $\{0\}$

9) $\{-2\}$

10) $\left\{\frac{1}{2}\right\}$

11) $\left\{\frac{5}{2}\right\}$

12) $\left\{\frac{3}{2}\right\}$

13) $\left\{\frac{5}{2}\right\}$

14) $\left\{\frac{4}{3}\right\}$

15) $\left\{-\frac{5}{2}\right\}$

16) $\left\{-\frac{7}{3}\right\}$

17) $\left\{-\frac{4}{3}\right\}$

18) $\left\{-\frac{2}{3}\right\}$

19) $\left\{\frac{2}{3}\right\}$

20) $\left\{\frac{1}{2}\right\}$

21) $\left\{\frac{1}{2}\right\}$

22) $\{-1\}$

23) $\{1\}$

24) $\{1\}$

Answers to Supplementary Exercise 9A

1) $\{0\}$
2) $\left\{\frac{7}{3}\right\}$
3) $\{-1\}$
4) $\{-3\}$
5) $\left\{-\frac{1}{3}\right\}$
6) $\{2\}$
7) $\left\{-\frac{10}{3}\right\}$
8) $\left\{\frac{8}{3}\right\}$
9) $\left\{-\frac{1}{2}\right\}$
10) $\left\{-\frac{4}{3}\right\}$
11) $\{-2\}$
12) $\left\{-\frac{5}{3}\right\}$
13) $\left\{\frac{4}{3}\right\}$
14) $\left\{-\frac{3}{2}\right\}$
15) $\left\{\frac{1}{3}\right\}$
16) $\left\{-\frac{4}{3}\right\}$
17) $\left\{-\frac{5}{3}\right\}$
18) $\left\{\frac{7}{3}\right\}$
19) $\left\{\frac{2}{3}\right\}$
20) $\left\{-\frac{5}{3}\right\}$
21) $\left\{-\frac{4}{3}\right\}$
22) $\{-1\}$
23) $\left\{-\frac{11}{3}\right\}$
24) $\left\{\frac{3}{2}\right\}$

Answers to Supplementary Exercise 10

1) $\left\{-\frac{7}{3}\right\}$

2) $\{-3\}$

3) $\{-3\}$

4) $\left\{-\frac{7}{2}\right\}$

5) $\left\{-\frac{7}{2}\right\}$

6) $\{-2\}$

7) $\left\{-\frac{11}{3}\right\}$

8) $\left\{-\frac{7}{2}\right\}$

9) $\{18\}$

10) $\left\{-\frac{83}{26}\right\}$

11) $\left\{\frac{2}{21}\right\}$

12) $\left\{-\frac{4}{3}\right\}$

13) $\left\{-\frac{3}{29}\right\}$

14) $\left\{-\frac{4}{3}\right\}$

15) $\left\{\frac{23}{18}\right\}$

16) $\{3\}$

Answers to Supplementary Exercise 10A

1) $\left\{-\frac{11}{3}\right\}$

2) $\{-3\}$

3) $\left\{\frac{5}{2}\right\}$

4) $\left\{-\frac{7}{2}\right\}$

5) $\{-3\}$

6) $\left\{-\frac{11}{3}\right\}$

7) $\left\{-\frac{7}{2}\right\}$

8) $\{3\}$

9) $\left\{\frac{24}{17}\right\}$

10) $\left\{-\frac{4}{7}\right\}$

11) $\left\{-\frac{16}{17}\right\}$

12) $\left\{-\frac{34}{7}\right\}$

13) $\{12\}$

14) $\left\{-\frac{5}{2}\right\}$

15) $\left\{-\frac{24}{37}\right\}$

16) $\left\{-\frac{13}{87}\right\}$