Beginning Algebra

Second Edition

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ABSTRACT. A good introduction to school algebra – one that is worth the student's time and effort – should leave the student believing that algebra is exactly as she knows it should be. The student's experience should naturally involve wonder and discovery while instilling the ideas, skills, understanding, and intellectual tendencies that will bring success in mathematics throughout high school and college. The author has taught school mathematics to grades 4 through various years of calculus. In doing so, he has seen children display intellectual sophistication that they are supposed to be too young to have. This text does not assume such sophistication, but provides opportunities for it to appear and be enjoyed. The author's deeply held belief that the student, although less experienced, is a peer in the quest for mathematical truth has produced a book respectful of the student's intellect, curiosity, wonder, and humanity. The text includes no fewer than 985 problems. Answers to all problems and full solutions to many are provided in the Appendix. Most of these problems are practice, but there are some that go deeper than mere practice and typically ask the student for a proof or a reasoned explanation.

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Preface

An equation means nothing to me unless it expresses a thought of God.
—Srinivasa Ramanujan

For thousands of years, our species has found intellectual satisfaction and aesthetic delight in the exploration of the mathematical universe. Discoveries, sometimes as surprising as they were certain, heightened the sense of wonder. Beginning algebra students should have a similar experience.

A good beginning in algebra leaves students skillful at manipulating expressions and equations. A very good beginning leaves students knowing that understanding a few fundamental ideas is more effective than memorizing an assortment of "math facts". Students in this latter group leave prepared for higher level courses in which the memorization approach will produce at best limited success.

In the category of helpful advice:

(1) Work examples along with the book. Think of an example as being the same as an exercise problem, except that you and the author do it together.

(2) Except for word problems, copy the problem onto your own paper. Doing so will speed your becoming fluent in the language of mathematics.

(3) Doing assignments in time for the next class prepares you to understand new material introduced in the next class. Joining the class discussion of the assignment helps you get the full benefit of having done the assignment.

The author wishes to thank the 2012-2013 7^{th} grade class at *Madison Country Day School* for their valuable insights and their patience during the first use of this book. These students discovered errors in the text that escaped the author and survived the several proof readings by adults. Students ever so politely pointed out such errors. The author hopes, probably in vain, to have included all such corrections in this revision of the book.

Note that the symbol *marks a problem at the boundary of what a student at this level should be able to work. The symbol ** identifies a problem beyond that boundary.

Ray Tenebruso

Chapter 1

Preliminary ideas

Some of the ideas you will encounter in mathematics will seem obvious to you. It is my responsibility to convince you that, sometimes, obvious ideas lead to spectacularly non-obvious conclusions.

1.1. The early bird gets the worm

Case 1. More birds than worms:

bird	*bird*	*bird*	*bird*	*bird*	*bird*	*bird*	*bird*
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
worm	*worm*	*worm*	*worm*	*worm*	*worm*	*worm*	

Case 2. More worms than birds:

bird	*bird*	*bird*	*bird*	*bird*	*bird*	*bird*	
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
worm	*worm*	*worm*	*worm*	*worm*	*worm*	*worm*	*worm*

Case 3. Exactly the same number of birds and worms!:

bird	*bird*	*bird*	*bird*	*bird*	*bird*	*bird*	*bird*
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
worm	*worm*	*worm*	*worm*	*worm*	*worm*	*worm*	*worm*

In this last case, there is a 1-1 correspondence of birds and worms which was lacking in the first two cases. Seeing that there is a 1-1 correspondence is all that is needed to know there are exactly as many birds as worms. Let us commemorate this obvious idea by making it our first theorem.

Theorem 1.1
Two collections have exactly the same number of members if and only if there is a 1-1 correspondence between the members of the two collections. ■

The symbol ■ signals the end of an example, proof, definition, or theorem. In this book, it is usually left out when the end of such an element is obvious.

Some new notation. When we write $1,2,3,\cdots,70$, we mean the "$\cdots$" to include the numbers from 4 to 69. When we write $1,2,3,\cdots$, we mean that this pattern continues without end.

1.2. Numbers and even numbers

Case 1. Stop at 10:

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ & 2 & & 4 & & 6 & & 8 & & 10 \end{array}$$

Half as many even numbers as numbers.

Case 2. Stop at 100:

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & 100 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots & \updownarrow \\ & 2 & & 4 & & 6 & & 8 & \cdots & 100 \end{array}$$

Half as many even numbers as numbers.

Case 3. Stop at 10000:

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & 10000 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots & \updownarrow \\ & 2 & & 4 & & 6 & & 8 & \cdots & 10000 \end{array}$$

Yup, half as many!

Case 4. Never stop:

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & \cdots \end{array}$$

There are *exactly the same number of even numbers as there are numbers!* We know this, because there is a 1-1 correspondence between the even numbers and the numbers.

Moral of story: even the most obvious of ideas can sometimes surprising consequences. So, it may be wise to show patience for obvious ideas.

Exercise 1.1

1. Using $\cdots$, write the numbers from 0 to 1000.
2. Using $\cdots$, write the numbers from 7 to 93.
3. Using $\cdots$, write the numbers from 5 on.
4. Are there the same number of odd numbers as even numbers? Show why your answer is correct.
5. Are there the same number of odd numbers as numbers? Show why your answer is correct.
6. Are there the same number of multiples of 5 as numbers? Show why your answer is correct.

1.3. Letters

There is nothing magical or mysterious about the use of letters in mathematics. The letter is merely a very brief name that we use instead of a long name or descriptive phrase.

Some school textbooks call letters "variables". This is unfortunate, because not every letter is a variable. We will call letters "letters".

Example 1.1

You know that

$$1+2=2+1,$$
$$1+3=3+1,$$
$$2+5=5+2.$$

You know much more than this, because you know this is true for every pair of numbers. How can we communicate this idea? Here is a convenient way.

Let a and b represent any numbers. Then $a+b=b+a$.

Example 1.2

The following statements are true.

$$(1+2)+3 = 1+(2+3).$$
$$(3+4)+5 = 3+(4+5).$$
$$(3+7)+8 = 3+(7+8).$$

Assert that this idea holds true for *every* triplet of numbers.

Solution

Let a, b, c represent any numbers. Then $a+(b+c) = (a+b)+c$. ■

In previous grades, you learned that the product of two fractions is a fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators, provided that no denominator is zero. For instance, $\frac{2}{3} \times \frac{5}{7} = \frac{2\times 5}{3\times 7}$. That was a mouthful. Using letters, the idea is communicated with simplicity.

Example 1.3

Write the rule for finding the product of fractions.

Solution

For any numbers a, b, c, d with b and d different from zero,

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}.$$

Example 1.4

Write the rule for canceling common factors in fractions.

Solution

For any numbers a, b, c with b and c different from zero, $\dfrac{a \times c}{b \times c} = \dfrac{a}{b}$.

Remark 1.1
The choice of letters is of no consequence. The statements "$a+b=b+a$" and "$x+y=y+x$" have exactly the same meaning. Each statement expresses the idea that the terms of a sum may be added in any order.

Exercise 1.2

1. Using the letter a, express the idea that a number multiplied by 1 is that same number.
2. Using the letter a, express the idea that a fraction in which the numerator and the denominator are identical is equal to 1.
3. Using the letter b, express the idea that zero added to a number results in the number.
4. Using the letters a, b, and c, show the addition of two fractions that have a common denominator.
5. Using the letters a and b, express the idea that the order in which two numbers are multiplied does not affect the product.
6. * Using the letter y, write all the even numbers.

1.4. Equality

1.4.1. The meaning of equality

The symbol "=" is very familiar. Even when you were little, you wrote "$2+1=3$" and "$2\times 3=6$". Have you ever wondered what it means when we write "$2+1=3$"? The author of this book recalls once having been told that "$2+1=3$" means that "$2+1$" and "3" are the same. This answer did not seem right, because, for starters, "$2+1$" and "3" don't even look the same. When he mentioned this, he was told "Well, you don't understand, it means that "$2+1$" and "3" are the same number!" But he was still confused, because as near as he could tell, the numbers 2 and 1 appeared on the left side of "$2+1=3$" but only the one number, 3, appeared on the right side.

Later, many years later, the author received an explanation of what "$2+1=3$" means that actually did make sense. It went like this, "$2+1$" is the name for a number. "3" is the name for a number. The statement "$2+1=3$" says that "$2+1$" and "3" name the same number. The statement "$2+1=3$" is true, because "$2+1$" and "3" do name the same number. The statement "$2+1=4$" is false, because "$2+1$" and "4" do not name the same number. Let's make a special note of this fact about equality.

Definition 1.1 (Equality)
$a=b$ means that a and b are names for the same object.

1.4.2. Two names for one person

The statement "Mark Twain wrote the book *Adventures of Huckleberry Finn*" is true. Many people know this. There are not quite as many people who know that "Samuel Longhorn Clemens wrote *Adventures of Huckleberry Finn*". But this statement is true, too. This is because "Mark Twain" and "Samuel Longhorn Clemens" are both names for the same person, who did write *Adventures of Huckleberry Finn*. This brings up an important idea. If "Mark Twain" and "Samuel Longhorn Clemens" both name the same person, then the statements "Mark Twain wrote the book *Adventures of Huckleberry Finn*" and "Samuel Longhorn Clemens wrote *Adventures of Huckleberry Finn*" are both true, regardless of which name is used for the author. You may use either name, as you wish. This idea is so important in mathematics, that it is given a special name. It is called the "Principle of Substitution".

1.4.3. Principle of Substitution.

If two expressions name the same object, one expression may be substituted for the other in any statement without changing the truth of the statement.

1.4.4. The nature of equality

However rich the idea of equality may be, three properties of equality are frequently used in mathematics. Let a, b, and c represent any numbers.

(1) Reflexive. $a = a$.
(2) Symmetric. $a = b$ implies $b = a$.
(3) Transitive. $a = b$ and $b = c$ implies $a = c$.

Example 1.5

The following illustrate the reflexive nature of equality.

(1) $5 = 5$.
(2) $3+4 = 3+4$.
(3) $(3+7)+8 = (3+7)+8$.
(4) $x+9 = x+9$.

Example 1.6

The following illustrate the symmetric nature of equality.

(1) $5+3 = 8$ implies $8 = 5+3$.
(2) $5+2 = 4+3$ implies $4+3 = 5+2$.
(3) $a+5 = b$ implies $b = a+5$.

Example 1.7

The following illustrate the transitive nature of equality.

(1) $5+3 = 8$ and $8 = 6+2$ implies $5+3 = 6+2$.
(2) $7+5\times 2 = 7+10$ and $7+10 = 17$ implies $7+5\times 2 = 17$.
(3) if $x+y = z$ and $z = 7+a$ then $x+y = 7+a$.

Example 1.8

State the property of equality that justifies each of the following.

(1) if $2+11 = 10+3$ and $10+3 = 13$ then $2+11 = 13$.
(2) if $a+8 = b+7$ and $b+7 = 23$ then $a+8 = 23$.
(3) if $b+c = 17$ then $17 = b+c$.

Solution

(1) Transitive.
(2) Transitive.
(3) Symmetric. ■

The Principle of Substitution and the facts that equality is reflexive, symmetric, and transitive lead to other facts about equality. Two such facts that you will use thousands of times are stated in the next theorem.

Theorem 1.2
Let a, b and c be any numbers.

(1) If $a = b$ then $a + c = b + c$, and
(2) If $a = b$ then $a \times c = b \times c$.

Proof. Let a, b and c be any numbers. Suppose that $a = b$.

$a + c = a + c$, equality is reflexive.

$a + c = b + c$, by substitution. We supposed $a = b$.

Therefore, if $a = b$ then $a + c = b + c$. ■

The proof for multiplication is similar, so it is left for an exercise.

Exercise 1.3

1. Use the symmetric quality of equality to rewrite $a + 101$.
2. Using the transitive property of equality, what conclusion follows from the statement $a + 3 = y$ and $y = b + 5$?
3. What property of equality does the following illustrate:
$a + b + c = 7 + 2$ so $7 + 2 = a + b + c$?
4. What property of equality justifies:
$a + b + c = d$ and $b + c = 9$, so $a + 9 = d$? [Hint: do not forget the principle of substitution.]
5. State in words, each of facts (1) and (2) of Theorem (1.2).
6. Let a, b and c be any numbers. Prove that
If $a = b$ then $a \times c = b \times c$. [Hint: see proof of theorem 1.2.]

1.4.5. What a difference a name makes!

You may be wondering: What difference can a name make? The answer is: A big difference! You have been using that big difference to your advantage for several years.

The value of the sum $\frac{1}{3}+\frac{2}{5}$ is not obvious. But, if we write

$$\frac{1}{3} \text{ using the name } \frac{5}{15}$$

and

$$\frac{2}{5} \text{ using the name } \frac{6}{15},$$

then $\frac{11}{15}$ is obviously the value of the sum.

Example 1.9

Find the sum $\frac{3}{7}+\frac{5}{8}$ and note when one name is substituted for another name.

$$\frac{2}{7}+\frac{5}{8}=\underbrace{\frac{16}{56}+\frac{35}{56}}_{\text{substitutions}}=\frac{51}{56}.$$

Exercise 1.4

1. If a names the number 3, then what number is $2+a\times 3$?
2. If a represents the number 7, then what number is $(5+a)\times 3$?
3. If a names the number 4, and b number 9, then what is $a+b+2$?
4. If a is 10, and b is 7, then what is $a+b$?. (Note: since phrases like "names the number" and "represents the number" are tiring to write and speak, we often just say, for example, "a is 7".)

1.5. Sets

The word **set** means a collection of objects. For example, when we speak of the set of Alice's stamps, we mean her collection of stamps. The objects in a set are called **members** or **elements** of the set. The elements of Alice's set of stamps are the stamps. Often, the members of a set have some quality in common. The members of Alice's set of stamps have in common that they are each a stamp belonging to Alice. Notice that a set has qualities of its own. For example, the set of Alice's stamps has the quality that it contains 100 stamps. And, the members of a set may have a quality that the set itself cannot be said to have. For instance, the set of Alice's stamps is not itself a stamp. And, although each of Alice's stamps might be blue in color, the set of her stamps certainly is not blue. The words "set", "collection", "class", and "aggregate" are used synonymously.

In mathematics, we are often concerned with sets that contain numbers as members. For example, we might speak of the set of even numbers or the set of odd numbers less than 7. Notice that, like Alice's set of stamps, the set of odd numbers less than 7 has the quality that it contains 3 numbers and this is a quality none of its members can possess. While the members of the set of odd numbers less than 7 are each odd, the set of them cannot be said to be "odd".

1.5.1. Set notation

To indicate a set of items, simply place set brackets "{" and "}" around the items in the set. The symbol $\{1,3,8\}$ is pronounced "The set whose members are $1,3,8$". Examples will make clear how set notation works.

Example 1.10

(1) $\{1,2,3,4,5\}$ means the set of numbers $1,2,3,4,5$.
(2) $\{3,6,9,12\}$ means the set of numbers $3,6,9,12$.
(3) The set of even numbers between 4 and 14 is written $\{6,8,10,12\}$.
(4) The set of prime numbers less than 17 is written $\{2,3,5,7,11,13\}$.
(5) The set of factors of 12: $\{1,2,3,4,6,12\}$.
(6) The set of prime factors of 12: $\{2,3\}$.
(7) $\{1,2,3,\cdots,100\}$ is the set of numbers from 1 to 100.
(8) $\{1,2,3,\cdots\}$ is the set of numbers $1,2,3,\cdots$.

1.5.2. Set membership

When we wish to say that a number is a member of a set, we use the sign "$\in$". We pronounce "$a \in \{a,b,c\}$" as "a is an element of the set whose members are a, b, c". The phrases "is an element of", "is a member of", "is contained in" or simply "is in" are synonymous.

Example 1.11

(1) $2 \in \{1,2,3,4,5\}$ is true, because 2 *is* a member of $\{1,2,3,4,5\}$.
(2) $4 \in \{1,2,3,4,5\}$ is true.
(3) $5 \in \{1,2,3,4,5\}$ is true, because 5 is in $\{1,2,3,4,5\}$.
(4) $9 \in \{3,6,9,12\}$ is true.
(5) $2 \in \{3,6,9,12\}$ is false.
(6) $\{6\} \in \{3,6,9,12\}$ is false, but
(7) $\{6\} \in \{3,6,9,12,\{6\}\}$ is true.
(8) $23 \in \{1,2,3,\cdots,100\}$ is true.
(9) $18 \in \{2,4,6,8,\cdots\}$ is true. ■

We can assert that 7 is not a member of $\{1,3,6,9,12,15\}$ by writing $7 \notin \{1,3,6,9,12,15\}$. Sets may be named for convenience. For example, if $\mathbb{A} = \{1,2,3,4,5,6\}$, then $2 \in \mathbb{A}$ and $7 \notin \mathbb{A}$ are true statements.

1.5.3. Special sets

Two special sets are the **Empty Set**, denoted by the symbol $\emptyset$ or by $\{\}$ and the **Universal Set**, often denoted by the symbol U. The Empty Set has no members and the Universal Set contains everything.

1.5.4. Subsets

When all the members of one set are contained in another set, we say the one set is a **subset** of the other. If $\mathbb{A}$ is a subset of $\mathbb{B}$, we write $\mathbb{A} \subseteq \mathbb{B}$. To deny that $\mathbb{A}$ is a subset of $\mathbb{B}$, write $\mathbb{A} \nsubseteq \mathbb{B}$. A set is a subset of itself, $\mathbb{A} \subseteq \mathbb{A}$. If all the members of $\mathbb{A}$ are in $\mathbb{B}$, but some member of $\mathbb{B}$ is not in $\mathbb{A}$, then $\mathbb{A}$ is called a **proper subset** of $\mathbb{B}$ and we write $\mathbb{A} \subset \mathbb{B}$. The empty set is a subset of every set; that is, for any set $\mathbb{A}, \emptyset \subseteq \mathbb{A}$.

Example 1.12

Let $\mathbb{A} = \{1,3,6,7,8,10,12\}$, and $\mathbb{B} = \{1,10\}$

(1) $3 \in \mathbb{A}$ is true.
(2) $10 \in \mathbb{B}$ is true.
(3) $\{10\} \subseteq \mathbb{B}$ is true.
(4) $\{3,7\} \nsubseteq \mathbb{B}$ is true.
(5) $\{3,7,10\} \subseteq \mathbb{A}$ is true.
(6) $\mathbb{B} \subseteq \mathbb{A}$ is true.
(7) $\mathbb{A} \nsubseteq \mathbb{B}$ is true.
(8) $\emptyset \subseteq \mathbb{A}$ is true.
(9) $\emptyset \subseteq \mathbb{B}$ is true.

Exercise 1.5

[Part 1] *True or false.*

1. $9 \in \{1,2,5,7,9,15,27\}$.
2. $9 \in \{2,4,8,16,32\}$.
3. $28 \in \{1,3,5,7,\cdots\}$.
4. $6 \notin \{1,3,5,7,\cdots\}$.
5. $\{2\} \in \{1,2,5,7,\cdots\}$.
6. $\{2\} \in \{1,\{2\},5,7,\cdots\}$.
7. $\{1,5,11\} \subseteq \{1,3,5,7,\cdots\}$.
8. $\emptyset \in \{1,2,5,7,\cdots\}$.
9. $\emptyset \subseteq \{1,2,5,7,\cdots\}$.
10. $\emptyset \subseteq \emptyset$.

[Part 2] *Insert* $\in$ *or* $\notin$ *to make each statement true.*

1. 4_____$\{1,2,4,6,8,10,\cdots\}$.
2. 30_____$\{1,3,6,9,12,\cdots\}$.
3. 11_____$\{2,4,6,8,\cdots\}$.
4. $\{7\}$_____$\{1,2,4,6,\{7\},8,10,\cdots\}$.

[Part 3] *Describe each set.*

1. $\{2,4,6,8,\cdots\}$.
2. $\{10,15,20,25,\cdots,105\}$.

1.6. Three fundamental ideas

When you think of people you know, or even of characters in a book, you look for certain qualities in those people that are interesting to you. For example, you might note that a certain friend is an especially kind person, that another friend is always honest, or that a character in a certain book is adventurous.

In your study of mathematics, you will discover that the objects of mathematics, for example numbers, have characters too. For example, the number 2 is even and the number 3 is odd. The number 9 may be divided by 3 without remainder, but 8 divided by 3 leaves a remainder.

Perhaps you have noticed that collections of people have character of their own, too. For example, a sports team you play on might be boisterous. But maybe the choir that you sing with is calm. Your mathematics class might be chatty, but your English class well behaved. The character of the group is determined by the members *and how they interact*.

When you find yourself in an unfamiliar group — maybe a new school or a new sports team — you observe for a while until you learn the character of the group.

Sets of mathematical objects have characters too. Once you know what to look for, you will find your way around an unfamiliar collection of numbers, just as you find your way around in a new group of people.

In order to discover the qualities that sets of numbers possess, we will examine some unusual cases. But, this will allow us to focus precisely on the qualities themselves.

Let us consider a very small collection of numbers, the set $\{0,1,2\}$. The interactions of these numbers are completely specified in Table (1.1). We agree to call the operation of the table, which is written with the little circle $\circ$, “circle”.

$\circ$	**0**	**1**	**2**
0	0	1	2
1	1	2	0
2	2	0	1

TABLE 1.1. $(\{0,1,2\},\circ)$

Here is how to read the table. To find the result of $1\circ 2$, go to “1” in the left most column, then read across that row until you get to the column headed by “2”, the result is “0”. We write “$1\circ 2 = 0$”.

Order of operations is shown by using parenthesis.

Example 1.13

Find $2 \circ (1 \circ 0)$.

Solution

$\circ$	**0**	**1**	**2**
0	0	1	2
1	1	2	0
2	2	0	1

Table 1.1 (Copied from page 13.)

$$\begin{aligned} 2 \circ (1 \circ 0) &= 2 \circ 1 \\ &= 0. \end{aligned}$$

Conclusion: $2 \circ (1 \circ 0) = 0$.

1.6.1. First fundamental idea.

If you study Table (1.1), you will notice that the operation $\circ$ performed on any two members of the set $\{0, 1, 2\}$ always results in a member of the set. This characteristic of a set of numbers is so important in mathematics that it has a special name. We say the set $\{0, 1, 2\}$ is *closed* under the operation $\circ$.

Definition 1.2 (Closed)
Let a and b be any members of the set $\mathbb{A}$ and $\star$ be an operation defined on set $\mathbb{A}$. We say set $\mathbb{A}$ is **closed** under the operation $\star$ if $a \star b$ always results in an element of the set $\mathbb{A}$.

1.6.2. Second fundamental idea.

Perhaps you have noticed that in Table (1.1)

$$\begin{aligned} 0 \circ 1 &= 1 \circ 0, \\ 0 \circ 2 &= 2 \circ 0, \\ 1 \circ 2 &= 2 \circ 1. \end{aligned}$$

Since this is true for *every* pair of elements in Table (1.1), we say that the operation $\circ$ is **commutative** on the set $\mathbb{A}$.

Definition 1.3 (Commutative)
Let a and b be any members of the set $\mathbb{A}$ and $\star$ be an operation defined on set $\mathbb{A}$. We say that $\star$ is **commutative** on $\mathbb{A}$ if $a \star b = b \star a$ for every pair of elements in $\mathbb{A}$.

1.6.3. Third fundamental idea.

This one is harder to notice than the other two and tedious to verify. Once you know what to look for, it is easier to see. Notice that in Table (1.1)

$$\begin{aligned} 1 \circ (1 \circ 2) &= (1 \circ 1) \circ 2), \\ 1 \circ (2 \circ 0) &= (1 \circ 2) \circ 0), \\ 2 \circ (1 \circ 2) &= (2 \circ 1) \circ 2). \end{aligned}$$

Since this is true for *every* triplet of elements in Table (1.1), we say that the operation $\circ$ is **associative** on the set $\mathbb{A}$.

Definition 1.4 (Associative)
Let a, b and c be any members of the set $\mathbb{A}$ and $\star$ be an operation defined on set $\mathbb{A}$. We say that $\star$ is **associative** on $\mathbb{A}$ if $a \star (b \star c) = (a \star b) \star c$ for every triplet of elements in $\mathbb{A}$. ■

Once you know that a set of numbers is closed, commutative, and associative under an operation, you may take advantage of these qualities when you work with those numbers and that operation.

Remark 1.2
The set of natural numbers, $\{1, 2, 3, \cdots\}$, is

(1) closed, commutative, and associative under the operation of addition, and
(2) closed, commutative, and associative under the operation of multiplication.

Example 1.14

Since the natural numbers are closed under addition, we can be certain that $2 + 3$ is a natural number and so is $1203945753 + 33271952378$.

Example 1.15

The natural numbers are commutative under addition, so

$$1203945753 + 33271952378 = 33271952378 + 1203945753.$$

Example 1.16

The natural numbers are associative under addition, so it is a sure bet that $2 + (3 + 5) = (2 + 3) + 5$.

Example 1.17

The set of natural numbers is not closed under the subtraction, because $2 - 3$ is not a natural number.

Example 1.18

Rewrite $2 + (3 + 5) + 7$ as $(3 + 2) + 5 + 7$.

Solution

$$\begin{aligned} 2 + (3 + 5) + 7 &= (2 + 3) + 5 + 7 && \text{addition is associative} \\ &= (3 + 2) + 5 + 7 && \text{addition is commutative} \end{aligned}$$

Example 1.19

Rewrite $2 \times (3 \times 5) \times 7$ as $3 \times 2 \times (7 \times 5)$.

Solution

$$\begin{aligned} 2 \times (3 \times 5) \times 7 &= 2 \times 3 \times (5 \times 7) && \text{multiplication is associative.} \\ &= 2 \times 3 \times (7 \times 5) && \text{multiplication is commutative.} \\ &= 3 \times 2 \times (7 \times 5). && \text{multiplication is commutative} \end{aligned}$$

Example 1.20

Rewrite $(3 + 8) + 1$ as $1 + (3 + 8)$.

Solution

$$(3 + 8) + 1 = 1 + (3 + 8), \text{ addition is commutative.} \quad \blacksquare$$

This last example requires you to see $(3+8)$ not as several symbols, but as one blob of stuff. Imagine that you see $(3+8)$ as $\blacklozenge$. So instead of $(3+8)+1$ you see $\blacklozenge+1$. Now you are more likely to see how commutativity applies: $\blacklozenge+1=1+\blacklozenge$.

With experience, you will sense when to see the details and when to see a blob.

Example 1.21

Rewrite $2+3+(5+7)$ as $(7+2)+3+5$.

Solution

$$
\begin{aligned}
2+3+(5+7) &= 2+3+(7+5) && \text{addition is commutative.}\\
&= 2+(3+7)+5 && \text{addition is associative.}\\
&= 2+(7+3)+5 && \text{addition is commutative.}\\
&= (2+7)+3+5 && \text{addition is associative.}\\
&= (7+2)+3+5. && \text{addition is commutative}
\end{aligned}
$$

Exercise 1.6

1. Rewrite $(2+7)+13$ as $(2+13)+7$.
2. Rewrite $(9\times 11)\times 6$ as $(9\times 6)\times 11$.
3. Rewrite $1+4+(7+12)$ as $12+(1+7)+4$.
4. Rewrite $3+(7+12)+(4\times 7)$ as $(4\times 7)+12+(7+3)$.
5. Which of the following are natural numbers: $2, 1001, \frac{1}{3}, 6, 2.5$?
6. Is $\frac{12}{3}$ a natural number?

Chapter 2

Integers

The purpose of this chapter is to introduce the integers. We begin with the natural numbers, because the integers are an extension of the natural numbers.

2.1. Natural numbers

Definition 2.1 (The natural numbers)
The numbers $1, 2, 3, \cdots$ are called "**natural numbers**". The symbol $\mathbb{N}$ is typically used to represent the set of natural numbers.

Example 2.1

The numbers $3, 731, 1002$ are natural numbers, because they are in the set $\{1, 2, 3, \cdots\}$. But the numbers $\frac{1}{2}, \frac{2}{5}, \frac{9}{8}$ are not natural numbers, because they are not in $\{1, 2, 3, \cdots\}$. ■

Someone is bound to ask whether $3/1$ and $10/2$ are natural numbers. We know that $3/1 = 3$ and $10/2 = 5$. Since $3/1$ names the number 3 and $10/2$ names the number 5, we conclude that $3/1$ and $10/2$ are natural numbers, although the names $3/1$ and $10/2$ obscure that fact.

When writing by hand, it is better to write fractions like this $\frac{2}{7}$ than like this $2/7$.

2.1.1. Properties of the natural numbers

The natural numbers are closed, commutative, and associative under the operations of addition and multiplication.

No one knows when any of these numbers were first discovered. We can imagine our ancestors happily counting sheep, goats, and various other items. Eventually, our ancestors learned to add, multiply, subtract, and even divide within the set of natural numbers.

A merchant could by subtraction know how many bushels of oats remained of 100 if 70 bushels were sold. Multiplication could tell the merchant how much money the 70 bushels would bring. All was well. For a while, anyway. Until someone asked how many sheep remain if a shepherd sells all 40 of a herd of 40 sheep.

Oops. No one knew the answer to $40-40$. Worse yet, no one knew of any number that could possibly be the answer. The numbers had run out. An unhappy state if ever there were one. Applying the ideas of the previous chapter, we would say that the natural numbers are not closed under subtraction.

The discovery of the number zero provided the answer to "What remains of 40 sheep if 40 are sold?" Numbers were extended to include 0. But, happiness would not have reigned long.

Some troublemaker must have asked What is 5 subtract 6? Oops. Out of numbers, again! Even with 0 included, the set of numbers $\{0,1,2,3,\cdots\}$ is not closed under subtraction.

2.2. Subtraction - one view

There will be another view of subtraction a few pages from now.

You might have first learned to add by *counting on*. And, first learned to subtract by *counting back*.

What is $5+3$? Counting on:

$$1,2,3,4,5,6,7,8,\cdots$$

What is $5-3$? Counting back:

$$1,2,3,4,5,6,\cdots$$

What is $5-6$? Counting back:

$$\blacksquare,0,1,2,3,4,5,6,\cdots$$

We do not yet know the number, or even that such a number exists. But if there is such a number, it *should* be at $\blacksquare$. Our goal, in the next few pages, is to discover the number for $\blacksquare$.

2.3. Two mathematical systems

A tiny mathematical system

$\circ$	**0**	**1**	**2**
0	0	1	2
1	1	2	0
2	2	0	1

Table 1.1 (Copied from page 13.)

The tiny system consisting of $\{0,1,2\}$ and the made-up operation $\circ$ defined by Table(1.1) is humble. It would hardly enable the poorest shepherd to count his sheep.

But observe that no matter which of the numbers $0,1,2$ we choose, 0 can be produced:

$$0 \circ 0 = 0,$$
$$1 \circ 2 = 0,$$
$$2 \circ 1 = 0.$$

A huge mathematical system

$+$	**0**	**1**	**2**	**3**	$\cdots$	
0	0	1	2	3	$\cdots$	no 0 in this row
1	1	2	3	4	$\cdots$	no 0 in this row
2	2	3	4	5	$\cdots$	no 0 in this row
3	3	4	5	6	$\cdots$	no 0 in this row
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	no 0 in these rows

TABLE 2.1. $(\{0,1,2,3,\cdots\},+)$

The huge system shown in Table 2.1 is familiar, because it is the numbers $0,1,2,3,\cdots$ together with ordinary addition. However, notice that there is no number that added to 1 results in 0, no number that added to 2 results in 0, and so on.

Tiny versus huge

The tiny system is algebraically richer than the huge system. Every member of $\{0,1,2\}$ under $\circ$ can be obtained from the other members by the operation $\circ$. A similar claim cannot be made for $\{0,1,2,\cdots\}$ under $+$.

2.4. Discovering the integers

Let us take another look at

$\circ$	**0**	**1**	**2**
0	0	1	2
1	1	2	0
2	2	0	1

Table 1.1 (Copied from page 13.)

Notice that for every member of $\{0,1,2\}$ there is another member such that the operation $\circ$ on the pair of members results in 0.

$$0 \circ 0 = 0.$$
$$1 \circ 2 = 0.$$
$$2 \circ 1 = 0.$$

The "other number" is called the **inverse** of the first number under the operation $\circ$. In $\{0,1,2\}$ under $\circ$,

The inverse of 0 is 0, because $0 \circ 0 = 0$.
The inverse of 1 is 2, because $1 \circ 2 = 0$.
The inverse of 2 is 1. because $2 \circ 1 = 0$.

If the huge set of numbers $\{0,1,2,3,\cdots\}$ is extended by including the inverse element under addition of each of $1,2,3,\cdots$, the resulting set is called the integers.

In the integers under addition, every number, including 0, is the sum of two different integers. 0 is the sum of any integer a and the inverse of a under addition. There is a very nice way to name the inverse elements in the integers. For addition

the inverse of 1 is -1,

the inverse of 2 is -2,
the inverse of 3 is -3.
$\vdots$

In general, the inverse under addition of a is written $-a$.

Definition 2.2 (Integers)
The **integers** are the numbers
$\cdots,-3,-2,-1,0,1,2,3,\cdots$.

(1) 0 is called the "**identity element**" for addition. $a+0=a$ and $0+a=a$, for any integer a.
(2) For each integer a, there exists another integer written $-a$ that is the inverse of a under addition. $a+(-a)=0$ and $(-a)+a=0$.
(3) The inverse of a under addition is usually called the **additive inverse** of a. ■

At the end of Section 2.2, we knew where the number equal to $5-6$ belonged.

$$\blacksquare, 0, 1, 2, 3, 4, 5, 6, \cdots$$

We just did not know the number. Now we do. We fill in ■ with the additive inverse of 1.

$$-1, 0, 1, 2, 3, 4, 5, 6, \cdots$$

In fact, we can continue filling in all the numbers to the left of 0.

$$\cdots,-6,-5,-4,-3,-2,-1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \cdots$$

2.4.1. Names of subsets of integers

Names used for the integers and important subsets of integers are

$\mathbb{Z}=\{\cdots,-3,-2,-1,0,1,2,3,\cdots\}$, the set of integers.
$\mathbb{Z}^+=\{1,2,3,\cdots\}$, the set of positive integers.
$\mathbb{Z}^-=\{\cdots,-3,-2,-1\}$, the set of negative integers.

The set of numbers $\{0,1,2,3,\cdots\}$ is called the set of "nonnegative integers" and has no special symbol that names it. Until part way through Chapter 3, the word "number" will be understood to mean an integer. Figure (2.1) illustrates this.

$$\underbrace{\underbrace{\cdots, -6, -5, -4, -3, -2, -1}_{\text{negative integers } \mathbb{Z}^-}, \underbrace{0, \underbrace{1, 2, 3, 4, 5, 6, \cdots}_{\text{positive integers } \mathbb{Z}^+}}_{\text{nonnegative integers}}}_{\text{integers}}$$

FIGURE 2.1. Important subsets of the integers

Example 2.2

The sum $2+(-2)=0$, because (-2) is the additive inverse of 2. We place parenthesis around -2 to emphasize that -2 is a single symbol that names a particular number.

Example 2.3

For each of the following, write an equation using the number given and its inverse on the left hand side and 0 on the right hand side.

(1) 2
(2) -2
(3) 100
(4) -17
(5) -5
(6) $a, a \in \mathbb{Z}$
(7) $-a, a \in \mathbb{Z}$

Solution

(1) $2+(-2)=0$.
(2) $(-2)+2=0$.
(3) $100+(-100)=0$.
(4) $(-17)+17=0$.
(5) $(-5)+5=0$.
(6) $a+(-a)=0$.
(7) $(-a)+a=0$.

Exercise 2.1

[Part 1] *For each of the following, write an equation using the number given and its inverse on the left hand side and 0 on the right hand side.*

1. 7.
2. 4.
3. 2090.
4. -33.
5. -51.
6. 8.
7. $x, x \in \mathbb{Z}$.
8. $-x, x \in \mathbb{Z}$.

[Part 2] *Questions in Part 2 refer to Table (2.2) that defines an operation* $\star$ *on the set* $\{1,2,3,4\}$.

$\star$	**1**	**2**	**3**	**4**
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

TABLE 2.2

1. What number plays the role that 0 had in Table 1.1 on page 13?
2. What is the inverse element of 3 under $\star$?
3. Does every element of $\{1,2,3,4\}$ have an inverse element?
4. Is $\{1,2,3,4\}$ commutative under $\star$?

[Part 3] *Questions in Part 3 refer to Table (2.3) that defines an operation* $\star$ *on the set* $\{a,b,c,d\}$.

$\star$	**a**	**b**	**c**	**d**
a	a	b	c	d
b	b	d	a	c
c	c	a	d	b
d	d	c	b	a

TABLE 2.3. For questions 13 – 16

1. What letter plays the role that 0 had in Table 1.1 on page 13?
2. What is the inverse element of b under $\star$?
3. Does every element of $\{a,b,c,d\}$ have an inverse element?
4. Is $\{a,b,c,d\}$ commutative under $\star$?

2.5. Addition and subtraction with integers

Having discovered the integers, it would be nice to know how they behave under addition and subtraction. If a and b represent positive integers ($a,b \in \mathbb{Z}^+$), there are 8 cases to consider.

case 1: $a+b$
case 2: $a-b$
case 3: $(-a)+b$
case 4: $(-a)-b$
case 5: $a+(-b)$
case 6: $a-(-b)$
case 7: $(-a)+(-b)$
case 8: $(-a)-(-b)$

Cases 1−4: $a+b, a-b, (-a)+b, (-a)-b$

The first four cases are easily handled by counting on and counting back. Just like in second grade. We illustrate cases 3 and 4 with numeric examples.

What is $-3+2$? Counting on:

$$\cdots, \text{-4}, \text{-3}, \text{-2}, \text{-1}, 0, 1, 2, 3, \cdots$$

so, $-3+2=-1$. ■

What is $-3-2$? Counting back:

$$\cdots, \text{-6}, \text{-5}, \text{-4}, \text{-3}, \text{-2}, \text{-1}, 0, 1, 2, 3, \cdots$$

so, $-3-2=-5$. ■

There are several more ways to think of subtraction. They are illustrated in the following example.

Example 2.4

Evaluate $3-8$ in several different ways.

Solution

(1) Imagine the number line. Counting back from 3 to 0 takes 3 steps. That leaves 5 steps to go. So $3-8=-5$.

$$\cdots,\text{-}7,\text{-}6,\text{-}\underbrace{5,\text{-}4,\text{-}3,\text{-}2,\text{-}1,}_{\text{5 steps}}\ \mathbf{0}\ \underbrace{1,\ 2,\ 3}_{\text{3 steps}},\ 4,\ 5,\cdots$$

So $3-8=-5$.

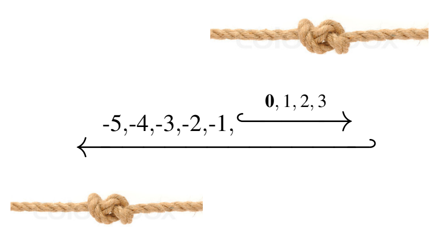
(2) Think of -8 as $-5-3$. Then substituting $-3-5$ for -8, produces

$$\begin{aligned} 3-8 &= 3-3-5 \\ &= -5. \end{aligned}$$

The diagram below goes well with this way of thinking.

$$\cdots,\text{-}7,\text{-}6,\text{-}5,\xleftarrow{-5}\mathbf{0}\xleftarrow{-3}3,\ 4,\ 5,\cdots$$

(3) Tug of war. 3 right, then 8 left. Ends at -5.



Would the tug of war have the same result if the order of events were: 8 left, then 3 right?

(4) This may be the easiest method.

(a) Judge whether the result will be positive or negative. For $3-8$ result will be **negative**,

(b) Find the positive difference of the numbers. Here it is $8-3=5$,

(c) Write the number from step (b) with a negative sign if indicated at step (a). $\therefore 3-8=-5$.

The symbol $\therefore$ means "Therefore".

Exercise 2.2

Subtract as required.

1. $6-9$.
2. $1-7$.
3. $2-3$.
4. $5-7$.
5. $-3-4$.
6. $-2-14$.
7. $0-6$.
8. $8-17$.
9. $-1-4$.
10. $12-28$.

Cases $5-8$ will require some work. However, once we figure out one of the cases, the remaining three cases will go quickly.

Case 5: $a+(-b)$
Let us explore several specific examples of this case.

Example 2.5

What number is $3+(-2)$?

Solution

It is in no way clear how to count on by -2. Instead, we reason as follows.

$$\begin{aligned} 3+(-2) &= (1+2)+(-2)), && \text{substitution, } 3=1+2. \\ &= 1+(2+(-2)), && \text{associative.} \\ &= 1+0, && \text{inverse elements.} \\ &= 1, && \text{identity element.} \end{aligned}$$

Conclusion, $3+(-2)=1$. Let's note that $3-2=1$, too.

Example 2.6

What number is $7+(-4)$?

Solution

$$
\begin{aligned}
7+(-4) &= (3+4)+(-4), && \text{substitution, } 7=3+4.\\
&= 3+(4+(-4)), && \text{associative.}\\
&= 3+0, && \text{inverse elements.}\\
&= 3, && \text{identity element.}
\end{aligned}
$$

Conclusion, $7+(-4)=3$. Similar to previous example, note that that $7-4=3$. ■

Could $a+(-b)$ mean $a-b$? Our two examples certainly point in that direction!

Let us see if we can use the same reasoning for *any* pair of positive numbers a and b.

Idea 1.
Let a and b be positive numbers. Then $a+(-b)=a-b$.

Reasoning. Let a and b be positive numbers. Suppose there is a number, call it c, such that $a-b=c$. This means that $a=c+b$. We are going to first substitute $c+b$ for a. At the end, we will substitute $a-b$ for c.

$$
\begin{aligned}
a+(-b) &= (c+b)+(-b), && \text{substitution, } a=c+b.\\
&= c+(b+(-b)), && \text{associative.}\\
&= c+0, && \text{inverse elements.}\\
&= c, && \text{identity element.}\\
&= a-b, && \text{substitution, } c=a-b.
\end{aligned}
$$

Therefore.

$$a+(-b)=a-b. \tag{2.1}$$ ■

In other words, adding the additive inverse of a positive number means subtracting the number. And, subtracting a positive number means adding the additive inverse of the number.

Example 2.7

Find the sum $10+(-3)$.

Solution

$$10+(-3)=10-3=7. \quad \text{Using Idea 1.}$$ ■

Not only have we disposed of Case 5, but we have a good start on Case 6. First, let's note that additive inverses come in pairs. In other words $-a$ is the additive inverse of a and a is the additive inverse of $-a$.

Case 6: $a-(-b)$

This is the subtraction of the integer $(-b)$. But, according to equation (2.1), that is accomplished by adding the additive inverse of $-b$. Since the additive inverse of $-b$ is b, $a-(-b)=a+b$. The familiar computation $a+b$ takes the place of the unfamiliar computation $a-(-b)$. This reasoning leads to Idea 2.

Idea 2. Let a and b represent any positive numbers.
Then $a-(-b)=a+b$. ■

In other words, subtracting the additive inverse of a positive number means adding the number. And, adding a positive number means subtracting the additive inverse of the number.

Case 7: $(-a)+(-b)$
Case 8: $(-a)-(-b)$

We treat cases 7 and 8 respectively using the following two ideas. The reasoning for these two ideas is left to the reader in the exercises for this section.

Idea 3. Let a and b represent any positive numbers.
Then $(-a)+(-b)=(-a)-b$.

Idea 4. Let a and b represent any positive numbers.
Then $(-a)-(-b)=(-a)+b$.

2.5.1. No one likes eight cases

The eight cases, though instructive, make this topic seem more complicated than it is. Let us see if we can pare them down.

All eight cases fell to the four ideas. Perhaps there is some common idea that is present in all four ideas If so, maybe all we really need is that common idea. The ideas are listed below for reference.

For a and b positive numbers,

(1) $a+(-b)=a-b$.

(2) $a-(-b)=a+b$.
(3) $(-a)+(-b)=(-a)-b$.
(4) $(-a)-(-b)=(-a)+b$.

In each of these four equations, subtraction appears on one side and addition on the other side. If we use the symmetric property of equality, we can make this very visible.

$$a-b=a+(-b). \tag{2.2}$$
$$a-(-b)=a+b. \tag{2.3}$$
$$(-a)-b=(-a)+(-b). \tag{2.4}$$
$$(-a)-(-b)=(-a)+b. \tag{2.5}$$
(2.6)

The idea that is common to all four almost jumps off the page! Subtraction *is* addition of the additive inverse. We commemorate this insight in the following definition.

Definition 2.3 (Subtraction)
For any numbers x and y,

$$x-y=x+(-y). \quad \blacksquare$$

The "other view" of subtraction, promised earlier.

It is interesting to ask where the four ideas acquired their common feature. Let's return to the reasoning provided for Idea 1. It began like this:

> **Reasoning.** Let a and b be positive numbers. Suppose there is a number, call it c, such that $a-b=c$. This means that $a=c+b$. We are going to first substitute $c+b$ for a. At the end, we will substitute $a-b$ for c.

The key idea is that

$$a-b=c \text{ means that } a=c+b.$$

When you were quite young, you answered questions like "What is $7-5$?" by saying "$7-5=2$, because $2+5=7$." You already knew that that "if b added to c produces a, then subtracting b from a produces c". That is the idea that defines subtraction.

Example 2.8

Evaluate $12-20$.

Solution

First notice that the result will be negative. Then $20-12=8$, so the result is $12-20=-8$.

Example 2.9

Evaluate $3-(-8)$.

Solution

$$\begin{aligned} 3-(-8) &= 3+8, && \text{add the additive inverse of } -8. \\ &= 11. \\ \therefore\ 3-(-8) &= 11. \end{aligned}$$

Example 2.10

Evaluate $3+(-8)$.

Solution

$$\begin{aligned} 3+(-8) &= 3-8 \\ &= -5. \\ \therefore\ 3+(-8) &= -5. \end{aligned}$$

Example 2.11

Evaluate $10-7\times 2$.

Solution

$$\begin{aligned} 10-7\times 2 &= 10-14 \\ &= -4. \\ \therefore\ 10-7\times 2 &= -4. \end{aligned}$$

Example 2.12

Evaluate $((-3)-(-7))\times 5$.

Solution

$$\begin{aligned} ((-3)-(-7))\times 5 &= ((-3)+7)\times 5. \\ &= 4\times 5. \\ \therefore\ ((-3)-(-7))\times 5 &= 20. \end{aligned}$$

Exercise 2.3

1. Provide reasoning for Idea 3, which says for any positive numbers a and b, $(-a)+(-b)=(-a)-b$.
2. Provide reasoning for Idea 4: for any positive numbers a and b, $(-a)-(-b)=(-a)+b$.

[Hint for (1) and (2): see reasoning for Ideas 1 and 2]

Exercise 2.4

Evaluate.

1. $4+(-2)+3$
2. $3+(-3)-4$
3. $2\times 2-2$
4. $1-(-2)+4$
5. $(2-1)\times 3$
6. $3-2-(-4)$
7. $4-(2-1)$
8. $-1-(-4-4)$
9. $4-(1-3)$
10. $-4+(-2)+1$
11. $-4-(3+3)$
12. $2\times 4-(-4)$
13. $-3-(-1-4)$
14. $4+(-4)-(-3)$
15. $4-(2-(-4))$
16. $-1-(1-(-3))$
17. $-1-2-(-1)$
18. $1-(4-(-3))$
19. $-1-2-3$
20. $-1+3-(-1)$
21. $-2+(-3)+3$
22. $-1-(-3+1)$
23. $-3+(-1)-2$
24. $3-2-4$
25. $1-(-1)-(-4)$
26. $-3-(-1-1)$
27. $4-(-3-1)$
28. $1+2-(-3)$
29. $-4-(2-3)$
30. $2-1-(-3)$

2.5.2. The additive inverse of the additive inverse

It is a fact that the additive inverse of the additive inverse of a number is that number. Any fact so much fun to say deserves to be a theorem.

Theorem 2.1
For any number a, $-(-a) = a$.

Proof. Let a represent any number.

$$\begin{aligned} -(-a) &= 0 + (-(-a)), && \text{identity element.} \\ &= (a + (-a)) + (-(-a)), && \text{inverse element.} \\ &= a + ((-a) + (-(-a)), && \text{associative.} \\ &= a + 0, && \text{inverse elements.} \\ &= a, && \text{identity element.} \end{aligned}$$

Therefore for any number a, $-(-a) = a$. ∎

2.5.3. Is $-a$ necessarily a negative number?

It is tempting to think that $-a$ is a negative number, because we see the symbol "$-$" in front of a. In nearly all of the proofs we have produced and the problems we have solved, the phrase "Let a represent any number" appears. If a is a positive number, for instance 5, then $-a$ is -5, the additive inverse of 5. In this case, a is negative number. But, a is supposed to represent *any* number. So, a represents negative as well as positive numbers. Suppose, for example, a represents -3. Then $-a$ is the additive inverse of -3 and the additive inverse of -3 is 3. In this case, $-a$ is a positive number. Moral: do not fall into the trap of thinking $-a$ must be a negative number. $-a$ is positive when a is negative and negative when a is positive. By the way, Who is -0? This should be the additive inverse of 0. And that is 0, because $0+0=0$. 0 is its own additive inverse. We write 0, not -0, for the additive inverse of 0.

This is surely worth a couple of points on the SAT.

Example 2.13

Several simplifications that use Theorem (2.1) follow.

(1) $-(-5) = 5$.
(2) $-(-(-5)) = -5$.
(3) $-(-(-(-5))) = 5$.
(4) $-(-(-(-(-5)))) = -5$.

Example 2.14

Suppose $a = -2$, what number is $-a$?

Solution

Since $a = -2$, $-a = -(-2) = 2$.

Exercise 2.5

[Part 1] *Simplify.*

1. $-(-7)$
2. $-(-113)$
3. $-(-9)$
4. $15 - (-(-(-3)))$

[Part 2] *Answer the following.*

1. In section (2.5), we discussed cases $a - b$ and $a + b$, for a and b positive numbers. If we let b represent *any* number, do we still need both $a - b$ and $a + b$? Explain.
2. Simplify $14 - a$, if $a = -7$.
3. Simplify $134 - (-a)$, if $a = -1$.

2.6. Multiplication with integers

Let us briefly review how we have arrived here. The natural numbers allowed one to pose the question "What is 5 subtract 6"? But the natural numbers could not provide the answer. This question led to the discovery of numbers that had previously gone unnoticed. Those numbers are the additive inverses of the natural numbers. We extended the natural numbers to the integers $-$the set that contains all the natural numbers, all their additive inverses, and 0. We then investigated how the operations of addition and subtraction should behave in the integers. It is now time to investigate multiplication in the integers.

2.6.1. First, some useful notation

There are a variety of ways to write multiplication. In levels above arithmetic, the sign "$\times$" is avoided. A few examples will get you started on the alternative notations for multiplication.

Let a and b be any numbers. Then multiplication may be indicated in the following ways.

Letter times letter

(1) ab means $a \times b$, using juxtaposition.
(2) $a \cdot b$ means $a \times b$, using dot.
(3) $(a)(b)$ means $a \times b$, using parentheses.

Number times letter or number times expression in parentheses.

(1) $3a$ means $3 \times a$.
(2) $3 \cdot a$ means $3 \times a$.
(3) $(3)(a)$ means $3 \times a$.
(4) $3(a)$ means $3 \times a$.
(5) $3(a+b)$ means $3 \times (a+b)$.

Number times number

(1) $3 \cdot 5$ means 3×5.
(2) $(3)(2)$ means 3×2.
(3) $3(2)$ means 3×2.

A variety of factors

(1) $3(x+y)$ means $3 \times (x+y)$.
(2) $7a(2x+3y) - 8x$ means $7 \times a \times (2 \times x + 3 \times y) - 8 \times x$.

This last example makes our new notation very appealing.

Exercise 2.6

[Part 1] *Rewrite each expression using juxtaposition instead of the "$\times$" symbol.*

1. $7 \times b$
2. $9 \times x$
3. $a \times d$
4. $2 \times a \times b$
5. $5 \times (x+2)$
6. $9a \times (3y+4)+2$
7. $5 \times (a+b)$
8. $7 \times (a+1)$
9. $7 \times (2 \times a+d)$
10. $2 \times (3 \times a+4)$
11. $6 \times (2 \times x-y)$
12. $-9 \times (5 \times x-2 \times y)$

[Part 2] *Rewrite the following using only the "$\times$" symbol for multiplication.*

1. $3(2a+5)$
2. $4abc$
3. xy
4. $xy(2y+5x)$
5. $3x(5x+3y+7)$

2.6.2. Distribution (rewriting a product as a sum)

Multiplication of integers will raise questions similar to those occasioned by addition of integers. For example, What is the product $(-5)(-6)$? Researching this and similar questions requires that we understand another fundamental idea. To introduce this idea, let us answer the question What is the product $5(3+4)$? There are two ways to perform this computation.

(1) first way

$$\begin{aligned} 5(3+4) &= 5(7) \\ &= 35. \end{aligned}$$

(2) second way

$$\begin{aligned} 5(3+4) &= 5(3)+5(4) \\ &= 15+20 \\ &= 35. \end{aligned}$$

Another example,

(1) first way

$$\begin{aligned} 2(5+7) &= 2(12) \\ &= 24. \end{aligned}$$

(2) second way

$$\begin{aligned} 2(5+7) &= 2(5)+2(7) \\ &= 10+14 \\ &= 24. \end{aligned}$$

The reader is perhaps thinking the author has lost his wits, because who would compute the second way when the first is easier?

The author would answer by saying "Everybody! Even the reader has been using the second way for years."

The following scheme for computing $5(37)$ is familiar to you.

$$\begin{array}{r} 37 \\ \times 5 \\ \hline 35 \\ 150 \\ \hline 185 \end{array}$$

Now let's annotate it.

$$\begin{array}{rl} 37 & \\ \times 5 & \\ \hline 35 & \leftarrow 5(7) \\ 150 & \leftarrow 5(30) \\ \hline 185 & \leftarrow 5(7)+5(30) \end{array}$$

This is exactly the "second way"! Thinking of 37 as $30+7$,

$$\begin{aligned} 5(37) &= \mathbf{5}(\mathbf{7}+\mathbf{30}) \\ &= \mathbf{5}(\mathbf{7})+\mathbf{5}(\mathbf{30}) \\ &= 35+150 \\ &= 185. \end{aligned}$$

The official name for what we have been calling the “second way” is “**distribution**”.

Distribution. Let a, b and c represent any numbers, then

$$a(b+c) = ab+ac. \quad \blacksquare$$

Eventually, we will collect the several “fundamental ideas” that we have met, christen them “Axioms”, and mention that those few ideas are the basis of all the algebraic procedures you will ever learn. Distribution will be among those ideas.

The verb “distribute” is often replaced by the verb “**expand**”. For example, the command “Expand $3(x+6)$” is obeyed by writing $3x+18$. In this example, $3x+18$ is called the “**expanded form**” of $3(x+6)$.

Example 2.15

Rewrite each of the following products by distribution.

(1) $2(a+b)$
(2) $5(2+y)$
(3) $3(4x+5)$
(4) $10(2a+3b)$

Solution

(1) $2a+2b$
(2) $10+5y$
(3) $12x+15$
(4) $20a+30b)$

Example 2.16

Expand the following.

(1) $3(x+y)$
(2) $4(3+x)$
(3) $2(3x-2)$
(4) $5(3a+2b)$

Solution

(1) $3x+3y$
(2) $12+4x$
(3) $6x-4$
(4) $15a+10b)$

Exercise 2.7

[Part 1] *Expand the following.*

1. $a(b+c)$
2. $3(a+2)$
3. $4(x+y)$
4. $2(y+5)$
5. $7(a+3b)$
6. $6(2+x)$
7. $5(2a+3b)$
8. $11(x+2)$

[Part 2] *It is a fact that multiplication distributes over the sum of any number of terms. You will not be able to prove this until some years from now. But we can prove it for a sum of three terms, a sum of four terms, and so on.*

1. Prove that $a(b+c+d) = ab+ac+ad$.
2. Prove that $a(b+c+d+e) = ab+ac+ad+ae$.

2.6.3. Sign of the product

Since there are three flavors of integers (negative, zero, positive), there are several cases to consider. If $a, b \in \mathbb{Z}^+$, the cases are:

Case 1: multiplication by 0, $a \cdot 0$ and $0 \cdot a$
Case 2: $a \cdot b$
Case 3: $-a \cdot b$ and $a \cdot (-b)$
Case 4: $(-a) \cdot (-b)$

Case 1. Everyone knows that the product of 0 and a number is 0. But, just out of interest, let's see if we can prove that this is so.

Theorem 2.2
Let a be any number. Then $a \cdot 0 = 0$ and $0 \cdot a = 0$.

Proof. Let a be any number.

$1 = 1,$	equality is reflexive.
$1 = 1+0,$	identity element.
$a \cdot 1 = a(1+0),$	theorem 1.2.
$a \cdot 1 = a \cdot 1 + a \cdot 0,$	distribution.
$a = a + a \cdot 0,$	identity element.
$(-a)+a = (-a)+(a+a \cdot 0),$	theorem 1.2.
$(-a)+a = (-a+a)+a \cdot 0,$	associative.
$0 = 0 + a \cdot 0,$	inverse elements.
$0 = a \cdot 0,$	identity element.
$a \cdot 0 = 0,$	equality is symmetric.

■

Well, we proved we can prove it. But it was tedious. For the time being, the proofs you see here and any you write will be tedious. That is because at this stage of your learning, you will benefit from the labor of seeing and writing each application of a theorem or fundamental idea. Soon, though, you will be so familiar with the theorems and fundamental ideas, that there will not be much benefit in continuing with this degree of detail. When that time is reached, the proof above will look more like this:

Proof. Let a be any number.

No step has been "skipped", but many have been combined.

$$a = a(1+0).$$
$$a = a + a \cdot 0.$$
$$0 = a \cdot 0.$$

■

Remark 2.1
The proof of theorem (2.2) is different than those you have seen before. Until this proof, we started with one side of an equation, say the LHS (Left Hand Side), and then we rewrote that side of the equation until it appeared exactly the same as the RHS (Right Hand Side). In this book, this kind of proof is called a "cross the road" proof. In the proof of theorem (2.2), we began with a true equation, then wrote a sequence of equivalent equations, until we produced the equation we desired to prove. The two strategies are outlined

below.

First strategy	Second strategy
$LHS =$	first equation
$=$	second equation
$=$	third equation
$=$	fourth equation
$\vdots$	$\vdots$
$= RHS$	desired equation

Case 2: $a \cdot b$. This you have known forever.

Case 3. $a \cdot (-b) = -(ab)$ and $(-a) \cdot (b) = -(ab)$.

The first part of case 3 says $a \cdot (-b) = -(ab)$. Showing that this is true amounts to proving the following theorem. The proof of the first part, $a \cdot (-b) = -(ab)$, is provided. The proof of the second part, $(-a) \cdot (b) = -(ab)$, is left as an exercise.

Theorem 2.3
Let a and b be any numbers. Then

$$a \cdot (-b) = -(ab) \text{ and } (-a) \cdot (b) = -(ab).$$

Proof.
First part, $a \cdot (-b) = -(ab)$

$$
\begin{aligned}
b + (-b) &= 0. \\
a(b + (-b)) &= 0. \\
ab + a \cdot (-b) &= 0, && \text{distribution.} \\
-(ab) + ab + a \cdot (-b) &= -(ab). \\
a \cdot (-b) &= -(ab).
\end{aligned}
$$

■

Exercise 2.8

1. Prove the other part of Theorem (2.2); that is, if a is any number, then $0 \cdot a = 0$. [Hint: Since the first part of Theorem (2.2) has been proved, you may use it.]
2. Prove the second part of Theorem (2.3); that is, for any numbers a and b, $(-a) \cdot (b) = -(ab)$. [Hint: see proof of the first part of Theorem (2.3).]

Case 4. $(-a) \cdot (-b)$

Theorem 2.4
Let a and b be any numbers. Then

$$(-a) \cdot (-b) = ab.$$

Proof. Let a and b be any numbers.

$$\begin{aligned}
b + (-b) &= 0. \\
-a(b + (-b)) &= 0. \\
(-a)b + (-a)(-b) &= 0, && \text{distribution.} \\
-(ab) + (-a)(-b) &= 0, && \text{theorem 2.3.} \\
(-a)(-b) &= ab.
\end{aligned}$$

■

Several examples illustrate the application of the last four theorems.

Example 2.17

Simplify each of the following.

(1) $3 \cdot (-5)$
(2) $-2 \cdot (-3)$
(3) $-5 \cdot (-5)$
(4) $-6 \cdot x$
(5) $-9(-y)$
(6) $-7(-1)$

Solution

(1) -15
(2) 6
(3) 25
(4) $-6x$
(5) $9y$
(6) 7

Example 2.18

Expand $3(x-4)$.

Solution

$$3(x-4) = 3x-(3)(4) \qquad \text{think "3 times } \textit{negative} \text{ 4".}$$
$$= 3x \overset{\star}{-} 12.$$

The ⋆ symbol marks the sign that causes trouble for beginners. Always check your work when distributing a negative number or distributing over a difference.

Example 2.19

Expand $-5(a+3)$.

Solution

$$-5(a+3) = -5a \overset{\star}{-} 15. \qquad \text{Think "} \textit{negative} \text{ 5 times 3".}$$

Example 2.20

Expand $-7(2a-6)$.

Solution

$$-7(2a-6) = -14a \overset{\star}{+} 42. \qquad \text{Think "} \textit{negative} \text{ 7 times } \textit{negative} \text{ 6".}$$

Example 2.21

Simplify each of the following.

(1) $5(a+3)$
(2) $3(x-4)$
(3) $7(x-5)$
(4) $-2(a+6)$
(5) $-3(y-1)$
(6) $-7(x-y)$
(7) $(a+3)(4)$
(8) $(x-5)(3)$
(9) $(x-2)(-8)$
(10) $(-5)(-a-3)$
(11) $-7(-y-4)$
(12) $-2(a-3)$

Solution

(1) $5a+15$
(2) $3x-12$
(3) $7x-35$
(4) $-2a-12$

(5) $-3y+3y$
(6) $-7x+7y$
(7) $4a+12$
(8) $3x-15)$
(9) $-8x+16$
(10) $5a+15$
(11) $7y+28$
(12) $-2a+6$

Example 2.22

Use distribution to rewrite each sum as a product.

(1) $2x+6$
(2) $3y+9$
(3) $15x-5y$
(4) $7x-21$
(5) $-2x-14$
(6) $-9x+27$

Solution

(1) $2(x+3)$
(2) $3(y+3)$
(3) $5(3x-y)$
(4) $7(x-3)$
(5) $-2(x+7)$
(6) $-9(x-3)$

Exercise 2.9

Expand.

1. $5(5n+5)$
2. $-4(1+5a)$
3. $-5(1+2k)$
4. $-2(a-5)$
5. $4(-5x-5)$
6. $-5(n+4)$
7. $5(1+k)$
8. $2(-3p+5)$
9. $2(5x-5)$
10. $4(5-5n)$
11. $-4(5m+3)$
12. $-3(-4+5r)$
13. $5(x+4)$
14. $4(1+n)$
15. $-3(3+4b)$
16. $2(1-4r)$
17. $5(1+2x)$
18. $4(x-1)$
19. $-3(4+4a)$
20. $-4(p+4)$
21. $-4(2+3x)$
22. $-5(-5n+3)$
23. $3(m+2)$
24. $-5(r+3)$
25. $5(4x+3)$
26. $5(4n+3)$
27. $2(1-2v)$
28. $3(-5x+2)$
29. $3(3n+2)$
30. $2(4a+2)$
31. $2(5k+1)$
32. $5(5+2x)$
33. $2(x+2)$
34. $2(-2+2n)$
35. $3(-5+m)$
36. $-2(4v+5)$
37. $5(3n-4)$
38. $2(-1+5)$
39. $-5(3-2m)$
40. $-2(4r+1)$

Exercise 2.10

Expand.

1. $-(2+2n)$
2. $-4(-2b+2)$
3. $-2(4+4n)$
4. $-2(x+3)$
5. $-4(x+1)$
6. $-3(5+a)$
7. $-4(5k+1)$
8. $-2(2p+1)$
9. $-4(-x+1)$
10. $-5(n-4)$
11. $-3(1+3m)$
12. $-4(3r-2)$
13. $-(5-3x)$
14. $-2(5-n)$
15. $-4(b+2)$
16. $-2(1-v)$
17. $-5(-4x-4)$
18. $-5(3-n)$
19. $-3(4a-1)$
20. $-5(5k-1)$
21. $-4(1+5x)$
22. $3(-1-2n)$
23. $-3(k-1)$
24. $-3(4-2p)$
25. $-3(1+x)$
26. $-3(5-4x)$
27. $-(4r-3)$
28. $-4(-x-3)$
29. $-2(-4n-3)$
30. $-3(v+2)$
31. $-4(5x-3)$
32. $-2(3x-3)$
33. $-4(4a-4)$
34. $-5(1-3p)$
35. $-5(5x-4)$
36. $-2(1-4n)$
37. $-5(-1-4m)$
38. $-3(3r-4)$
39. $-5(x+3)$
40. $-3(n-5)$

2.6.4. Questions you may wish to discuss in class

(1) Are the parentheses in the expression $(x+3)+(a+2)$ necessary?
(2) Must one always work a sum from left to right?
(3) Suppose $10+8-3+2$ is rewritten as $10+8+(-3)+2$. Then must the computation be performed left to right?
(4) Must you perform the computation $10+8-3+2$ from left to right?
(5) Suppose $10+8-3+2$ is rewritten as $10+(8-3)+2$. Then must the computation be performed left to right?
(6) Suppose a classmate says "It is OK to rewrite in a different order an expression that includes subtraction, but you have to move a number and the '$-$' sign in front of it together. For example, $9+10-7+2 = 9-7+10+2$. The '-7' moved as a block." Is this correct?
(7) Using distribution, $5(2+3) = 10+15$. Does this violate the rules you learned about order of operations?

The author hopes your class concludes that the answers to section (2.6.4) #2 and #4 are "No". In practice, people who do mental arithmetic well, take advantage of the fact that that expressions including only sums and differences need not be evaluated left to right. In the following example, first try to find pairs of numbers that add to ten or five, then add the tens and fives.

Example 2.23

Find the sum: $7+2+9+9+6+7+3+4+3+8+1$.
Solution.

$$\cancel{7}+2+9+9+6+7+\cancel{3}+4+3+8+1 \quad \text{think one 10}$$
$$\cancel{7}+\cancel{2}+9+9+6+7+\cancel{3}+4+3+\cancel{8}+1 \quad \text{think two 10's}$$
$$\cancel{7}+\cancel{2}+\cancel{9}+9+6+7+\cancel{3}+4+3+\cancel{8}+\cancel{1} \quad \text{think three 10's}$$
$$\cancel{7}+\cancel{2}+\cancel{9}+9+\cancel{6}+7+\cancel{3}+\cancel{4}+3+\cancel{8}+\cancel{1} \quad \text{think four 10's}$$
$$\cancel{7}+\cancel{2}+\cancel{9}+9+\cancel{6}+\cancel{7}+\cancel{3}+\cancel{4}+\cancel{3}+\cancel{8}+\cancel{1} \quad \text{think five 10's}$$

Perhaps the reader will make up some similar sums for amusement. Do think back to this later when combining like terms is discussed.

Five tens plus 9 is 59. Multiple lines were used to show the sequence. In practice, it would be a single line with multiple cancellations and mental additions.

2.7. Like terms

First some terminology. We call items in a sum (or in a difference) "**terms**". For example.

Yup. Pun.

(1) "$3a+2a+5a+a$". Four terms: $3a, 2a, 5a$ and a.
(2) "$3a+2a+5b+8x$". Four terms: $3a, 2a, 5b$ and $8x$.
(3) "$3a+2a+11a$". Three terms: $3a, 2a$ and $11a$.

(4) “$3a - 12a + 11a$”. Three terms: $3a, (-12a)$ and $11a$.
(5) “$7a + 2a$’. Two terms: $7a$ and $2a$.
(6) “$7a \times 2a$”. One term.

A sum or difference of one or more terms is called a **polynomial**. Polynomials consisting of one, two or three terms are often called by special names shown in Table (2.4).

Number of terms	Special name	Examples
1	monomial	$2a$, $3x$, 7, a
2	binomial	$x+2$, $b-7$, $3x+3y$
3	trinomial	$a+b+c$, $2x-y+1$

TABLE 2.4. Table (Polynomials having special names)

Perhaps you have heard the expression “You cannot add apples and oranges.” Someone is sure to point out that one can add 3 apples and 6 oranges to get 9 fruit. Granted. But if we wish the sum to be some number of apples or some number of oranges, then we are at a loss as to what is the sum of 3 apples plus 6 oranges. Similarly, the sum of 3 a’s plus 6 b’s is neither a number of a’s nor a number of b’s.

On the other hand,

3 apples plus 7 apples is 10 apples,
3 horses plus 7 horses is 10 horses,
3 planets plus 7 planets is 10 planets,
$3a$ plus $7a$ is $10a$,
$3b$ plus $7b$ is $10b$.

We call the terms “3 apples” and “ 10 apples” “**like terms**”. The following are “unlike terms”,

3 horses and 7 sheep,
3 planets and 7 oven-mitts,
$3a$ and $7b$,
$3x$ and $7y$.

Rule. *You may always combine like terms by addition or subtraction. You may never combine unlike terms by addition or subtraction.* Add apples to apples, or some number of a’s to some other number of a’s, but do not add apples to oranges or *a*'s to *b*'s.

No doubt the reader would be better satisfied with an explanation of the like terms rule that is based on the mathematics we have covered rather some old tale about apples and oranges.

Let's try this: $7a$ and $2a$ may be added, because $7a+2a=(7+2)a=9a$. We used distribution. This goes for apples, too. $7\,apples+2\,apples=(7+2)\,apples=9\,apples$.

$7a$ and $2b$ do *n*ot add. Using distribution to obtain "(sum of numbers) $\times$ letter" is impossible, because the same letter does not appear in both terms. $7a+2b=(7+2)\times ?=?$. The following should make this reasoning clear.

$$7a+2a=(7+2)a \tag{2.8a}$$

$$=9a. \tag{2.8b}$$

but

$$7a+2b=(7+2)\underline{\;???\;} \tag{2.9a}$$

$$=\;??? \tag{2.9b}$$

Equation (2.8b) is equation (2.8a) simplified. The key equation, (2.8a), can only be obtained by distribution.

Equation (2.9b) would be equation (2.9a) simplified. But, the key equation, (2.9a), cannot be obtained.

Now that you are convinced that like terms add and subtract, but unlike terms do not, let us consider a few examples.

Example 2.24

The following are simplified by combining like terms.

(1) $8a+3a-2a=9a$.
(2) $6a-11a+3a=-2a$.
(3) $5x+4x=9x$. ■

When several kinds of like terms appear in an expression, each kind is simplified independently of the other kinds. The expression $6a+8b+9a+7a+2b$ contains two kinds of like terms. One kind is in a. The other kind is in b. The expression $6a+8b+9a+7a+2b$ simplifies to $22a+10b$.

We consider a variety of examples of simplification.

Example 2.25

Simplify. [Note: only one kind of term is present.]

(1) $3a+2a$
(2) $6a-5a+3a-10a$
(3) $(2x+5x)+(9x-x)$

Solution

(1) $5a$
(2) $-6a$. No harder than (1), merely more computations.
(3) $(2x+5x)+(9x-x)=2x+5x+9x-x=15x$. The parentheses are unnecessary.

Example 2.26

Simplify. [Note: two kinds of terms are present.]

(1) $3a+2b$
(2) $9a-3a+7b-12b$
(3) $(2x+3)+(9x-1)$

Solution

(1) $3a+2b$ is already simplified.
(2) $6a-5b$
(3) $11x+2$. 3 and -1 are like terms, because each contains the same letter. Namely, no letter.

Example 2.27

Simplify. [Note:several kinds of terms are present.]

(1) $5a+2b+3c$
(2) $12a-3b+5c+10b-a+5c$
(3) $(4x+3y+8z-7)+(3x+2z+9y)$

Solution

(1) $5a+2b+3c$ is already simplified.
(2) $11a+7b+10c$
(3) $(4x+3y+8z-7)+(3x+2z+9y)=7x+12y+10z-7$. ■

There is a good way of keeping track of terms as you combine them. *Look* at the expression $3a+9a+5b+4b+2c+10c$. The first term is in a. Scan across combining terms in a. Keep track of each term you use by *lightly and neatly* striking through it.

If you have ever done a "word search" in history class, you have already done something similar to, but harder than, ferreting out these like terms.

$$
\begin{aligned}
&3a+5b+9a+2c+4b+10\\
&\quad=\cancel{3a}+5b+\cancel{9a}+2c+4b+10c && \}\text{ think } 12a\\
&\quad=\cancel{3a}+5b+\cancel{9a}+\cancel{2c}+4b+\cancel{10c} && \}\text{ think } 12a+12c\\
&\quad=\cancel{3a}+\cancel{5b}+\cancel{9a}+\cancel{2c}+\cancel{4b}+\cancel{10c} && \}\text{ think } 12a+12c+9b\\
&\quad=12a+9b+12c && \}\text{ done}
\end{aligned}
$$

The several lines above show the sequence of computations. In practice, the work would probably look like this:

$$\cancel{3a}+\cancel{5b}+\cancel{9a}+\cancel{2c}+\cancel{4b}+\cancel{10c}=12a+9b+12c.$$

Example 2.28

Simplify.

(1) $3a+2a+8b+9b$
(2) $6a-5a+3b-b$
(3) $12a+3a-3b-5b$
(4) $15b-2a+10b+3a$
(5) $3x+2x+25y+x+9x+16y$
(6) $3a+5b+2a+9c-a-2c$
(7) $(3x+5y)+(2x+8y)$
(8) $(x+5y+3z)+(2x+5z)$
(9) $3x+2+(9x+5)$
(10) $2x+3y$
(11) $2x+8+9y+7$
(12) $x+7+2y-x$

Solution

(1) $5a+17b$
(2) $a+2b$
(3) $15a-8b$
(4) $a+25b$
(5) $15x+41y$
(6) $4a+5b+7c$
(7) $5x+13y$
(8) $3x+5y+8z$
(9) $12x+7$
(10) $2x+3y$, already simplified
(11) $2x+9y+15$,
(12) $2y+7$

Exercise 2.11

Simplify. If an expression is already simplified, say so.

1. $4a - 2a + 5$
2. $3a - 2b + 9b + 12a$
3. $(6x + 5y) + (2x - 3y + 2)$
4. $9a + 2b - 12c + 7 - 2a + 8c - 3b + 2$
5. $(2c + 7b - 5) + (5a + c - 8)$
6. $3x + 7y + 2x + 8z + 2y - 9$
7. $-5x + 2z + 7y - 8 + 3y + 8z - 5x$
8. $2a + 2x - 8c - 8 + 3x - 5x$
9. $2a + 2x + 12c - 7$
10. $5x + 5y + 5z + 5$

2.7.1. More like terms

A single term may be, and in fact often will be, the product of several letters. For example, $3abc$ is a single term. So is $3ab$. Now, are these two like terms? The answer is "no", because like terms must have exactly the same letters. While $3abc$ and $3ab$ are not like terms, $3abc$ and $5abc$ are like terms.

$3abc$	is	one term
$3abc + 9ac$	contains	two unlike terms.
$3abc + 9abc$	contains	two like terms.
$3abc + 9ab - 6bc$	contains	three unlike terms.
$3abc + 9ab - 6abc + 2ab$	contains	two kinds of like terms.

Example 2.29

Simplify each of the following.

(1) $3ab + 2ab + 8b + 9b$
(2) $6ac - 5a + 3ac - 6a$
(3) $12a + 3b - 3ab$
(4) $15ab - 2c + 10ab + 2c + 5$
(5) $3xy + 2xyz + 25y + x + 9y + 16xy$
(6) $3abd + 5abc + 2ab + 9bc$
(7) $2xyz + 3xy - xyz + 9xy - 2$

Solution

(1) already simplified
(2) $9ac - 11a$
(3) already simplified
(4) $25ab + 5$
(5) $x + 19xy + 2xyz + 34y$
(6) already simplified
(7) $xyz + 12xy - 2$ ■

There is a nice notation for writing the product of several identical factors. For example, the product $3 \cdot 3$ is written 3^2 and $5 \cdot 5 \cdot 5$ is written 5^3. Similarly, the product of seven factors of x is written x^7. The following are all different terms and cannot be combined by addition or subtraction: $3x^2$, $3x^3$, $3x^4$. But, $3x^2$ and $5x^2$ are like terms, so $3x^2 + 5x^2 = 8x^2$. It is always true that like terms contain exactly the same letters with exactly the same exponents.

In 3^6, 3 is called the base and 6 is called the exponent.

Example 2.30

Simplify each of the following.

(1) $5a^3 + 2a^3$
(2) $5a^3 + 2a^4$
(3) $2a^4 + 5a^3 + 10a^4 + 2a$
(4) $2a^4b^3 + 5a^4b^3$
(5) $7a^4b^3 + 2a^4b^2$
(6) $(3x^2 + 7x + 3) + (9x^2 + 3x)$

Solution

(1) $7a^3$
(2) Already simplified
(3) $12a^4 + 5a^3 + 2a$
(4) $7a^4b^3$
(5) Already simplified
(6) $12x^2 + 10x + 3$

Exercise 2.12

Simplify. If an expression is already simplified, say so.

1. $2a^5 + 6a^5$
2. $7x^3y + 9x^3y$
3. $2x^3y + 2x^2y$
4. $4a^3b^3c^2 + 2a^3b^3c^2$
5. $4a^3b^3c^2 + 2a^3b^3c^3$
6. $2x^2 + 2x^2y$

2.7.2. Simplifying expressions that involve distribution

Many expressions will involve like terms and distribution. Such expressions are simplified by first distributing and then combining like terms.

Example 2.31

Simplify each of the following.

(1) $3(x+5)+6x$
(2) $2(a+b)+5a+9b$
(3) $3(a+2b)+8b$
(4) $4(a-3)+6$
(5) $5(-x-4)+3(x-2)$
(6) $3(x^2+y)+3x^2+5y$
(7) $2(a+3b)+5(a+b)$
(8) $3(x^2+x+5)+7(x^2+x+1)$
(9) $a(a+6)+a^2+3a+5$
(10) $2a(a+5)+3a^2+8a-2$
(11) $2a(b+c)+5(ab+ac)$
(12) $3(2b+5c)-6b-15c$

Solution

(1) $3x+15+6x=9x+15$.
(2) $2a+2b+5a+9b=7a+11b$.
(3) $3a+14b$.
(4) $4a-6$.
(5) $7x+y+11z+10xy$.
(6) $-2x-26$.
(7) $7a+11b$.
(8) $10x^2+10x+22$.
(9) $2a^2+9a+5$.
(10) $5a^2+18a-2$.
(11) $7ab+tac$.
(12) 0.

Exercise 2.13

Simplify the following expressions. If an expression is already simplified, say so.

1. $3(a+5)+7a-12$
2. $5(a+b)+3a+9$
3. $7(2a+b)+4a-2b+1$
4. $2(x+3)+5(x-1)$
5. $5(-x+2)+3(x-1)$
6. $6(3-2y)+2(y+7)$
7. $11(-2x-1)+2(3x+5)$
8. $2(3a+2b-5)+5(2a+b-1)$
9. $3(a^2+a+7)+5(a^2+a+6)$
10. $b(a+b+2)+a(2a+3b-1)$
11. $4(a^3b+a^2b+b)+3a^3+5a^2b+3b-7$

Exercise 2.14

Simplify.

1. $10(m-9)-2$
2. $10(1-5r)+3r$
3. $-(5+10x)-10x$
4. $3n+3(-6+10)$
5. $-5+8(-2+9b)$
6. $9(v-7)-9v$
7. $-4x+8(1+10x)$
8. $-4(a-2)+10$
9. $-6(3+2r)+5$
10. $-7+4(5+9k)$
11. $-3(1+4x)+3x$
12. $8(v+7)-9$
13. $1+7(x+7)$
14. $-2(6m-1)-9$
15. $5+10(8v-1)$
16. $-9(9n-3)+5$
17. $5(-1-2b)-4b$
18. $-2n+2(-8+n)$
19. $5(9-5k)-1$
20. $-8+4(-10+10x)$
21. $-5(b+6)-6$
22. $-8a+10(1+10a)$
23. $-3(3x+2)+2$
24. $6n+4(-1-3n)$
25. $-3(2+x)-8x$
26. $5(-4n+1)-2$
27. $-3+6(1-k)$
28. $10(p+3)+10$
29. $-2x+9(-2-6x)$
30. $4(8-2n)-7$
31. $-7+3(5m-1)$
32. $7(2-9r)+5$
33. $-(-2k-10)+1$
34. $-3+3(1-n)$
35. $-10(b+4)-9$
36. $-4(8v-10)-7v$
37. $6(-10-10x)-10x$
38. $10(n+7)+5$
39. $10(a+10)+9a$
40. $-7(10k+2)+2k$
41. $8(8p+4)+2(8p+9)$
42. $-(3x-6)+2(-8+x)$
43. $-4(n+2)+6(-4n+1)$
44. $6(4-2m)+10(1-m)$
45. $-7(-3r-4)+7(1-5r)$
46. $6(6+v)+2(1-5v)$
47. $7(-6n+9)+3(n+7)$
48. $-4(1+6b)+7(7b-1)$
49. $8(5x+10)+8(x+4)$
50. $-10(5m+1)+6(8-7m)$
51. $4(-8n-5)+3(-9n-8)$
52. $-7(2n-7)+5(-4-6n)$
53. $2(x+2)+7(5x+10)$
54. $-9(7x-7)+7(2+5x)$
55. $2(-10v+8)+7(v-2)$
56. $-4(2+n)+3(8n-5)$
57. $4(-1-10k)+2(k+6)$
58. $-8(7a+3)+7(10a+6)$
59. $-2(b-10)+3(1+9b)$
60. $-10(8n+10)+3(n+8)$

2.7.3. Expressions that cause trouble

We have avoided these two bad-actors for as long as we could, but their time has arrived. They are expressions like $10-2(x+3)$ and its even worse cousin, $10-(x+3)$. Needless to say, a whole army of these can be produced merely by substituting different numbers for 10, 2, and 3.

Suppose we wish to rewrite without parentheses the expression

$$10-2(x+3). \tag{2.10}$$

The presence of the leading 10 is complicating matters. If the 10 were absent, the expression would be

$$-2(x+3).$$

This would cause no trouble. We know

$$-2(x+3)=-2x-6.$$

So, now we are essentially done. We need only realize that equation (2.10) is equivalent to

$$10-2(x+3)=10-2x-6. \tag{2.11}$$

Replacing 10 in expression (2.10) with any number other than 0, produces no end of fiends like expression (2.10). But, we can take care of them all by generalizing the work that led to equation (2.11). We do so.

Let A, a, b and c be any numbers. Then,

$$-a(b+c)=-ab-ac. \qquad \text{Distribution} \tag{2.12a}$$

$$A-a(b+c)=A-ab-ac. \qquad \text{Theorem 1.2} \tag{2.12b}$$

And this shows that

$$A-a(b+c)=A-ab-ac, \tag{2.13}$$

which we state as a theorem.

Theorem 2.5
For any numbers A, a, b and c, $A-a(b+c)=A-ab-ac$.

Example 2.32

What about the worse cousin, $10-(x+3)$?

Well, theorem 2.5 is true for all numbers, so it is true when $a = 1$. That is,

$$A - 1(b + c)$$

which, by theorem 2.5, equals $A - 1b - 1c = A - b - c$. ■

When you work with expressions like these, be on the lookout for mistakes with the signs. It almost seems that the mission of these expressions is to fiendishly tempt beginners into making sign mistakes. Always pause a moment to check work when these expressions are involved.

The following two examples should be carefully compared to each other.

Example 2.33

Simplify $16 - 3(x + 4)$.

Solution

$$\begin{aligned} A - a(b + c) &= A - ab - ac, & A = 16, a = 3, b = 1, \mathbf{c} = \mathbf{4}. \\ 16 - 3(x + 4) &= 16 - 3x - 12. \\ &= 4 - 3x. \end{aligned}$$

Example 2.34

Simplify $16 - 3(x - 4)$.

Solution

$$\begin{aligned} A - a(b + c) &= A - ab - ac. \\ 16 - 3(x + 4) &= 16 - 3x - (3)(-4), & A = 16, a = 3, b = 1, \mathbf{c} = (-\mathbf{4}). \\ &= 16 - 3x - (-12) \\ &= 16 - 3x + 12 \\ &= 28 - 3x. \end{aligned}$$

In the following examples, the first column is what you write when applying theorem (2.5). The second column is what you might think based on theorem (2.5).

Example 2.35

Simplify $20-2(x+6)$

Solution

$$\begin{aligned} 20-2(x+6) &= 20-2x-12 \quad \textit{think: } (-2)(x) \text{ is} -2x, (-2)(6) \text{ is } -12.. \\ &= 8-2x. \end{aligned}$$

Example 2.36

Simplify $20-2(x-6)$.

Solution

$$\begin{aligned} 20-2(x-6) &= 20-2x+12 \quad \textit{think: } (-2)(x) \text{ is} -2x, (-2)(-6) \text{ is } 12. \\ &= 32-2x. \quad \blacksquare \end{aligned}$$

And for the worse cousin.

Example 2.37

Simplify $8-(x+2)$.

Solution

$$\begin{aligned} 8-(x+2) &= 8-x-2 \qquad \textit{think: } (-1)(x) \text{ is} -x, (-1)(2) \text{ is } -2. \\ &= 6-x. \end{aligned}$$

Example 2.38

Simplify $8-(x-2)$

Solution

$$\begin{aligned} 8-(x-2) &= 8-x+2 \qquad \textit{think: } (-1)(x) \text{ is} -x, (-1)(-2) \text{ is } 2. \\ &= 10-x. \end{aligned}$$

Example 2.39

Simplify $-1-(3x-5)$

Solution

$$\begin{aligned} -2-(3x-5) &= -2-3x+15 \quad \textit{think: } (-1)(3x) \text{ is} -3x, (-1)(-5) \text{ is } 5. \\ &= -3x+3. \end{aligned}$$

Example 2.40

Simplify each of the following.

(1) $15-2(x+5)$
(2) $20-3(x-2)$
(3) $9-2(3x+1)$
(4) $6-3(-x-2)$
(5) $-5-4(x-2)$

Solution

(1) $5-2x$.
(2) $22-3x$.
(3) $6-6x$.
(4) $12+3x$.
(5) $13-4x$.

Exercise 2.15

Simplify.

1. $12-3(a+2)$
2. $7-5(y+1)$
3. $6-3(x-2)$
4. $4-8(x-7)$
5. $-2-(3x+8)$
6. $-5-2(x+5)$
7. $-1-(x-1)$
8. $-1-(x+1)$
9. $12-2(5a-3)$
10. $3+2(x-7)$
11. $2-9(1-x)$
12. $-7-3(8-x)$
13. $2+5(b-3)$
14. $8+2(x-5)$
15. $1-(x-3)$
16. $-20-10(x-20)$
17. $7-(x+7)$
18. $21-(2x-21)$

Exercise 2.16

Simplify

1. $-(-9n-4)+10n$
2. $2a-8(-6-5a)$
3. $-6(8k+8)-7$
4. $-3(5+8x)+6$
5. $6x-6(x-3)$
6. $-8(1+7n)+10n$
7. $2-6(-4+7m)$
8. $-p-6(7p-9)$
9. $7x-(3+6x)$
10. $-7(-n+6)-2n$
11. $-(1+2m)+4m$
12. $-2(2r-2)-10$
13. $3x-9(10x-3)$
14. $-7(7-3n)-6n$
15. $-4(9b-3)+7$
16. $-10(1-9y)+5$
17. $10x-4(x+8)$
18. $3-(5-4x)$
19. $-2(2+9a)-5$
20. $8k-10(10k+9)$
21. $-8(7+8p)-1$
22. $-9-5(-3x+8)$
23. $-3(-8n+8)-9n$
24. $3m-10(-4m+7)$
25. $-5-8(-r+7)$
26. $-5(3+7x)+8$
27. $4n-4(n-2)$
28. $9b-6(b+2)$
29. $4-6(5v-2)$
30. $-4(x-2)-4x$
31. $9n-(5-2n)$
32. $9-9(a-3)$
33. $-8-6(3-3k)$
34. $-4(10-3x)+5x$
35. $4x-(7x-4)$
36. $-4-9(5n+9)$
37. $-6(10m+9)+9$
38. $3p-6(1-7p)$
39. $-8x-10(x+9)$
40. $-7(-8n+8)+5$

The following examples involve expressions that are more complicated, but require no new ideas or techniques.

Example 2.41

Simplify $8-3(x+3)+2(2x+9)$.

Solution

$$\begin{aligned}8-3(x+3)+2(2x+9) &= 8\overbrace{-3x-9}^{\text{1st dist}}\underbrace{+4x+18}_{\text{2nd dist}}\\ &= x+17.\end{aligned}$$

Example 2.42

Simplify $7(a-2)-5(2a+3)$.

Solution

$$\begin{aligned}7(a-2)-5(2a+3) &= \overbrace{7a-14}^{\text{1st dist}}\underbrace{-10a-15}_{\text{2nd dist}}\\ &= -3a-29.\end{aligned}$$

Example 2.43

Simplify $9-3(a-5)-8(a+1)$.

Solution

$$\begin{aligned}9-3(a-5)-8(a+1) &= 9\overbrace{-3a+15}^{\text{1st dist}}\underbrace{-8a-8}_{\text{2nd dist}}\\ &= -11a+16.\end{aligned}$$

Example 2.44

Simplify $14-(a+6)-5(a-3)+4(a-2)$.

Solution

$$\begin{aligned} 14-(a+6)-5(a-3)+4(a-2) &= 14\overbrace{-a-6}^{\text{1st dist}}\underbrace{-5a+15}_{\text{2nd dist}}\overbrace{+4a-8}^{\text{3rd dist}} \\ &= -2a+15. \end{aligned}$$

Have the simplifications in these last examples been “harder” than those of previous examples? These examples involved no new ideas, strategies, or techniques. Perhaps they are no harder than previous “simplify” problems. They are, however, *more complicated* than the simplifications in earlier examples. You do the same old stuff, just more of it on these last four problems.

Exercise 2.17

Simplify.

1. $-8(7m-4)-4(3+4m)$
2. $-3(r+5)-5(r-6)$
3. $-(-6x-3)-7(1+7x)$
4. $-8(n+5)-5(5n+7)$
5. $-4(b-7)-5(6+7b)$
6. $-(2v-7)-4(1+7v)$
7. $-6(4x+3)-4(-3x+6)$
8. $-8(x-8)-3(1+6x)$
9. $-4(8a-3)-6(a-4)$
10. $-8(-6k+2)-8(1-8k)$
11. $-2(7p-8)-2(3p+5)$
12. $-8(1-5x)-2(8+5x)$
13. $-5(3-5n)-3(1+3n)$
14. $-8(-7+5m)-5(m-1)$
15. $-7(3r-2)-8(3-5r)$
16. $-8(1-2x)-8(7x-5)$
17. $-(4n+1)-7(8n-5)$
18. $-7(-6+b)-8(3b+3)$
19. $-4(1-3v)-6(v-6)$
20. $-8(3x+2)-4(1+5x)$
21. $-5(n-8)-6(n-8)$
22. $-3(-5+a)-5(5a+1)$
23. $-8(1-6k)-5(4k+1)$
24. $-7(2x+5)-(x+4)$
25. $-5(1+8x)-3(1-x)$
26. $-5(n+3)-2(7n+6)$
27. $-4(8m+1)-(-2+7m)$
28. $-(1-7p)-(-6-8p)$
29. $-7(-3+3x)-2(7x+7)$
30. $-7(n+2)-(2n+7)$
31. $-4(1+2b)-2(b+6)$
32. $-2(7-5r)-4(1+r)$
33. $-3(5x-8)-8(4x+6)$
34. $-7(n+5)-8(5n+6)$
35. $-3(b-7)-7(1-4b)$
36. $-7(6v-2)-(v-8)$
37. $-8(3x-2)-6(-5+5x)$
38. $-5(3x+8)-6(8x-4)$

Chapter 3

Rational numbers

The purpose of this chapter is to introduce the rational numbers. We begin with the integers, because the rational numbers are an extension of the integers.

3.1. Integers

The discovery of the integers gave our ancestors mathematical riches beyond those of the natural numbers, while retaining the wealth of the natural numbers. Over a period of more than 2000 years, mathematically insightful humans discovered ever deeper qualities of the integers. Recently, 1994, Andrew Wiles proved a theorem known as "Fermat's Last Theorem". The theorem resisted proof for over 300 years. In proving Fermat's Last Theorem, Wiles discovered a surprising connection between the integers and another area of mathematics. The study of the integers is called "Number Theory".

3.1.1. Division with integers

In many cases, the result of the division of an integer by an integer is an integer. For example, $15 \div 3 = 5$. In other cases, dividing an integer by an integer does not result in an integer. For example, $15 \div 2$ is not an integer. Based on our past experience, we naturally wonder if the integers can be extended in such a way that every division results in a number.

The number 1 plays the same role in multiplication as the number 0 does in addition. Let a represent any integer. Just as $a + 0 = a$ so too does $a \cdot 1 = a$. We called 0 the identity element for addition. We call 1 the identity element for multiplication.

Every integer, except 1, can be written as the product of two distinct integers. For example, $3 = 3 \cdot 1$. But no pair of distinct integers exists whose product is 1.

In a previous section, we discovered that 0, the identity element for addition, is the sum of a number and its additive inverse. Perhaps each number has a multiplicative inverse so that the product of the number and its multiplicative inverse is 1, the identity element for multiplication.

You have for some time known that

$$2 \cdot \frac{1}{2} = 1$$

$$3 \cdot \frac{1}{3} = 1$$

$$4 \cdot \frac{1}{4} = 1.$$

And, in general, for any integer a other than 0,

$$a \cdot \frac{1}{a} = 1.$$

The number $\frac{1}{a}$ is the multiplicative inverse of a, because the product of a and $\frac{1}{a}$ is 1.

When the integers are extended by including the multiplicative inverse of every integer, except 0, the result is the set of rational numbers. We defined the integers by listing the members $\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$.

It is a little inconvenient to list the rational numbers, but we do so for the positive rational numbers in table (3.1). Notice that the positive integers are included in table (3.1). They appear in the first column.

$$\begin{array}{cccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

TABLE 3.1. the positive rational numbers

The following definition of the rational numbers is usually given in elementary textbooks.

Definition 3.1 (The Rational Numbers)
A number that can be written in the form $\frac{a}{b}$ where a and b are integers and b is not 0 is called a **rational number**. The set of all such numbers is called the set of **rational numbers** and is often denoted by the symbol $\mathbb{Q}$. ■

In this book, we will use the terms "rational number" and "fraction" interchangeably.

Remark 3.1
The integers are contained in the set of rational numbers. Every integer is a rational number, but some rational numbers are not integers. In other words, the set of integers is a proper subset of the set of rational numbers, $\mathbb{Z} \subset \mathbb{Q}$.

Definition 3.2 (Irrational Number)
A number that is not a rational number is called an **irrational number**. ■

This will be important in subsequent courses. For now it will not be used.

Example 3.1

(1) $\frac{2}{3}$ is a rational number, because 2 and 3 are integers.

(2) $\frac{119}{107}$ is a rational number, because 119 and 107 are integers.

(3) 5 is a rational number, because it can be written $\frac{5}{1}$ and 5 and 1 are integers.

(4) $\frac{-2}{7}$ is a rational number, because -2 and 7 are integers.

(5) $\frac{9}{-11}$ is a rational number, because 9 and -11 are integers.

(6) $\frac{\sqrt{2}}{5}$ is an irrational number, because $\sqrt{2}$ is not an integer.

Numbers like $\sqrt{2}$ will not be discussed in this book.

(7) $\frac{0}{2}$ is a rational number, because 0 and 2 are integers.

(8) $\frac{1}{0}$ is not a rational number, because the denominator 0 violates definition 3.1.

(9) $\frac{0}{0}$ is not a rational number, because the denominator is 0.

As for the last two items, we will soon discuss the fact that the mix of symbols $\frac{a}{0}$, where a is any number, is meaningless, or, as is usually said, "undefined".

Someone is bound to ask whether

$$\frac{\frac{2}{5}}{\frac{7}{9}}$$

is a rational number. The answer to this question will have to wait for a few pages until we discuss the ideas and techniques needed to provide an answer to it.

3.2. Division

Of course

$$6 \cdot 2 = 12. \tag{3.1}$$

Suppose we multiply both sides by $\frac{1}{2}$,

$$6 \cdot 2 \cdot \frac{1}{2} = 12 \cdot \frac{1}{2}$$

Since 2 and $\frac{1}{2}$ are multiplicative inverses, and 1 is the identity element for multiplication,

$$6 \cdot 1 = 12 \cdot \frac{1}{2}$$

$$6 = 12 \cdot \frac{1}{2}.$$

Noting that $6 = 12 \div 2$

$$12 \div 2 = 12 \cdot \frac{1}{2} \tag{3.2}$$

Equation (3.2) quite clearly suggests that dividing by a number and multiplying by the multiplicative inverse of that number have the same meaning, because they each name the same number. Whenever we can write an equation like equation (3.1), we can obtain an equation like equation (3.2). And, equation (3.1) simply equates the number 12 with the product of its factors, 2 and 6.

You probably used the idea of equation (3.1) when you first learned about division. Suppose when you were first learning about division, you were asked

"What is

$$15 \div 5?\text{''}$$

you may thought

$$5 \times \blacksquare = 15.$$

and answered

“3”.

You knew that multiplication and division are related operations.

Let us mention two more multiplication facts.

$$2 \cdot 3 = 6$$
$$2 \cdot 4 = 8.$$

Knowing these two multiplication facts, you could perform the following two divisions

$$6 \div 2 = 3$$
$$8 \div 2 = 4.$$

But what about

$$7 \div 2 = \blacksquare?$$

If the equations are placed in the following order

$$2 \cdot 3 = 6 \tag{3.3}$$
$$2 \cdot \blacksquare = 7 \tag{3.4}$$
$$2 \cdot 4 = 8, \tag{3.5}$$

then it is obvious there is no integer equal to $7 \div 2$. Before the discovery of the rational numbers, this was as far as any human could go, because there is no integer that, substituted for $\blacksquare$, would make equation (3.4) true. But, there is a rational number that makes equation (3.4) true. We can find that number like this.

Suppose there exists a number, call it a, for which

$$7 \div 2 = a. \tag{3.6}$$

This would mean that

$$2 \cdot a = 7.$$

Multiply both sides by $\frac{1}{2}$,

$$\frac{1}{2}\cdot 2\cdot a=\frac{1}{2}\cdot 7.$$

Since 2 and $\frac{1}{2}$ are multiplicative inverses, and 1 is the identity element for multiplication,

$$1\cdot a=\frac{1}{2}\cdot 7$$

$$a=\frac{1}{2}\cdot 7. \tag{3.7}$$

Equation (3.6) says $a = 7\div 2$ and we substitute that value of a into equation (3.7) to obtain

$$7\div 2=\frac{1}{2}\cdot 7. \tag{3.8}$$

Equation (3.8) suggests that dividing by a number and multiplying by the multiplicative inverse of that number have the same meaning. This is similar to what we found when we considered $12\div 2$, but in this example 2 is not a factor of 7.

If you believe that the essential feature of division is expressed in equations (3.2) and (3.8), then our next definition will strike you as being exactly right.

Definition 3.3 (Division)
Provided $b \neq 0$, the expressions

$$a\div b,\quad \frac{a}{b},\quad a\cdot\frac{1}{b}$$

have identical meaning; that is, the expressions name the same number. If $b = 0$, then none of these expressions is meaningful. ■

Example 3.2

Find each quotient using the same reasoning as was used to rewrite equation (3.6) as equations (3.7).

(1) $14\div 7$
(2) $39\div 7$

Solution

(1)

$$14 \div 7 = x$$

means that

$$7 \cdot x = 14.$$

Multiply both sides by $\frac{1}{7}$,

$$\frac{1}{7} \cdot 7 \cdot x = \frac{1}{7} \cdot 14.$$

Since 7 and $\frac{1}{7}$ are multiplicative inverses, and 1 is the identity element for multiplication,

$$1 \cdot x = \frac{1}{7} \cdot 14$$

$$x = \frac{1}{7} \cdot 14$$

$$x = 2.$$

$$\therefore \quad 14 \div 7 = 2.$$

(2)

$$39 \div 7 = x$$

means that

$$7 \cdot x = 39.$$

Multiply both sides by $\frac{1}{7}$,

$$\frac{1}{7} \cdot 7 \cdot x = \frac{1}{7} \cdot 39.$$

Since 7 and $\frac{1}{7}$ are multiplicative inverses, and 1 is the identity element for multiplication,

$$1 \cdot x = \frac{1}{7} \cdot 39$$

$$x = \frac{1}{7} \cdot 39$$

$$x = \frac{39}{7}$$

$$\therefore \quad 39 \div 7 = \frac{39}{7}.$$

Example 3.3

Find the quotient of $231 \div 53$ using the definition of division.

Solution

$$231 \div 53 = 231 \cdot \frac{1}{53}.$$

$$\therefore \quad 231 \div 53 = \frac{231}{53}.$$

Exercise 3.1

[Part 1] *Find each quotient by imitating Example (3.2).*

1. $27 \div 3$
2. $31 \div 6$
3. $25 \div 3$
4. $137 \div 9$

[Part 2] *Use the definition of division to find the following quotients.*

1. $9 \div 3$
2. $17 \div 8$
3. $52 \div 13$
4. $5 \div 12$

3.2.1. Division by zero is undefined

There are several ways to see this is true. We mention three of them.

3.2.1.1. Contradiction

Division by zero is meaningless. To see why this is so, suppose it is not so. That is, suppose, for the sake of argument, that the expression $\frac{a}{0}$ does equal a number. We will call that number c. Then,

$$\frac{a}{0} = c.$$

By theorem (1.2),

$$\frac{a}{0} \cdot 0 = c \cdot 0.$$

Then

$$\frac{a}{\emptyset} \cdot \emptyset = c \cdot 0.$$

Remember, we are supposing that division by zero is defined.

By theorem (2.2),

$$a = 0.$$

We have just shown that if we suppose $\frac{a}{0}$ is meaningful, every number must equal 0! This is about as false as false can be. So, our supposition that $\frac{a}{0}$ is meaningful must have been wrong.

3.2.1.2. No multiplicative inverse

According to our definition, division by 0 is multiplication by the multiplicative inverse of 0. But any number times 0 is 0, not 1. This means 0 has no multiplicative inverse. So, division by 0 is meaningless.

3.2.1.3. Third grade facts

This might be the best explanation for why division by zero is undefined. You have known for a long time that $6 \div 2 = 3$ means that $2 \times 3 = 6$ and that $15 \div 3 = 5$ means that $3 \times 5 = 15$. In general, $a \div b = c$ means that $b \times c = a$. Now, consider $a \div 0 = c$, where $a \neq 0$. This would mean that $0 \times c = a$. But for every number c, $0 \times c = 0$, not a.

3.3. Summary

3.3.1. Numbers

The natural numbers were extended to the integers, and the integers were extended to the rational numbers. Until the rational numbers are extended to what are known as the real numbers, the word “number” in this book will denote a rational number. If for some reason we wish to limit a discussion to a subset of the rational numbers, that will be made clear at the time.

3.3.2. Definitions

In mathematics, a definition tells what a thing *is*, by telling us the essential qualities such a thing must possess. It often requires considerable insight, intuition, and effort to discover the correct definition of a mathematical object. For example, the area of mathematics called “calculus” depends on a single definition. The mathematical quest for that definition was handed down through generations of mathematicians. Generations −because the correct definition took over two hundred years to discover!

3.3.3. Axioms

We have used several ideas that in this book have been called "fundamental" - or sometimes "basic" - ideas. Mathematicians realized that only a few of these basic ideas were needed to provide the foundation of algebra. These few basic ideas were singled out and called "**axioms**". All the rules of algebra follow from the axioms. Since the axioms are the starting place, no proof is given for them. The axioms are supposed to be obvious ideas.

3.3.4. Axioms of the rational numbers

Addition and multiplication are commutative

For every number x and every number y,

$$x \cdot y = y \cdot x \qquad\qquad x + y = y + x$$

Addition and multiplication are associative

For every x, every y, and every z,

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \qquad\qquad (x + y) + z = x + (y + z)$$

Identity elements

For every x,

$$x \cdot 1 = x \qquad\qquad x + 0 = x$$

Inverse elements

For every x, there exists a y for which,

$$x \cdot y = 1, \ \text{ provided } x \neq 0 \qquad\qquad x + y = 0$$

Multiplication is distributive over addition

For every x, every y, and every z,

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

3.3.5. Theorems

A theorem is a statement of a mathematical fact that has been proved. The proof of a theorem uses only axioms, definitions, and previously proved theorems. Proof guarantees that the theorem is as true as the axioms and definitions used in its proof.

3.3.6. Division and subtraction

Notice that the axioms are all about addition and multiplication. There is no mention of subtraction and division. As we realized in Chapter 2, subtraction is *adding* the multiplicative inverse of a number and division is *multiplying* by the multiplicative inverse of a number.

3.3.7. Any number

We very often use the phrase "Let a represent any number." When we say this, what qualities do we suppose a to posses? Answer: only the qualities that every number is known to possess. And those common qualities are exactly the qualities stated in the axioms. Of course a number has certain qualities that make it the special number it is. For example, 2 is the only even prime number $-$that's quite a distinction. It is certainly not a quality that 2 has in common with all other numbers. When we say "a represents *any* number", we ignore any attributes that might make a number special.

Remark 3.2

Theorems in Chapter 2 are valid for rational numbers. The word "number" that appears in the theorems of Chapter 2 should now be understood to refer to a rational number.

3.4. Negative and Positive Rational Numbers

The sign of a sum, difference, or product of rational numbers is determined exactly as it would be for integers. The sign of the quotient of rational numbers is determined just as it would be for the product. This should be no surprise, since division is defined as multiplication by the multiplicative inverse.

3.4.1. Writing positive and negative fractions

The following are three equivalent forms of one fraction.

$$-\frac{3}{5} = \frac{-3}{5} = \frac{3}{-5}.$$

While each of the above forms is mathematically correct, some are better style than others. As you gain experience, you will always write good style. Don't lose any sleep over it now.

Example 3.4

Bad style	Good style
$-\frac{7}{3}\cdot\frac{1}{-4}$	$\frac{-7}{3}\cdot\frac{-1}{4}$
$-\frac{5}{3}\cdot-\frac{1}{4}$	$\frac{-5}{3}\left(-\frac{1}{4}\right)$
$\frac{11}{5}--\frac{2}{7}$	$\frac{11}{5}-\left(-\frac{2}{7}\right)$
$\frac{11}{5}--\frac{2}{7}$	$\frac{11}{5}-\frac{-2}{7}$
$\frac{9}{13}+-\frac{15}{16}$	$\frac{9}{13}+\left(-\frac{15}{16}\right)$
$\frac{9}{13}+-\frac{15}{16}$	$\frac{9}{13}+\frac{-15}{16}$

3.4.2. Product and Quotient of Positive and negative fractions

Example 3.5

Find the product of $\frac{3}{7}$ and $\frac{-5}{2}$.

Solution

The product is computed just as in earlier grades: $\frac{\text{product of numerators}}{\text{product of denominators}}$. Be careful to get the signs correct.

$$\begin{aligned}\frac{3}{7}\cdot\frac{-5}{2}&=\frac{3(-5)}{7\cdot 2}\\&=\frac{-15}{14}.\end{aligned}$$

Example 3.6

Find the product of $\frac{-5}{8}$ and $\frac{3}{-7}$.

Solution

Recall that $\frac{3}{-7} = \frac{-3}{7}$. Then,

$$\begin{aligned}\frac{-5}{8} \cdot \frac{3}{-7} &= \frac{-5}{8} \cdot \frac{-3}{7} \\ &= \frac{(-5)(-3)}{8 \cdot 7} \\ &= \frac{15}{56}.\end{aligned}$$

Example 3.7

Each of the following shows the LHS simplified to the RHS.

(1) $\frac{2}{3} \cdot \frac{-5}{7} = \frac{-10}{21}$.

(2) $\frac{2}{5}\left(-\frac{3}{7}\right) = \frac{-6}{35}$.

(3) $\frac{-6}{5} \cdot \frac{-5}{11} = \frac{6}{11}$.

(4) $\left(-\frac{2}{3}\right)\left(\frac{-1}{5}\right) = \frac{2}{15}$.

(5) $\left(-\frac{7}{3}\right)\left(\frac{1}{-4}\right) = \frac{7}{12}$.

Example 3.8

Find the product $\left(\frac{8}{15}\right)\left(\frac{-21}{64}\right)$.

All mathematics you learned in previous years still applies. Cancel common factors before you multiply.

Solution

$$\begin{aligned}\left(\frac{8}{15}\right)\left(\frac{-21}{64}\right) &= \left(\frac{\cancel{8}^{1}}{\cancel{15}_{3}}\right)\left(\frac{\cancel{-21}^{-7}}{\cancel{64}_{8}}\right) \\ &= \frac{-7}{24}.\end{aligned}$$

Example 3.9

Find the product $\left(\frac{3}{-6}\right)\left(\frac{-15}{-12}\right)$.

Solution

$$\begin{aligned}\left(\frac{3}{-6}\right)\left(\frac{-15}{-12}\right) &= \left(\frac{-3}{6}\right)\left(\frac{-(-15)}{12}\right)\\ &= \left(\frac{-3}{6}\right)\left(\frac{15}{12}\right)\\ &= \left(\frac{\cancel{-3}^{-1}}{\cancel{6}_2}\right)\left(\frac{\cancel{15}^5}{\cancel{12}_4}\right)\\ &= \frac{-5}{8}.\end{aligned}$$

Example 3.10

Find the quotient $\frac{-2}{5} \div (-23)$.

Solution

$$\begin{aligned}\frac{-2}{5} \div (-23) &= \frac{-2}{5} \cdot \frac{-1}{23}\\ &= \frac{2}{115}.\end{aligned}$$

Example 3.11

Find the quotient $\frac{-2}{5} \div (5) \div (-7)$.

Solution

$$\begin{aligned}\frac{-2}{5} \div (5) \div (-7) &= \frac{-2}{5} \cdot \frac{1}{5} \cdot \frac{-1}{7}\\ &= \frac{2}{175}.\end{aligned}$$

Example 3.12

Find the product $\frac{-a}{3} \cdot \frac{-2}{b} \cdot \frac{7}{5}$.

Solution

$$\frac{-a}{3}\cdot\frac{-2}{b}\cdot\frac{7}{5}=\frac{(-a)(-2)(7)}{(3)(b)(5)}$$

$$=\frac{14a}{15b}.$$

Exercise 3.2

Evaluate.

1. $\frac{-3}{4}\cdot\frac{5}{7}$
2. $\frac{-12}{5}\cdot\frac{-3}{11}$
3. $\left(\frac{-7}{11}\right)\left(\frac{-2}{4}\right)$
4. $\left(\frac{3}{4}\right)\left(\frac{-24}{27}\right)$
5. $-\frac{6}{9}\cdot\frac{-12}{2}$
6. $-4\cdot\frac{-7}{3}$
7. $-5\cdot\frac{-11}{5}\cdot\frac{-13}{33}$
8. $\frac{3}{8}\left(\frac{-12}{21}\right)$
9. $9\cdot\frac{2}{-3}\cdot\frac{-1}{16}$
10. $\frac{2}{-9}\cdot\frac{-3}{5}$
11. $\frac{2}{11}\cdot\frac{-3}{-2}$
12. $\frac{4}{5}\div(-3)$
13. $\frac{-3}{8}\div(-2)$
14. $-\frac{5}{9}\div(-10)$
15. $\frac{-12}{5}\div(6)$
16. $\frac{1}{-5}\div(-4)$
17. $\frac{-a}{2}\cdot\frac{-b}{3}$
18. $\frac{-3a}{2}\cdot\frac{-6}{5}$
19. $\frac{3x}{5}\cdot\frac{-4}{y}$
20. $\frac{-4x}{7}\cdot\frac{-14}{2}$
21. $\frac{3a}{2}\cdot\frac{-7}{3a}$
22. $\frac{5a}{12}\div(-5b)$
23. $-4\cdot\frac{-a}{5}\div 2a$
24. $-a\div b\div c$
25. $-7a\div b\div c$

3.4.3. Sum and Difference of Positive and Negative Fractions

We will call fractions that have the same denominator “like fractions”. As you have known for some time, like fractions may be combined by addition and subtraction. This sounds quite similar to the rule discussed section (2.7) which says that “like terms” may be added and subtracted. The similarity is real. The fraction $\frac{a}{b}$ can always be rewritten as $a \cdot \frac{1}{b}$. Provided $b \neq 0$,

$$\frac{a}{b} = a \cdot \frac{1}{b}.$$

If we think of $\dfrac{1}{b}$ as indicating a “basic unit”, then $a \cdot \dfrac{1}{b}$ denotes a quantity a of such units. For example,

(1) $\dfrac{2}{3}$ denotes 2 thirds, just as “2 apples” denotes two apples.

(2) $\dfrac{8}{5} = 8 \cdot \dfrac{1}{5}$ which indicates 8 of the basic unit the fifth.

(3) $\dfrac{7}{5}$ denotes 7 fifths.

In section (2.7) we used distribution to provide a mathematically compelling reason for the common wisdom that “you can add apples to apples but not apples to sheep”. We can use distribution in the context of fractions, too. Fifths may be added to fifths, because

$$\begin{aligned}\frac{3}{5} + \frac{4}{5} &= 3 \cdot \frac{1}{5} + 4 \cdot \frac{1}{5} \\ &= (3+4)\frac{1}{5} \\ &= (7)\frac{1}{5} \\ &= \frac{7}{5}.\end{aligned}$$

Fifths may not be added to sevenths, because

$$\begin{aligned}\frac{3}{5} + \frac{4}{7} &= 3 \cdot \frac{1}{5} + 4 \cdot \frac{1}{7} \\ &= (3+4) \cdot ???\end{aligned}$$

Of course, if $\frac{3}{5}$ and $\frac{4}{7}$ are written with a common denominator, the basic units will match so that the addition can be performed.

$$\begin{aligned}\frac{3}{5}+\frac{4}{7} &= \frac{21}{35}+\frac{20}{35} \\ &= 21\cdot\frac{1}{35}+20\cdot\frac{1}{35} \\ &= (21+20)\frac{1}{35} \\ &= (41)\frac{1}{35} \\ &= \frac{41}{35}.\end{aligned}$$

Will we, given any two fractions, always be able to write them with a common denominator (common basic unit) so that they may be added or subtracted? The answer is “Yes”. A few pages from now, we will prove that this is so.

3.4.4. Proofs of facts we already knew

Some of the motivation for obtaining proofs of facts we already know is to bolster our confidence in the system of axioms, definitions, and theorems by noting that the system produces the results it *should* produce.

As long as we are in the category “proofs of stuff I’ve known for years”, we may as well do a few more. The reader will be asked to prove the next two theorems as exercises.

Theorem 3.1
For any number a other than 0, $\dfrac{a}{a}=1$.

Theorem 3.2
For any number a, $a=\dfrac{a}{1}$.

Theorems (3.1) and (3.2) are used in the proofs of the next several theorems.

Theorem 3.3
For any numbers a and b with neither 0, $\dfrac{1}{a}\cdot\dfrac{1}{b}=\dfrac{1}{ab}$.

Proof.

$$\begin{aligned}
\text{LHS} &= \frac{1}{a}\cdot\frac{1}{b} \\
&= \frac{ab}{ab}\cdot\frac{1}{a}\cdot\frac{1}{b} && \text{Theorem (3.1)} \\
&= \frac{1}{ab}\cdot a\cdot\frac{1}{a}\cdot b\cdot\frac{1}{b} && \text{definition division} \\
&= \frac{1}{ab}\cdot 1\cdot 1 && \text{inverse elements} \\
&= \frac{1}{ab} \\
&= \text{RHS}
\end{aligned}$$

■

Theorem 3.4
For any numbers a,c and $b \neq 0, d \neq 0$, $\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$.

Proof.

$$\begin{aligned}
\text{LHS} &= \frac{a}{b}\cdot\frac{c}{d} \\
&= a\cdot\frac{1}{b}\cdot c\cdot\frac{1}{d} && \text{definition division} \\
&= ac\cdot\frac{1}{b}\cdot\frac{1}{d} && \text{multiplication commutative} \\
&= ac\cdot\frac{1}{bd} && \text{Theorem (3.3)} \\
&= \frac{ac}{ab} && \text{definition division} \\
&= \text{RHS}
\end{aligned}$$

■

The next theorem answers the question "Will we, given any two fractions, always be able to write them with a common denominator so that they may be added or subtracted?"

Theorem 3.5
For any numbers a,b,c,d with $b \neq 0$ and $d \neq 0$,

$$\frac{a}{b}+\frac{c}{d} = \frac{ad+bc}{bd}.$$

Proof.

$$\begin{aligned}\frac{a}{b}+\frac{c}{d} &= \frac{a}{b}\cdot 1+\frac{c}{d}\cdot 1\\ &= \frac{a}{b}\cdot\frac{d}{d}+\frac{c}{d}\cdot\frac{b}{b}\\ &= \frac{ad}{bd}+\frac{bc}{bd}\\ &= \frac{ad+bc}{bd}.\end{aligned}$$

■

Theorem (3.6) similar to Theorem (3.5). Its proof is left as an exercise.

Theorem 3.6
For any numbers a,b,c,d with $b \neq 0$,

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}.$$

Exercise 3.3

1. Prove Theorem (3.1): $\frac{a}{a}=1, a\neq 0$.
2. Prove Theorem (3.2): $a=\frac{a}{1}$.
3. Provide justifications for the steps in the proof of theorem (3.5).
4. Prove Theorem (3.6): $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}, \quad b\neq 0$.

The next theorem may not be in the category of "stuff I've known forever". It is about the multiplicative inverse of the rational number $\frac{a}{b}$.

Theorem 3.7
For any numbers a, b where $a \neq 0$ and $b \neq 0$,

$$\frac{1}{\frac{\text{a}}{b}} = \frac{b}{a}.$$

Proof. The right hand side of this equation is just the left hand side multiplied by the number 1 in a fancy form. That is easier to see if you think of $\frac{b}{a}$ as a blob.

$$\begin{aligned}
\frac{1}{\frac{a}{b}} &= \frac{1}{\frac{a}{b}} \cdot \frac{\frac{b}{a}}{\frac{b}{a}} \\
&= \frac{\frac{b}{a}}{\frac{a}{b} \cdot \frac{b}{a}} \\
&= \frac{\frac{b}{a}}{\frac{ab}{ab}} \\
&= \frac{\frac{b}{a}}{1} \\
&= \frac{b}{a}.
\end{aligned}$$

■

Remark 3.3
Theorem (3.7) says that the multiplicative inverse of $\dfrac{a}{b}$ is $\dfrac{b}{a}$. This means that, when neither a, b, nor d are 0,

The number $\frac{b}{a}$ is sometimes called the **reciprocal** of the number $\frac{a}{b}$.

$$\frac{c}{d} \div \frac{a}{b} = \frac{c}{d} \cdot \frac{b}{a}.$$

3.4.5. A leftover question

We now consider a question from the beginning of section (3.1.1). Is

$$\frac{\frac{2}{5}}{\frac{7}{9}} \tag{3.9}$$

a rational number?

Our definition of a rational number says that the expression (3.9) is rational, *if* it can be written as an integer over an integer. Now, expression (3.9) is just a fraction divided by a fraction.

This means

$$\begin{aligned} \frac{\frac{2}{5}}{\frac{7}{9}} &= \frac{2}{5} \div \frac{7}{9} \\ &= \frac{2}{5} \cdot \frac{9}{7} \\ &= \frac{18}{35}. \end{aligned}$$

which is a rational number.

Example 3.13

Simplify $\dfrac{\frac{9}{16}}{\frac{27}{8}}$

Solution

$$\begin{aligned} \frac{\frac{9}{16}}{\frac{27}{8}} &= \frac{9}{16} \div \frac{27}{8} \\ &= \frac{9}{16} \cdot \frac{8}{27} \\ &= \frac{1}{6}. \end{aligned}$$

3.5. Rational expressions

A **rational expression** is the quotient of two polynomials. Several examples are:

(1) $\frac{x}{5}$

(2) $\frac{x-5}{7}$

(3) $\frac{2x+9}{y}, \quad y \neq 0$

(4) $\frac{5b+2c-8}{a+2}, \quad a \neq -2$

Just as we simplify rational numbers, so too we simplify rational expressions.

Example 3.14

Simplify $\frac{2a}{3} + \frac{a}{2}$.

Solution

Just as in previous years, the first task is to get a common denominator.

$$\begin{aligned}\frac{2a}{3} + \frac{a}{2} &= \frac{2a}{3}\left(\frac{2}{2}\right) + \frac{a}{2}\left(\frac{3}{3}\right) \\ &= \frac{4a}{6} + \frac{3a}{6} \\ &= \frac{7a}{6}.\end{aligned}$$

Example 3.15

Simplify $\frac{2a+5}{3} + \frac{a+1}{5}$.

Solution

$$\begin{aligned}\frac{2a+5}{3} + \frac{a+1}{5} &= \frac{2a+5}{3}\left(\frac{5}{5}\right) + \frac{a+1}{5}\left(\frac{3}{3}\right) \\ &= \frac{5(2a+5)}{15} + \frac{3(a+1)}{15} \\ &= \frac{5(2a+5)+3(a+1)}{15} \\ &= \frac{10a+25+3a+3}{15} \\ &= \frac{13a+28}{15}.\end{aligned}$$

Example 3.16

Simplify $\frac{7a+5}{4}+\frac{3a-6}{2}$.

Solution

$$\begin{aligned}
\frac{7a+5}{4}+\frac{3a-6}{2} &= \frac{7a+5}{4}+\frac{3a-6}{2}\left(\frac{2}{2}\right)\\
&= \frac{7a+5}{4}+\frac{2(3a-6)}{4}\\
&= \frac{7a+5+2(3a-6)}{4}\\
&= \frac{7a+5+6a-12}{4}\\
&= \frac{13a-7}{4}.
\end{aligned}$$

Example 3.17

Simplify $\frac{2a+3b}{5}+\frac{6a-7}{10}$.

Solution

$$\begin{aligned}
\frac{2a+3b}{5}+\frac{6a-7}{10} &= \frac{2a+3b}{5}\left(\frac{2}{2}\right)+\frac{6a-7}{10}\\
&= \frac{4a+6b}{10}+\frac{6a-7}{10}\\
&= \frac{4a+6b+6a-7}{10}\\
&= \frac{10a+6b-7}{10}.
\end{aligned}$$

Example 3.18

Simplify $\frac{a+4}{3}-\frac{5a-2}{5}$.

Solution

$$\begin{aligned}
\frac{a+4}{3}-\frac{5a-2}{5} &= \frac{a+4}{3}\left(\frac{5}{5}\right)-\frac{5a-2}{5}\left(\frac{3}{3}\right)\\
&= \frac{5(a+4)}{15}-\frac{3(5a-2)}{15},
\end{aligned}$$

the next step will help you to avoid a popular mistake,

$$\begin{aligned}&= \frac{5(a+4)}{15} + \frac{-3(5a-2)}{15}\\&= \frac{5a+20-15a+6}{15}\\&= \frac{-10a+26}{15}. \qquad \blacksquare\end{aligned}$$

There are still no new ideas or techniques, even in example (3.18). But example (3.18) involves so much that you have only just learned, that you should not be surprised if you make mistakes. Remember, you are pushing against the boundaries of your knowledge and skill. With time and practice, your hand will work expressions like example (3.18) while you hold a conversation on a different topic

Example 3.19

Simplify $\frac{5}{3}\left(\frac{a}{2}+\frac{2}{3}\right)-\frac{a}{9}$.

Solution

$$\begin{aligned}\frac{5}{3}\left(\frac{a}{2}+\frac{2}{3}\right)-\frac{a}{9} &= \frac{5a}{6}+\frac{10}{9}-\frac{a}{9}\\&= \frac{15a}{18}+\frac{20}{18}-\frac{2a}{18}\\&= \frac{13a+20}{18}.\end{aligned}$$

Remark 3.4 (Fraction style)
Typeset in a book, there is little chance that $\frac{3}{8}a$ would be confused with $\frac{3}{8a}$. Written by hand, the distinction is not always so clear. That is why the handwritten $\frac{3a}{8}$ is usually better than the handwritten $\frac{3}{8}a$. And just try to decide whether $3/8a$ means $\frac{3a}{8} = \frac{3}{8}\,a$ or $\frac{3}{8a}$.

Exercise 3.4

Simplify. It the expression is already in simplest form, say so.

1. $\dfrac{\frac{3}{8}}{\frac{15}{16}}$

2. $\dfrac{\frac{12}{5}}{\frac{21}{10}}$

3. $\dfrac{\frac{121}{7}}{\frac{22}{28}}$

4. $\dfrac{\frac{13-3}{7}}{\frac{2+3}{3}}$

5. $\dfrac{\frac{21-5}{2+5}}{\frac{3+5}{17-10}}$

6. $\dfrac{\frac{20}{2}+\frac{5}{2}}{\frac{3}{4}+\frac{7}{4}}$

7. $\dfrac{\frac{a-b}{3}}{\frac{a+b}{6}}$

8. $\dfrac{\frac{2x}{9y}}{\frac{3}{5y}}$

9. $\dfrac{\frac{7y}{4x}}{\frac{28y}{20x}}$

10. $\dfrac{a-7}{13}+\dfrac{a+2}{13}$

11. $\dfrac{a-7}{5}+\dfrac{a+2}{3}$

12. $\dfrac{2x-11}{4}+\dfrac{3x-1}{2}$

13. $\dfrac{5a+3}{3}+\dfrac{a-7}{5}+\dfrac{2}{3}$

14. $\dfrac{8x-3}{3}-\dfrac{x-1}{3}$

Exercise 3.5

Simplify. It the expression is already in simplest form, say so.

1. $\frac{a+3}{10}-\frac{a+6}{5}$
2. $\frac{2b-7}{2}-\frac{2b+5}{3}$
3. $\frac{7y-2}{6}-\frac{y-2}{3}$
4. $\frac{2x-1}{8}-\frac{x-3}{6}$
5. $\frac{a-3}{4}-\frac{-3a-2}{2}$
6. $\frac{2a-7}{9}-\frac{-5a-3}{3}$
7. $\frac{-x-3}{15}-\frac{-x+7}{3}$
8. $\frac{-3x-4}{7}-\frac{-2x+1}{4}$
9. $\frac{-x-1}{9}-\frac{-x-2}{6}$
10. $\frac{-5x-7}{3}-x-2$
11. $\frac{11x+2}{7}-\frac{1}{3}-\frac{4x-3}{21}$
12. $\frac{-3x-20}{12}-\frac{-2x-3}{4}-\frac{5x}{6}$
13. $x-\frac{5x-2}{4}-\frac{-3x+2}{3}$
14. $\frac{a-12}{9}-\frac{2a-5}{3}-\frac{2a-7}{3}$
15. $\frac{3b-1}{36}-\frac{b-1}{6}+\frac{5}{18}$
16. $\frac{b+2}{11}+\frac{5b-2}{22}-4-\frac{b-1}{11}$
17. $\frac{b+1}{-2}+\frac{b-3}{8}+\frac{2-b}{4}$
18. $\frac{6a+7}{24}-\frac{b-3}{-12}+\frac{3-a}{-6}$
19. $\frac{b+3}{6}-\frac{b+2}{12}+\frac{b}{3}+\frac{2b}{6}$
20. $\frac{-x-2}{3}-\frac{2x+1}{4}-1$
21. $\frac{2}{3}\cdot\frac{2x+1}{4}-\frac{x-1}{6}$
22. $-\frac{2}{5}\cdot\frac{a-2}{3}-\frac{a-4}{3}$
23. $\frac{1}{2}\cdot\frac{3x+5}{2}+\frac{2}{3}\cdot\frac{x-5}{3}$
24. $\frac{3}{2}\cdot\frac{4a-1}{2}-\frac{2}{3}\cdot\frac{7a-8}{2}$
25. $x-\frac{1}{3}\cdot\frac{x+13}{2}-\frac{1}{2}\cdot\frac{2x-3}{3}$
26. $\frac{\frac{x-3}{2}}{\frac{-3}{4}}+\frac{-x-2}{6}$

Exercise 3.6

Simplify.

1. $-3(-2x+2)$
2. $-3(2+2n)$
3. $2(m-3)$
4. $3(r+2)$
5. $-2n+\frac{n}{2}$
6. $x-\frac{7}{3}+\frac{3}{2}x+\frac{1}{2}$
7. $-\frac{7}{2}v+\frac{3}{2}v$
8. $-b+\frac{5}{2}-\frac{1}{2}b$
9. $x+\frac{1}{2}-x$
10. $-\frac{10}{3}n-\frac{1}{2}+n+\frac{1}{3}$
11. $-\frac{4a}{3}-\frac{5a}{3}$
12. $\frac{k}{2}-\frac{1}{2}+k+\frac{3}{2}$
13. $\frac{8}{3}\left(\frac{3p}{2}-\frac{5}{2}\right)$
14. $\frac{1}{3}\left(-\frac{1}{3}x+1\right)$
15. $\frac{4}{3}\left(n+\frac{4}{3}\right)$
16. $\frac{1}{3}\left(\frac{2m}{3}+\frac{5}{2}\right)$
17. $-\left(\frac{r}{2}+2\right)$
18. $-\left(\frac{3x}{2}+\frac{1}{2}\right)$
19. $\frac{-11}{3}\left(\frac{4n}{3}-\frac{3}{2}\right)$
20. $-\left(\frac{4b}{3}-\frac{8}{3}\right)$
21. $\frac{2}{3}\left(-2v+\frac{5}{2}\right)-\frac{v}{3}$
22. $\frac{-5}{3}\left(\frac{-7x}{2}+\frac{3}{2}\right)+\frac{4x}{3}$
23. $\frac{4}{3}\left(\frac{-n}{3}-\frac{7}{2}\right)+\frac{1}{2}$
24. $\frac{1}{3}-\frac{5}{3}\left(\frac{a}{2}-1\right)$
25. $\frac{-3}{2}\left(v-\frac{7}{3}\right)+\frac{1}{3}$
26. $\frac{-5}{3}-\frac{11}{3}\left(x+\frac{3}{2}\right)$
27. $\frac{-3}{2}\left(\frac{a}{2}-\frac{2}{3}\right)-\frac{3}{2}$
28. $\frac{2}{3}-3\left(\frac{n}{3}-\frac{8}{3}\right)$
29. $\frac{3}{2}\left(\frac{4k}{3}-\frac{4}{3}\right)+\frac{5}{2}$
30. $-3\left(p+\frac{2}{3}\right)+\frac{5}{3}$
31. $\frac{-2}{3}\left(\frac{-11x}{3}-1\right)-x$
32. $3\left(k+\frac{2}{3}\right)-\frac{2}{3}$
33. $-2\left(\frac{2m}{3}+\frac{5}{3}\right)+\frac{2m}{3}$
34. $\frac{4r}{3}+\frac{1}{2}\left(\frac{5r}{2}+\frac{1}{3}\right)$
35. $-2x-\frac{1}{2}\left(2x-\frac{5}{3}\right)$
36. $\frac{-3n}{2}+1+n$

Exercise 3.7

Simplify.

1. $2p+\frac{1}{2}\left(\frac{-3p}{2}-\frac{11}{3}\right)$
2. $\frac{-3}{2}-\frac{11}{3}\left(\frac{2n}{3}-1\right)$
3. $\frac{-7}{3}\left(x-\frac{5}{3}\right)+\frac{x}{2}$
4. $\frac{4}{3}\left(\frac{x}{3}+1\right)+x$
5. $\frac{-5}{3}-\frac{2}{3}\left(\frac{4r}{3}-1\right)$
6. $\frac{-11}{3}\left(x+\frac{10}{3}\right)+\frac{5x}{3}$
7. $\frac{3}{2}\left(n-\frac{3}{2}\right)+\frac{n}{3}$
8. $-1-2\left(\frac{b}{3}+2\right)$
9. $v+\frac{5}{3}\left(\frac{5v}{2}-\frac{7}{2}\right)$
10. $-\frac{3}{2}+\frac{2}{3}\left(\frac{-x}{3}+\frac{7}{3}\right)$
11. $\frac{7}{3}\left(\frac{5x}{3}-\frac{7}{3}\right)-\frac{5x}{2}$
12. $\frac{-5a}{2}-\frac{7}{3}\left(a-\frac{1}{2}\right)$
13. $\frac{-k}{3}-\frac{3}{2}\left(\frac{k}{2}+1\right)$
14. $\frac{8p}{3}+2\left(\frac{p}{3}-\frac{7}{2}\right)$
15. $\frac{-7x}{2}+\frac{1}{3}\left(-2x-\frac{4}{3}\right)$
16. $\frac{-3}{2}\left(-2x-\frac{11}{3}\right)+\frac{1}{3}\left(\frac{-5x}{2}-\frac{1}{2}\right)$
17. $\frac{-4}{3}\left(\frac{m}{3}+\frac{5}{3}\right)-\frac{4}{3}\left(m+\frac{4}{3}\right)$
18. $\frac{1}{2}\left(r-\frac{1}{2}\right)-\frac{5}{2}\left(r+\frac{1}{2}\right)$
19. $\frac{5}{3}\left(\frac{3x}{2}+\frac{5}{2}\right)+\frac{1}{2}\left(\frac{x}{3}+\frac{1}{2}\right)$
20. $\frac{7}{3}\left(n-\frac{1}{2}\right)+\frac{2}{3}\left(\frac{-7n}{3}+\frac{4}{3}\right)$
21. $\frac{-4}{3}\left(b+\frac{8}{3}\right)-\frac{8}{3}\left(\frac{b}{2}-2\right)$
22. $\frac{3}{2}\left(v+\frac{1}{2}\right)-\frac{11}{3}\left(\frac{3v}{2}+\frac{1}{2}\right)$
23. $\frac{3}{2}\left(\frac{-3n}{2}+2\right)-2\left(\frac{n}{3}+1\right)$
24. $2\left(\frac{-4x}{3}+\frac{4}{3}\right)-\left(\frac{-2x}{3}+\frac{4}{3}\right)$
25. $\frac{-4}{3}\left(a+\frac{3}{2}\right)-\frac{8}{3}\left(2a+\frac{3}{2}\right)$
26. $-2\left(\frac{4}{3}k-2\right)+\frac{4}{3}\left(-\frac{3}{2}k+\frac{1}{3}\right)$
27. $2\left(x+\frac{5}{3}\right)+\frac{2}{3}\left(2x+\frac{2}{3}\right)$
28. $\frac{1}{2}\left(\frac{5}{3}x-\frac{3}{2}\right)+\frac{4}{3}\left(\frac{1}{2}x+1\right)$
29. $-\frac{10}{3}\left(x-\frac{10}{3}\right)+\frac{1}{3}\left(x+\frac{5}{3}\right)$
30. $\frac{-7}{2}\left(m-\frac{3}{2}\right)+\frac{3}{2}\left(m+\frac{3}{2}\right)$

Chapter 4

Equations

4.1. The idea of an equation

An **equation** is a statement. “A=B” states that the expression on the left hand side (LHS) and the expression on the right hand side (RHS) are names for the same object.

A statement can be either true or false. The statement “A=B” is true, if “A” and “B” (the LHS and the RHS) do name the same object. Otherwise, the statement is false.

The equation

$$8 = 8 \tag{4.1}$$

is so obviously true that is seems silly to even mention that it is true.

The statement

$$6+2 = 8 \tag{4.2}$$

is just as obviously true as equation (4.1) once we simplify the LHS,

$$\begin{aligned} LHS &= 6+2 \\ &= 8 \\ &= RHS. \end{aligned}$$

The statement

$$6+2 = 5+3 \tag{4.3}$$

is almost as obviously true as equation (4.2) once we simplify both the LHS and the RHS,

$$\begin{aligned} LHS &= 6+2 \\ &= 8. \end{aligned}$$

$$RHS = 5 + 3$$
$$= 8.$$

$$\therefore\ LHS = RHS.$$

On the other hand, it is not quite so obvious that

$$\frac{1}{2}\left(\frac{1}{2} - \frac{1}{3}\right) - \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2}\right) = \frac{1}{2} - \frac{1}{3} \tag{4.4}$$

is true. But it is true, because

Can you say which of the three properties of equality is the backbone of this demonstration that equation (4.4) is true?

$$LHS = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3}\right) - \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2}\right)$$
$$= \frac{1}{12} + \frac{1}{12}$$
$$= \frac{1}{6}.$$

$$RHS = \frac{1}{2} - \frac{1}{3}$$
$$= \frac{1}{6}.$$

$$\therefore LHS = RHS.$$

Sometimes we wish to show that a statement is false. For example, show that "$7 \cdot 12 = 123 - 38$" is false.

$$LHS = 7 \cdot 12$$
$$= 84.$$

$$RHS = 123 - 38$$
$$= 85.$$

But $LHS \neq RHS$, so "$7 \cdot 12 = 123 - 38$" is false.

The reader can verify that when 2 is substituted for x, the following equation is true.

$$3x + 12 = 18. \tag{4.5}$$

A variety of words and expressions are employed to say that a given number when substituted for the unknown makes an equation true. Speaking of equation (4.5), you may say

(1) "2 is the solution of equation (4.5)."
(2) "2 solves equation (4.5)."
(3) "2 satisfies equation (4.5)."
(4) "The number that satisfies equation (4.5) is 2."

Exercise 4.1

For each equation and proposed value, show that the value is (or is not) a solution of the equation.

1. $5x+8=23, \quad x=3.$
2. $7x-9=26, \quad x=5.$
3. $5a-4=20, \quad a=6.$
4. $\dfrac{2y}{5}+\dfrac{3}{2}=3, \quad y=\dfrac{15}{4}.$

4.2. Solving equations

We have a surefire way to test whether or not a stated value of the unknown satisfies an equation. But, how do we find the number to test? We need a method that will *make obvious* the number that satisfies an equation. Several ideas will be helpful as we search for such a method. They are introduced below.

Definition 4.1 (Equivalent equations)
Equations are said to be **equivalent** when they have exactly the same solution. ■

For example, the equations $x+8=12$ and $18-x=14$ and $12\div 3=x$ are equivalent equations because the number 4 is the solution to all of them.

Remark 4.1
Equivalent equations may be said to be different forms of the same equation. For example, $x+5=2$ and $x=-3$ are forms of the same equation.

Remark 4.2
It is a fact, though it will be many years before you will be able to prove it,

that an equation obtained from a prior equation by correct application of any combination of axioms, definitions, and theorems is equivalent to the prior equation.

We indicate that equations are equivalent by the symbols "$\Longleftrightarrow$" , "$\equiv$', or "iff". Each of the following is true:

(1) "$x+3=0 \Longleftrightarrow x+9=6$"
(2) "$2x=8 \equiv 7+x=11$"
(3) "$x+8=20$ iff $x+24=36$."

Equivalence and equality are two different relations. Expressions are said to be "equal", for example $3+7=20\div 2$. Equations are said to be "equivalent", for example $2x+1=7 \Longleftrightarrow 6x+3=21$. Like equality, equivalence is reflexive, symmetric, and transitive.

We should note that contrary to what you may read and hear, the letter x in the equation $x+4=9$ is not a *variable*. It is called a *letter* and it represents an *unknown* quantity, not a varying quantity.

With vocabulary and notation taken care of, let's return to our search for a method to find the solution of $13x-117=104$.

The author hopes that the value of x that makes $13x-117=104$ true is not obvious to you. If it is, perhaps you will humor the author and pretend that it is not. Our strategy is to begin with the equation $13x-117=104$ and produce a chain of equivalent equations

$$\begin{aligned} 13x-117=104 &\Longleftrightarrow \\ &\vdots \\ &\Longleftrightarrow \\ &\vdots \\ &\Longleftrightarrow x=17. \end{aligned}$$

Until the last equation $x=17$ is produced. The solution of the last equation of the chain,

$$x=17,$$

is, as they say, "obvious to even the most casual observer". The value of x that makes $x=17$ true is 17. No doubt about it!

The good news is that you have already acquired the ideas and techniques needed to carry out the strategy we have outlined. Here goes.

Example 4.1

Solve $13x-117=104$ for the unknown.

Solution

Add 117 to both sides,

$$13x - 117 = 104 \iff 13x = 104 + 117 \tag{4.6}$$
$$\iff 13x = 221, \tag{4.7}$$

multiply both sides by $\frac{1}{13}$,

$$\iff x = \frac{221}{13} \tag{4.8}$$
$$\iff x = 17. \tag{4.9}$$

$\therefore$ the value of x that satisfies equation (4.6) is 17. ■

Since each equation of the chain connecting the first equation (4.6) to the last equation (4.9) was obtained by application of only axioms, definitions, and theorems,

$$13x - 117 = 104 \iff x = 17.$$

But this means that the solution of the last equation, where it is obvious, must be the solution to the first equation where it was non-obvious.

Remark 4.3 (Checking answers)
Great, but how do we know that each link of the chain is sound? Maybe a mistake was made and has gone undetected? If an error had occurred, it would mean that the value $x = 17$ is not necessarily the solution. Now that's a pesky thought, isn't it? But, that is why checking answers is advised. And we know how to check. Just substitute 17 for x in the equation $13x - 117 = 104$, then verify that in fact $LHS = RHS$. We do so.

$$\begin{aligned} \text{LHS} &= 13(17) - 117 \\ &= 221 - 117 \\ &= 104. \\ \text{RHS} &= 104. \\ \text{LHS} &= \text{RHS}. \quad \blacksquare \end{aligned}$$

As you progress to higher levels, many of the problems you work will be quite tedious to check by substitution. In those cases, most people check by searching their work for an error or by repeating the solution. That is in mathematics class. In practice, the amount of checking depends on what is at stake. You can bet that an engineering equation to determine the required diameter of steel cable that holds up a suspension bridge is checked *hundreds* or more times, regardless of the inconvenience.

Equation	Degree	Common Name
$3x+7=0$	1	linear
$5x^2+3x-17=0$	2	quadratic
$5x^3+3x^2-8x+1=0$	3	cubic
$x^4+4x^3-9x^2+x-7=0$	4	quartic
$7x^5-3x^4+7x^3-2x^2-5x+17=0$	5	quintic

TABLE 4.1. Table (Equations of various Degrees)

4.3. Linear equation

You will best appreciate what a linear equation is when you learn about other equations that are not linear. In spite of that, we say that a "**linear equation**" is an equation in which the unknown is of degree 1. An exponent of 1 is usually not written, because $x^1 = x$. The degree of the equation is determined by the greatest exponent to which the unknown is raised. Examples of equations of various degrees appear in Table (4.1).

No matter how complicated a linear equation may be, it can always be rewritten in the form

$$ax+b=0,$$

where x is the unknown. This is the background for the following definition.

Definition 4.2 (Linear equation)
An equation is a **linear equation** if it can be written in the form

$$ax+b=c$$

where x, a, b and c are any numbers and $a \neq 0$. ■

4.3.1. Addition and subtraction only

These equations may be solved merely by adding or subtracting from both sides of the equation.

Example 4.2

Solve for y, if $y+21=2$.

Solution

$$y+21=2 \Longleftrightarrow y=-19.$$

Example 4.3

Solve for x, if $3+x=-20$.

Solution

$$3+x=-20 \Longleftrightarrow x=-23.$$

Example 4.4

Solve for x, if $5x=12+4x$.

Solution

$$5x=12+4x \Longleftrightarrow x=12.$$

Example 4.5

Solve for x, if $x+\frac{2}{5}=3$.

Solution

$$\begin{aligned} x+\frac{2}{5}=3 &\Longleftrightarrow x=3-\frac{2}{5} \\ &\Longleftrightarrow x=\frac{13}{5}. \end{aligned}$$

Example 4.6

Solve for a, if $a+\frac{3}{7}=-1$.

Solution

$$\begin{aligned} a+\frac{3}{7}=-1 &\Longleftrightarrow a=-1-\frac{3}{7} \\ &\Longleftrightarrow a=\frac{-10}{7}. \end{aligned}$$

Exercise 4.2

Solve.

1. $-1 = 2n - 3n$
2. $2b + 1 - 2 = 5$
3. $-4 = r - 3 - 1$
4. $6 = x + x$
5. $-2 = -1 - 2n + 3$
6. $2 = b + 3 + 1$
7. $-v - 2v = 3$
8. $2x + 3 + 1 = 0$
9. $5 = -2x - 3x$
10. $2 = 2a - 1 - 3$
11. $8 = -2k - 2k$
12. $-2p + p = -2$
13. $2 = 3 + 2x - 1$
14. $-2 = -3n + n$
15. $-1 = -m + 2 - 2m$
16. $3r + r = 0$
17. $-7 = x - 2 - 2$
18. $3n + 3 + 3n = 3$
19. $-2 = b - 2b$
20. $5 = 1 + v + 1$
21. $-3 = x - 2x$
22. $-3 = 2n - 1 + 2$
23. $a + 3 + 2 = 2$
24. $2k - k = 0$
25. $x + 2 + x = 4$
26. $-x + 3x = -4$
27. $-2n + n = 2$
28. $-m + 2m = -2$
29. $1 = 3p - 2 - 2p$
30. $-3 = -2x + 3x$

4.3.2. Multiplication and division only

The word "**coefficient**" means the number that multiplies a letter.

(1) In the expression $3x + 5$, 3 is the coefficient of x.

(2) In the expression $9a - 2$, 9 is the coefficient of a.

(3) In the expression $-7 - 105y$, -105 is the coefficient of y.

(4) In the expression $-7 - \frac{2}{3}y$, $\frac{-2}{3}$ is the coefficient of y.

There is more to say about the idea of a coefficient, but we have said enough for now.

Equations that involve only multiplication and division may be solved merely by multiplying both sides of the equation by the multiplicative inverse of the coefficient of the unknown.

Example 4.7

Solve for x, if $5x = 35$.

Solution

Since the coefficient of x is 5, multiply both sides by $\frac{1}{5}$.

$$\begin{aligned} 5x = 35 &\iff \frac{1}{5}(5x) = \frac{1}{5}(35) \\ &\iff x = \frac{35}{5} \\ &\iff x = 7. \end{aligned}$$

Example 4.8

Solve for x, if $12x = 13$.

Solution

Multiply both sides by $\frac{1}{12}$.

$$\begin{aligned} 12x = 13 &\iff \frac{1}{\cancel{12}_1}(\cancel{12}^1 x) = \frac{1}{12}(13) \\ &\iff x = \frac{13}{12}. \end{aligned}$$

Example 4.9

Solve for x, if $\frac{x}{3} = \frac{2}{5}$.

Solution

The coefficient of x is $\frac{1}{3}$, so multiply both sides by 3.

$$\begin{aligned} \frac{x}{3} = \frac{2}{5} &\iff 3 \cdot \frac{x}{3} = 3 \cdot \frac{2}{5} \\ &\iff x = \frac{6}{5}. \end{aligned}$$

Example 4.10

Solve for x, if $\frac{3x}{5} = 45$.

Solution

Multiply both sides by $\frac{5}{3}$.

$$\frac{3x}{5} = 45 \iff \frac{5}{3}\left(\frac{3x}{5}\right) = \frac{5}{3}(45)$$
$$\iff x = 75.$$

Example 4.11

Solve for x, if $\frac{x}{2012} = \frac{2}{2013}$.

Solution

Multiply both sides by 2012.

$$\frac{x}{2012} = \frac{2}{2013} \iff (\cancel{2012}) \cdot \frac{x}{\cancel{2012}} = (2012) \cdot \frac{2}{2013}$$
$$\iff x = \frac{4024}{2013}.$$

Exercise 4.3

Solve.

1. $5x = 30$
2. $3x = 2$
3. $\frac{x}{5} = 30.$
4. $\frac{x}{3} = 45.$
5. $\frac{x}{2} = \frac{3}{7}.$
6. $\left(\frac{2}{3}\right)x = \frac{1}{5}.$
7. $\frac{5x}{6} = 35.$
8. $\frac{x}{113} = \frac{112}{113}.$
9. $\frac{7x}{15} = \frac{4}{11}.$
10. $\left(\frac{6}{13}\right)x = \frac{3}{26}.$
11. $\left(\frac{23}{1137}\right)x = \frac{1}{1137}.$
12. $\frac{10x}{1331} = \frac{3}{1332}.$

Example 4.12

Solve for x, if $0.3x = 12$.

Solution

Rewriting 0.3 as $\frac{3}{10}$ makes the arithmetic easier.

$$\begin{aligned} 0.3x = 12 &\iff \frac{3x}{10} = 12 \\ &\iff x = \frac{10 \cdot \cancel{12}^{4}}{\cancel{3}} \\ &\iff x = 40. \end{aligned}$$

Example 4.13

Solve for x, if $0.35x = 11$.

Solution

Solution 1. If you are not required to answer with a decimal, then rewriting 0.35 as $\frac{35}{100}$ makes the arithmetic easier.

$$\begin{aligned} 0.35x = 11 &\iff \frac{35x}{100} = 11 \\ &\iff x = \frac{11 \cdot \cancel{100}^{20}}{\cancel{35}_{7}} \\ &\iff x = \frac{220}{7}. \end{aligned}$$

Solution 2. If you are required to answer with a decimal, then you would continue Solution 1 by writing $\frac{220}{7}$ as a decimal.

$$\begin{aligned} &\iff x = \frac{220}{7} \\ &\iff x = 31.4286. \end{aligned}$$

If a decimal answer is required, it might seem easier to keep 0.35 in decimal form. Then, This only seems easier. To finish, you end up computing $1100 \div 35$. This is no easier than $220 \div 7$.

$$\begin{aligned} 0.35x = 11 &\iff \frac{35x}{100} = 11 \\ &\iff x = \frac{11}{0.35} \\ &\iff x = 31.4286. \quad \blacksquare \end{aligned}$$

Usually clearing decimals makes the arithmetic easier. Several examples of clearing decimals follow.

Example 4.14

(a) $0.3x = 0.8 \iff (10)(0.3)x = (10)(0.8) \iff 3x = 8.$
(b) $0.27x = 0.5 \iff (100)(0.27)x = (100)(0.5) \iff 27x = 50.$
(c)

$$\begin{aligned} 36.4x = 0.71 &\iff (100)(36.4)x = (100)(0.71) \\ &\iff 3640x = 71. \end{aligned}$$

Exercise 4.4

Rewrite without decimals.

1. $0.25x = 0.72$
2. $0.801x = 0.4$
3. $1.31x = 0.602$
4. $11.91x = 1.3$
5. $0.001x = 0.5$

4.3.3. Several operations

Solving linear equations in one unknown uses nearly all the algebra you have learned so far. Even so, it is a process that will become routine for you. The usual strategy is

(1) simplify both sides of the equation including clearing fractions,
(2) add or subtract to get all terms with the unknown on one side of the equation, numbers on the other side,

(3) if the coefficient of the unknown is not already 1, then multiply the equation by the reciprocal of the coefficient of the unknown.

Example 4.15

Solve $15x+7=9x-8$.

Solution

$$\begin{aligned} 15x+7=9x-8 &\iff 15x=9x-15, && \text{subtract } 7\\ &\iff 6x=-15, && \text{subtract } 9x\\ &\iff x=\frac{-15}{6}, && \text{multiply by } \frac{1}{6}\\ &\iff x=\frac{-5}{3}. \end{aligned}$$

We have been conscientious in stating that each equation in the chain of equations is equivalent ($\iff$) to the preceding one. In practice, the equivalence is not usually stated, it being understood that a chain of equivalent equations is intended. We will follow that practice.

Example 4.16

Solve $3x-17=5x+3$.

Solution

$$3x-17=5x+3$$

$$3x=5x+20$$

$$-2x=20 \tag{4.10}$$

$$x=-10. \tag{4.11}$$

■

The reader should think the annotation for each line of the solution. For instance, when passing from equation (4.10) to equation (4.11), think "multiply by negative one-half" or "divide by negative two".

Equations often involve distribution.

Same strategy.
Simplify
Add (Subtract)
Multiply (Divide).

Example 4.17

Solve $10+3(2x+4)=x-13$.

Solution

$$
\begin{aligned}
10+3(2x+4) &= x-13 \\
10+6x+8 &= x-13, && \text{simplify} \\
6x+18 &= x-13, && \text{simplify} \\
6x &= x-31, && \text{add or subtract} \\
5x &= -31, && \text{add or subtract} \\
x &= \frac{-31}{5}, && \text{mutiply or divide.} \quad \blacksquare
\end{aligned}
$$

In all the previous examples we followed the routine introduced at the start of section (4.3.3): (1) simplify, (2) add (subtract), (3) multiply (divide). The next example is more complicated, but not harder, because it just more of what we have already been doing.

Example 4.18

Solve $6-3(x-7) = -(5x+13)+20$.

Solution

$$
\begin{aligned}
6-3(x-7) &= -(5x+13)+20 \\
6-3x+21 &= -5x-13+20 \\
-3x+27 &= -5x+7 \\
-3x &= -5x-20 \\
2x &= -20 \\
x &= -10.
\end{aligned}
$$

Exercise 4.5

Solve.

1. $79 = 5(1-4n)-6$
2. $5(-3a-3)+3a = -63$
3. $-114 = -6(5k+4)$
4. $-6(4p-1) = -90$
5. $-74 = -2(6x+1)$
6. $-5(1+6n)-6n = 67$
7. $-70 = 5(-5+3m)$
8. $-6(5+3r) = -102$
9. $4(-5+4x) = -100$
10. $2(5n+3)+3n = 71$
11. $96 = 6(-2b+4)$
12. $63 = 3(-4+5r)$
13. $-64 = -4(4x+4)$
14. $-3(1+6n) = 69$
15. $6+6(3-3a) = -84$
16. $5(6+4v) = 130$
17. $2(x+2)+4(-3x+6) = 38$
18. $2(-4x-6)-2(3+6x) = -58$
19. $-6(n-1)+4(n-6) = -16$
20. $37 = -(k-6)-6(1+6k)$
21. $-4(3p-6)-5(1-p) = 26$
22. $-42 = 4(x+4)-6(5x+1)$
23. $6(1-2n)+5(4-3n) = -55$
24. $27 = -(m+1)-3(6m-3)$
25. $-2(3r+3)+3(5-2r) = -51$
26. $28 = -4(1+3x)+4(x-2)$
27. $4 = -3(2n+4)+4(n+3)$
28. $-29 = 5(b-1)+6(-4+5b)$
29. $30 = 2(-6v-3)-2(2v+6)$
30. $-6(5x+2)+4(6x-6) = -36$

Exercise 4.6

Solve.

1. $68=-4(1-6k)$
2. $-5(2+5p)=-110$
3. $-2+5(2x-3)=-67$
4. $-77=4(-4n+2)-5$
5. $-140=-5(1-5m)+2m$
6. $6(1-3r)=78$
7. $6(1+5x)=-114$
8. $-6(6-2n)+3=-81$
9. $-3(4+5b)=78$
10. $3(-6v+1)=93$
11. $144=4(-6x+5)+4$
12. $-72=-4(2+4n)$
13. $-79=4a+5(1+2a)$
14. $6(4-5v)=-66$
15. $-4(-1+4x)-1=-93$
16. $-3(4x-5)-3x=75$
17. $-6(n+4)+4(n+5)=-2$
18. $-14=4(1-k)+6(-3+5k)$
19. $-4(3+4p)-2(-5p-2)=-2$
20. $24=4(5-4x)-6(2-4x)$
21. $17=6(n+3)-(n+6)$
22. $36=6(5+6m)-6(m+4)$
23. $6(-3r-2)-(-4+r)=30$
24. $-37=3(1+5x)+5(2x+2)$
25. $-2(-5+2n)-3(-1+2n)=-27$
26. $-5(-5b+2)+2(5b+3)=-4$
27. $3(1+4v)-4(1-2v)=59$
28. $-2(2x-2)-3(1-5x)=-10$
29. $-(5n+3)+6(2n-3)=-42$
30. $-(-3b+4)-6(b+5)=-49$

Exercise 4.7

Solve.

1. $m+5(-3m+6) = 4m+6(-m-1)$
2. $-n-2(2+3n) = -2(n-3)$
3. $2(6+3x) = 3(x-1)+6$
4. $-8(r+2) = 2(2r+4)$
5. $-7(n-7)-3(-n+8) = -7-n+5$
6. $-7(4v+6) = 7(-5v-4)$
7. $5b+6b = 2(b-2)-(b+6)$
8. $-2(1-3a)+6(-2a+7) = a-2a$
9. $-5(a-1)+8a = 2(a-3)$
10. $5(5-3x)-8(x+7) = -3x+7-x$
11. $6(-k-2) = 6(7k-2)$
12. $-7x-(x+6) = 2(-3x-5)$
13. $2(-7+8x) = 2x+2(2-2x)$
14. $-3(1+7n) = 6(1-2n)$
15. $-6-6(2m-8) = -3(1+3m)$
16. $-6(x+1) = -5(x+2)$
17. $-6p+6p = 4(-3p+5)-8(6+2p)$
18. $-6(-n-5)+6 = 6(1+6n)$
19. $6(3+8m) = 6(m-4)$
20. $-6(1+4x) = -5(5+5x)+4$
21. $-3(1-2r)+5r = 6(r-8)$
22. $2-7(-4+5x) = -5(x-6)$
23. $-7(r-8)-8 = -5(r-4)$
24. $-3(1+b) = 4b-(-1+6b)$
25. $-8+4(7x-4) = 8(-7+4x)$
26. $-4(7-2n)-3n = 7(n-6)$
27. $4(6-3a) = 4(4a-3)$
28. $4(4k+7)+3 = -8(-k+6)+7$
29. $7(p+5) = 5(1+3p)-2$
30. $3x+5(3-2x) = -4+4(4-x)$
31. $-4(2n+6) = -6-5(8+2n)$
32. $-6(m-4) = -(2m+8)$
33. $4(3r-2)+8(2-2r) = 4r-6+r+5$
34. $6(-4-6x) = -8(4x+5)$
35. $-8(5n+5)+6(8n+7) = n+6n$
36. $4(1-b) = 4(b+5)$
37. $-(1+v) = -8(5v+6)+8$
38. $-8-5(1+x) = -3(x-3)$
39. $7(n-4) = 8+4(n-6)$
40. $-2(n-8) = 8(n+7)$

Example 4.19

Solve $\dfrac{5x}{6}+14=4x+3.$

Solution

Take advantage of the fact that an equation may be multiplied by any number. Use this fact to clear fractions. In this example, multiply both sides by 6.

$$\frac{5x}{6}+14=4x+3.$$

$$\not{6}\left(\frac{5x}{\not{6}}\right).=6(4x-11) \qquad \text{Clear fractions.} \tag{4.12}$$

$$5x=24x-66. \qquad \text{Simplify.}$$

$$-19x=-66. \qquad \text{Subtract.}$$

$$x=\frac{66}{19}. \qquad \text{Multiply.}$$

When clearing fractions, a distribution is usually introduced. So, equation (4.12) is typical.

Example 4.20

Solve $\dfrac{7x}{3}=11x-4.$

Solution

$$\frac{7x}{3}=11x-4.$$

$$\not{3}\left(\frac{7x}{\not{3}}\right)=3(11x-4).$$

$$7x=33x-12.$$

$$-26x=-12.$$

$$x=\frac{6}{13}.$$

Example 4.21

Solve $-\frac{11}{12} = -\frac{7}{2}n + \frac{1}{3} + \frac{2}{3}n.$

Solution

$$
\begin{aligned}
-\frac{11}{12} &= -\frac{7}{2}n + \frac{1}{3} + \frac{2}{3}n \\
12\left(-\frac{11}{12}\right) &= 12\left(-\frac{7}{2}n + \frac{1}{3} + \frac{2}{3}n\right) \\
-11 &= \cancel{12}^{6}\left(-\frac{7}{\cancel{2}_1}n\right) + \cancel{12}^{4}\left(\frac{1}{\cancel{3}_1}\right) + \cancel{12}^{4}\left(\frac{2}{\cancel{3}_1}n\right) \qquad (4.13) \\
-11 &= -42n + 4 + 8n \\
-15 &= -34n \\
\frac{-15}{-34} &= n \\
n &= \frac{15}{34}.
\end{aligned}
$$

Example 4.22

Solve $3 - \frac{2}{3}x + \frac{7}{12} = -\frac{3}{4}x + \frac{2}{3} + \frac{5}{6}x.$

Solution

$$
\begin{aligned}
3 - \frac{2}{3}x + \frac{7}{12} &= -\frac{3}{4}x + \frac{2}{3} + \frac{5}{6}x \\
12\left(3 - \frac{2}{3}x + \frac{7}{12}\right) &= 12\left(-\frac{3}{4}x + \frac{2}{3} + \frac{5}{6}x\right) \qquad (4.14) \\
36 - 8x + 7 &= -9x + 8 + 10x \\
-8x + 43 &= x + 8 \\
-9x &= -35 \\
x &= \frac{35}{9}.
\end{aligned}
$$

"How much work *should* I show?" –Enough for a good student at your level to follow it.

In example (4.22) we did the distribution and arithmetic indicated in equation (4.14) mentally. Less clutter makes the work easier to follow than at equation (4.13) where less was done mentally. You may find that you make *fewer* mistakes by doing more algebra and arithmetic mentally.

Exercise 4.8

Solve.

1. $-\frac{3}{2} = r - \frac{11}{3} + \frac{5}{3}$
2. $-\frac{1}{3}x - \frac{4}{3}x = -\frac{25}{9}$
3. $\frac{-23}{6} = \frac{5b}{3} + \frac{3}{2} - 2$
4. $\frac{61}{9} = \frac{-7n}{2} + \frac{3}{2} + \frac{n}{3}$
5. $\frac{v}{3} - \frac{4v}{3} = \frac{7}{2}$
6. $\frac{1}{3} = \frac{4x}{3} - 1$
7. $\frac{3x}{2} + \frac{7x}{2} = \frac{15}{2}$
8. $0 = \frac{3a}{2} + 1 - 1$
9. $\frac{23}{3} = \frac{-7k}{3} - \frac{3k}{2}$
10. $\frac{17}{12} = \frac{3p}{2} + \frac{4p}{3}$
11. $\frac{11x}{3} + 1 - \frac{1}{2} = \frac{29}{3}$
12. $\frac{-2n}{3} + n = \frac{1}{2}$
13. $\frac{-41}{12} + x - \frac{11x}{3} = \frac{-7x}{2} - \frac{4}{3}$
14. $-\frac{16}{3} - \frac{1}{2}n = -n - \frac{7}{2}n$
15. $\frac{7}{3} - m = -\frac{4}{3}m + \frac{3}{2}$
16. $-5 - \frac{r}{2} = \frac{3r}{2} - \frac{1}{3}$
17. $\frac{-3x}{2} + 1 = \frac{5}{3} - x$
18. $\frac{5}{9} + n + \frac{3}{2} + \frac{7n}{3} = n + \frac{1}{2}$
19. $\frac{-5b}{3} - 3b = \frac{-32}{9} + \frac{2b}{3}$
20. $v + \frac{1}{2} = \frac{1}{4} + \frac{3v}{2}$
21. $\frac{-x}{2} - x - \frac{3}{2} = \frac{-5x}{2} + 1 - 2$
22. $\frac{-8n}{3} + 1 - 3 = \frac{-5}{6} - \frac{3n}{2}$
23. $\frac{3b}{2} + 1 = 2b + \frac{1}{2}$
24. $\frac{2b}{3} + \frac{3}{2} = \frac{19}{6} - b$

Exercise 4.9

Solve.

1. $0 = -2n - \frac{5n}{3}$
2. $r - \frac{7r}{2} = \frac{-35}{6}$
3. $2x + 2x = -4$
4. $\frac{3m}{2} + \frac{5}{3} - \frac{3}{2} = \frac{-13}{3}$
5. $\frac{n}{3} - \frac{4}{3} - \frac{n}{2} = \frac{-23}{18}$
6. $3 = 3b - \frac{3b}{2}$
7. $\frac{-41}{9} = \frac{-r}{3} + 1 + 2r$
8. $\frac{-3x}{2} + \frac{5x}{2} = \frac{8}{3}$
9. $\frac{-7}{6} = \frac{n}{3} - \frac{1}{3} + \frac{4n}{3}$
10. $\frac{a}{2} + \frac{5a}{2} = -4$
11. $-v - \frac{8v}{3} = \frac{22}{3}$
12. $x - \frac{1}{2} + \frac{x}{2} = -3$
13. $\frac{13}{9} - 2x = \frac{-5x}{3} + 1$
14. $\frac{-7n}{3} + 1 + 2n = \frac{21}{4} + \frac{5n}{2}$
15. $\frac{-35}{18} + k = \frac{k}{2} - 2$
16. $\frac{2p}{3} - \frac{8}{9} = p + \frac{p}{3}$
17. $5 - \frac{4x}{3} = -x - \frac{10x}{3}$
18. $\frac{n}{2} + 1 = \frac{-1}{6} + n$
19. $\frac{5}{18} + \frac{4r}{3} = r + \frac{1}{2}$
20. $\frac{5m}{3} - \frac{5}{2} - 2m$
    $= \frac{7}{9} + \frac{4m}{3} + \frac{3}{2} - 2$
21. $\frac{-4x}{3} + 2x = \frac{25}{9} + \frac{4x}{3} - \frac{11}{3} - \frac{4x}{3}$
22. $\frac{5b}{3} + \frac{3}{2} = -2b - \frac{13}{6}$
23. $\frac{2n}{3} + \frac{4}{3} = \frac{56}{9} + 2n$
24. $\frac{-3v}{2} + \frac{31}{12} = \frac{v}{3} - \frac{3}{2} - \frac{5}{3}$

4.3.4. Round up the usual suspects

There are a couple of troublemakers that sometimes show up when solving equations. Let's get acquainted with them before we continue solving linear equations.

The following two expressions tempt students into making invalid computations:

$$a\left(\frac{b}{a}+d\right), \tag{4.15}$$

$$\frac{ab+cd}{a}. \tag{4.16}$$

The invalid computations are these.

$$a\left(\frac{b}{a}+d\right)=\cancel{a}\left(\frac{b}{\cancel{a}}+d\right). \tag{4.17}$$

$$\frac{ab+cd}{a}=\frac{\cancel{a}b+cd}{\cancel{a}}. \tag{4.18}$$

To show that the computation (4.17) is invalid, it is enough to find one example where the computation results in a false statement.

$$\begin{aligned}\cancel{3}^1\left(\frac{7}{\cancel{3}_1}+5\right)&=7+5\\&=12. \quad \textbf{(FALSE)}\end{aligned}$$

Because,

$$\begin{aligned}3\left(\frac{7}{3}+5\right)&=\cancel{3}\left(\frac{7+15}{\cancel{3}}\right)\\&=22.\end{aligned}$$

To show that the computation (4.18) is invalid, consider

$$\begin{aligned}\frac{(2)(3)+(5)(6)}{2}&=\frac{(\cancel{2}^1)(3)+(5)(6)}{\cancel{2}_1}\\&=3+30\\&=33. \quad \textbf{(FALSE)}\end{aligned}$$

Because,

$$\begin{aligned}\frac{(2)(3)+(5)(6)}{2}&=\frac{6+30}{2}\\&=16.\end{aligned}$$

Exercise 4.10

[Part 1]

1. Show that the following computation is invalid.
$$\frac{1}{a}(a+b) = \frac{1}{\not{a}}(\not{a}+b).$$

2. Show that the following computation is valid. [Hint: use distribution in numerator.]
$$\frac{1}{a}(ab+ad) = \frac{1}{\not{a}}(\not{a}b+\not{a}d).$$

[Part 2] *Simplify each of the following. If an expression is already simplified, say so.*

1. $\frac{1}{2}(4a+1)$
2. $\frac{1}{3}(3a+3)$
3. $10-\frac{2}{5}(3a+5)$
4. $11-\frac{7}{3}(6a+3)$
5. $\frac{7+3x}{7}$
6. $\frac{6x+5}{3}$
7. $\frac{4+2a}{2}$
8. $\frac{4+4a}{2}$
9. $\frac{2+4a}{2}$
10. $\frac{11+11b}{11}$
11. $\frac{2+2b}{2}$
12. $\frac{2+32b}{3}$
13. $\frac{27+9b}{3}$
14. $\frac{81-9b}{3}$

4.3.5. Linear equations with subtle features

Let us solve the following two linear equations side-by-side so that we might compare the solutions:

$$\frac{2}{3}x+5=x-2 \tag{4.19}$$

$$\frac{2}{3}(x+5)=x-2. \tag{4.20}$$

Solution

$\frac{2}{3}x+5=x-2$	$\frac{2}{3}(x+5)=x-2$
(4.21) $3\left(\frac{2}{3}x+5\right)=3(x-2)$	$\cancel{3}\left(\frac{2}{\cancel{3}}\right)(x+5)=3(x-2)$
$2x+15=3x-6$	$2x+10=3x-6$
$-x=-21$	$-x=-16$
$x=21.$	$x=16.$

Notice the difference at equations (4.21). In the equation in the right hand column,

$$3\left(\frac{2}{3}\right)$$

is a product of two fractions and cancellation is possible.

But, in the equation in the left hand column

$$3\left(\frac{2}{3}x+5\right)$$

is a product of a fraction and the sum $\frac{2}{3}x+5$. As such, cancellation is not possible.

Now, $3\left(\frac{2}{3}x+5\right)$ may be rewritten

$$3\left(\frac{2x+15}{3}\right).$$

Although the fraction in the parentheses has a complicated numerator, the expression *is* never-the-less a product of fractions, so the cancellation

$$\cancel{3}\left(\frac{2x+15}{\cancel{3}}\right)$$

is valid.

Exercise 4.11

Solve.

1. $\frac{-637}{18} = 2k - \frac{7}{2}\left(\frac{-10k}{3} + 1\right)$
2. $\frac{-310}{9} = -\frac{10}{3}\left(-3x + \frac{4}{3}\right)$
3. $\frac{-11}{3}\left(2a - \frac{8}{3}\right) = \frac{286}{9}$
4. $\frac{-7}{2}\left(\frac{8n}{3} - 1\right) = \frac{217}{6}$
5. $\frac{-127}{4} = -3\left(\frac{-7p}{2} - \frac{5}{3}\right)$
6. $\frac{4}{3} - \frac{7}{2}\left(\frac{-10x}{3} + \frac{8}{3}\right) = -\frac{94}{3}$
7. $\frac{1837}{54} = \frac{-11}{3}\left(\frac{8x}{3} + \frac{1}{2}\right)$
8. $\frac{-7}{2}\left(\frac{-10m}{3} + 1\right) + \frac{1}{2} = -\frac{263}{6}$
9. $\frac{-1}{2}\left(x - \frac{5}{3}\right) = \frac{-1}{3}\left(\frac{4x}{3} + \frac{1}{2}\right)$
10. $\frac{7}{3}\left(n + \frac{3}{2}\right) = \frac{-10}{3} + \frac{4}{3}\left(\frac{2n}{3} + \frac{5}{3}\right)$
11. $\frac{4}{3} + \frac{1}{2}\left(\frac{3x}{2} + 1\right) = \frac{-10}{3}\left(x - \frac{2}{3}\right)$
12. $2b + \frac{1}{3}\left(\frac{b}{2} + \frac{5}{2}\right) = 2\left(\frac{4b}{3} + 1\right) - \frac{1}{2}$
13. $\frac{-5}{2}\left(\frac{-3n}{2} + \frac{1}{2}\right) = \frac{-7}{2}\left(n + \frac{4}{3}\right) + \frac{8}{3}$
14. $\frac{-3}{2}\left(\frac{v}{2} + 1\right) = \frac{-1}{2}\left(v + \frac{7}{2}\right)$
15. $\frac{1}{2} = \frac{3}{2}\left(\frac{b}{2} - \frac{5}{3}\right) - \frac{7}{2}\left(\frac{3b}{2} - \frac{5}{2}\right)$
16. $\frac{-4k}{3} - \frac{5}{2}\left(\frac{-2k}{3} - 2\right) = \frac{4}{3}\left(k + \frac{3}{2}\right)$

Exercise 4.12

Solve.

1. $\frac{-205}{6} = 3\left(\frac{5k}{2} - 1\right) + k$
2. $\frac{5x}{3} + \frac{7}{2}\left(\frac{-10x}{3} + 1\right) = \frac{67}{2}$
3. $3\left(\frac{-10x}{3} - \frac{7}{2}\right) = \frac{-71}{2}$
4. $\frac{-7}{2}\left(\frac{-7p}{2} + 1\right) - \frac{4}{3} = \frac{-1145}{24}$
5. $\frac{-11}{3}\left(\frac{-10m}{3} + 1\right) - \frac{m}{2} = \frac{-233}{6}$
6. $-\frac{286}{9} = -\frac{11}{3}\left(-\frac{5}{2}x - \frac{1}{2}\right)$
7. $\frac{141}{4} = \frac{-5a}{2} + 2\left(\frac{-7a}{2} + 1\right)$
8. $-\frac{7}{2}\left(\frac{8}{3}n + \frac{3}{2}\right) = -\frac{133}{4}$
9. $\frac{4}{3}\left(\frac{-8b}{3} + \frac{5}{2}\right) = \frac{-5}{3}\left(b + \frac{1}{2}\right) + \frac{3}{2}$
10. $2\left(\frac{-10r}{3} + \frac{1}{3}\right) = -r - \left(r - \frac{10}{3}\right)$
11. $\frac{-10b}{3} - b = \frac{8}{3}\left(\frac{-b}{2} + \frac{1}{2}\right) + 2\left(\frac{4b}{3} + 2\right)$
12. $-\left(\frac{8x}{3} + 1\right) + \frac{5x}{2} = 2\left(\frac{x}{2} + \frac{7}{3}\right)$
13. $\frac{n}{2} - n = \frac{1}{2}\left(\frac{5n}{2} + 2\right) - \frac{3}{2}\left(\frac{3n}{2} - \frac{10}{3}\right)$
14. $\frac{7}{3}\left(\frac{-5b}{3} - \frac{1}{2}\right) = \frac{-11}{3}\left(\frac{4b}{3} + 1\right)$
15. $-\left(\frac{-5x}{2} + \frac{3}{2}\right) = \frac{-11}{3}\left(x + \frac{3}{2}\right)$
16. $\frac{5x}{2} - \left(\frac{-3x}{2} + 1\right) = -2x - \frac{11}{3}\left(x + \frac{2}{3}\right)$

Exercise 4.13

Solve.

1. $-\left(\frac{-8n}{3}-\frac{8}{3}\right)=2\left(\frac{-3n}{2}+1\right)$
2. $2\left(\frac{3x}{2}+\frac{3}{2}\right)=\frac{5}{2}\left(\frac{x}{3}+\frac{2}{3}\right)-\frac{7x}{3}$
3. $-2b-\left(\frac{2b}{3}-\frac{4}{3}\right)=\frac{2}{3}\left(\frac{5b}{2}+1\right)$
4. $-\left(v-\frac{1}{3}\right)+\frac{2}{3}=\frac{1}{3}\left(\frac{-3v}{2}+1\right)-\frac{5}{3}$
5. $\frac{-1}{2}\left(\frac{-x}{3}+1\right)=\frac{1}{3}+\frac{2}{3}\left(-x+\frac{2}{3}\right)$
6. $\frac{-3}{2}\left(\frac{-n}{2}+\frac{1}{3}\right)+\frac{3}{2}=2\left(\frac{-3n}{2}-\frac{3}{2}\right)$
7. $2\left(3a+\frac{1}{2}\right)=3\left(\frac{3a}{2}-1\right)X$
8. $\frac{-7x}{2}+\frac{5}{3}\left(\frac{3x}{2}-2\right)=\frac{-10}{3}\left(2x-\frac{7}{2}\right)$
9. $2\left(\frac{4k}{3}+2\right)-\frac{4}{3}\left(k+\frac{3}{2}\right)=\frac{-7k}{2}-1+\frac{1}{3}$
10. $\frac{-11}{3}+\frac{3}{2}\left(x+\frac{2}{3}\right)=\frac{5}{2}\left(\frac{3x}{2}-1\right)+\frac{5}{2}$
11. $\frac{-3}{2}\left(\frac{-3n}{2}-\frac{3}{2}\right)+2=2\left(n-\frac{5}{2}\right)$
12. $\frac{5}{2}\left(\frac{5k}{3}+1\right)=\frac{5}{3}-\frac{10}{3}\left(\frac{k}{2}+1\right)$
13. $\frac{4}{3}\left(\frac{10n}{3}+\frac{4}{3}\right)+\frac{7}{2}\left(\frac{4n}{3}+1\right)=\frac{n}{3}-\frac{10n}{3}$
14. $\frac{-4}{3}\left(m+\frac{5}{2}\right)=\frac{-7}{2}\left(m+\frac{7}{3}\right)$
15. $\frac{1}{3}\left(x-\frac{2}{3}\right)=\frac{5}{3}\left(\frac{4x}{3}+1\right)$
16. $\frac{r}{2}+\frac{1}{2}-\frac{11}{3}r+\frac{8}{3}=2\left(r-\frac{1}{3}\right)+\frac{3}{2}\left(r+\frac{3}{2}\right)$
17. $\frac{-3}{2}\left(\frac{n}{3}+\frac{1}{2}\right)=\frac{-3}{2}\left(\frac{-7n}{2}+\frac{8}{3}\right)$
18. $\frac{-1}{2}\left(\frac{-3b}{2}+\frac{2}{3}\right)=\frac{5b}{2}+\frac{1}{2}\left(\frac{-3b}{2}+1\right)$
19. $-1+\frac{2}{3}\left(v+\frac{3}{2}\right)=\frac{5v}{3}-\frac{3}{2}\left(\frac{5v}{3}-\frac{3}{2}\right)$
20. $2\left(x-\frac{4}{3}\right)-2\left(\frac{5x}{2}+1\right)=\frac{5x}{2}-\frac{3x}{2}$
21. $2x+\frac{1}{3}\left(2x+\frac{1}{2}\right)=\frac{2}{3}\left(\frac{x}{3}+\frac{1}{3}\right)$

4.4. Rational expressions revisited

4.4.1. Factoring out $-1, -2, \cdots$

Students are sometimes surprised at what can be accomplished by recognizing -1 as a factor. Initially one might think that the expression

$$\frac{a-1}{1-a}$$

cannot be simplified. But, it does simplify.

Example 4.23

Simplify $\frac{a-1}{1-a}$, for $a \neq 1$.

Solution

$$\begin{aligned}\frac{a-1}{1-a} &= \frac{-1(1-a)}{1-a} \\ &= -1\left(\frac{1-a}{1-a}\right) \\ &= -1\left(\frac{\cancel{1-a}^{1}}{\cancel{1-a}_{1}}\right) \\ &= (-1)(1) \\ &= -1.\end{aligned}$$

Example 4.24

Simplify $\frac{2b-2}{1-b}, b \neq 1$.

Solution

$$\begin{aligned}\frac{2b-2}{1-b} &= \frac{-2(1-b)}{b-1} \\ &= -2\left(\frac{1-b}{1-b}\right) \\ &= -2\left(\frac{\cancel{1-b}^{1}}{\cancel{1-b}_{1}}\right) \\ &= (-2)(1) \\ &= -2.\end{aligned}$$

Exercise 4.14

Simplify.

1. $\dfrac{x-1}{1-x}$, $x \neq 1$
2. $\dfrac{1-b}{b-1}$, $b \neq 1$
3. $\dfrac{5-5a}{5a-5}$, $a \neq 1$
4. $\dfrac{8-8x}{2x-2}$, $x \neq 1$
5. $\dfrac{7-7y}{3-3y}$, $y \neq 1$
6. $-\dfrac{12-12x}{3-3x}$, $x \neq 1$
7. $21-\dfrac{24-24x}{2-2x}$, $x \neq 1$

4.5. A closer look at linear equations

Most of this section is Exercise (4.15). It may not be the easiest exercise. It will give you the opportunity to further examine the ideas you have acquired. You might have to grapple with some of the questions for possibly a long time. Patience and perseverance are desirable qualities in mathematics and they are qualities a person develops. You may gain more from this exercise by being stuck and from class discussion than by using the solutions to see how to do the problems.

Theorem 4.1
Every linear equation in one unknown has a solution in the rational numbers and this solution is unique. ■

Theorem (4.1) makes two statements in a compact sentence. The two statements are:

(1) Every linear equation in one unknown has a solution in the rational numbers.
(2) The solution mentioned in (1) is unique. There are no other solutions.

Exercise 4.15

[Part 1]

1. The equation $\frac{3}{7}x+2=0$ as written does not appear to be a linear equation. Why? [Hint: you need definition (4.2)].

2. Rewrite the equation $\frac{2}{5}x+\frac{1}{2}=3x-6$ in the form required by definition (4.2) and conclude that the equation is linear.

3. Solve the equation $3(2x-7)+5=2(3x-11)+12$.
   (a) What do you think the strange result means?
   (b) Does the result obtained contradict theorem (4.1)?

4. Solve the equation $3(2x-7)-9=2(3x-11)-8$.
   (a) What do you think this strange result means?
   (b) Does the result obtained contradict theorem (4.1)?

5. Make up several equations in which the unknown, call it x, occurs with an exponent of 1 and show that each can be rewritten in the form $ax+b=0$.

6. *Theorem (4.1) says that every linear equation in one unknown has a solution in the rational numbers. Prove this.

7. **Theorem (4.1) also says that the solution of every linear equation in one unknown is unique. Prove this.

8. **Is the inverse of a rational number other than 0 under multiplication unique? Or, could some rational number have more than one distinct inverse under multiplication? Prove that your answer is correct. [This question was raised by a seventh grader.]

[Part 2] *Questions* $1-5$ *refer to the following two tables that define operations* $\oplus$ *and* $\star$ *on the set* $\{0,1,2,3,4,5,6\}$.

$\star$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

$\oplus$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

Both operations $\oplus$ are commutative and associative. For $\oplus$ the identity element is 0 and each element has a unique inverse element. For $\star$ the identity element is 1 and each element other than 0 has a unique inverse element. We may use juxtaposition to indicate the operation $\star$. So *ab* means $a\star b$. Furthermore, the operation $\star$ distributes over $\oplus$; that is, $a\star(b\oplus c) = a\star b\oplus a\star c$.

Solve each of the following equations.

1. $3x\oplus 5 = 2$
2. $5x\oplus 4 = 3$
3. $2(x\oplus 5) = 6$
4. $4(3x\oplus 1) = 2x\oplus 5$
5. $3(5x\oplus 4) = 2(2x\oplus 3)$

Chapter 5

Applications

5.1. Introduction

Some people believe that the reason to study mathematics is that it is useful for accomplishing practical tasks. The author of this book believes that the reason to study mathematics is the intellectual and aesthetic satisfaction that human beings find in it. He hopes you agree with him. That said, it is a fact that mathematics is enormously useful. And its utility ranges from baking a cake to positioning a space vehicle on Mars. There is not a bridge you cross that does not owe its existence to applied mathematics. Additionally, it is kind of fun to discover how easily certain problems are solved once a little mathematics is used. Admittedly, the problems you will solve in this book are much more humble than designing the Brooklyn Bridge, but you have got to start someplace, right?

Translating facts given in words or pictures into mathematical expressions and equations is an acquired skill. Like most skills, it takes practice. Sometimes you will be frustrated. But, persevere. There is no other path to success.

5.2. Words to equations

Example 5.1

Suppose there are 12 more girls than boys enrolled at a certain school. If there are 252 students, how many boys and how many girls are enrolled?

It is easy to draw a picture that represents the information provided.

Once the bar representing the number of boys is drawn, the bar representing the number of girls can be drawn in only one way.

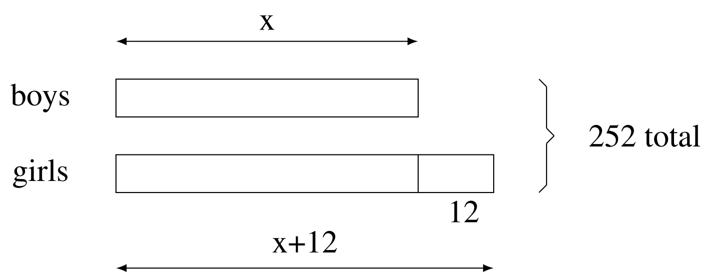


FIGURE 5.1. Boys and girls at a school.

From the picture, we can see that the quantity $x+(x+12)$ must be equal to 252.

$$x+(x+12)=252.$$

Solving for x,

$$2x+12=252$$
$$x=120.$$

So, we conclude that at this school there are 120 boys and 132 girls.

■

We can work the problem of Example (5.1) without the picture, but we will have to be careful to say exactly what the letter "x" represents. Here goes.

Example 5.2

Suppose there are 12 more girls than boys enrolled at a certain school. If there are 252 students, how many boys and how many girls are enrolled?

Solution

Once you declare that "x = the number of boys", the rest of the story can only unfold one way.

Let x represent the number of boys.

Then $x+12$ represents the number of girls.

Hence, $x+(x+12)=252$.

So, $x = 120$.
Therefore, there are 120 boys and 132 girls.

Remark 5.1
Our goal is to minimize the amount of "insight" or "inspiration" required to solve word problems. You should not have to face a word problem worrying that you might not "see" it. Or, even worrying that you will not know how to start. To achieve this goal, you must practice the process illustrated in these examples even when the problem is so simple for you that the answer is almost immediately obvious. ■

In Example (5.2), once x is chosen to represent the number of boys, the number of girls can only be written in one way, $x+12$. After that, there is only one way to write the total number of boys and girls, $x+(x+12)$. Since the total number of students is 252, $x+(x+12)=252$.

It was your choice of x to represent the number of boys that determined how every other piece of information would be written.

There is nothing magic about using a letter to represent a quantity. In fact, when you see the letter, it may be best to say of think what it represents. For instance in Example (5.2) when we write "$x+12$", we might say or think "The number of boys plus 12." The reason to write the letter "x", instead of the phrase "the number of boys" is merely convenience: imagine using the phrase "the number of boys" in the equations instead of "x". Awkward, right?

Suppose you are working Example (5.2) and have just declared "Let x represent the number of boys." After this declaration the letter x and the phrase "the number of boys" are interchangeable.

It may try your patience, but here is Example (5.2) with notes about what to think as you write.

Example 5.3

Suppose there are 12 more girls than boys enrolled at a certain school. If there are 252 students, how many boys and how many girls are enrolled?

Solution

Let x represent the number of boys.

Think: I choose to write x in place of writing "The number of boys".

Then $\mathbf{x+12}$ represents the number of girls.

Think: The number of girls is 12 more than the number of boys.

Hence, $\mathbf{x+(x+12)=252}$.

Think: The number of boys plus the number of girls totals 252.

So, $\mathbf{x=120}$.

Think: The number of boys is 120.
Therefore, there are 120 boys and 132 girls.

Example 5.4

Suppose there are twice as many girls as boys in a certain class. If there are 27 students, how many boys and girls are in the class?

We can, as before, draw a picture.

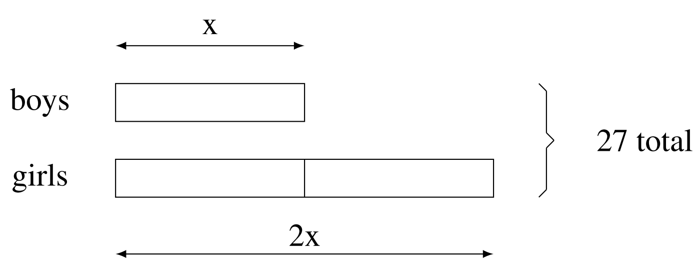


FIGURE 5.2. Boys and girls at a school.

From the picture, we can see that the quantity $x+2x$ must be equal to 27. We could write

$$x+2x=27.$$

Solving for x,

$$3x=27$$
$$x=9.$$

So, we would conclude that there at this school there are 9 boys and 18 girls. ■

We can work the problem of example (5.4) without the picture. We do so in the next example.

Example 5.5

Suppose there twice as many girls as boys in a certain class. If there are 27 students, how many boys and girls are in the class?

Solution

Let x represent the number of boys.
Then,
$2x$ represents the number of girls,

$x + 2x = 27$, [*What are you thinking?*]
$x = 9$.
Therefore, there are 9 boys and 18 girls.

Example 5.6

Janice sold $^3/5$ as many cupcakes as did Brian. Together they sold 80 cupcakes. How many cupcakes did each person sell?

Solution

Let x represent the number of cupcakes Brian sold.
Then,
$\frac{3}{5}x$ represents the number of cupcakes Janice sold, [*Thinking what?*]

$x + \frac{3}{5}x = 80$.

$x = 50$.

Therefore, Brian sold 50 cupcakes and Janice sold 30 cupcakes.

Example 5.7

John's bag of oranges contained 7 less than 3 times the quantity in Alicia's bag. If together the bags contained 53 oranges, how many oranges were contained in each bag?

Solution

Let $x =$ the number of oranges in Alicia's bag.
Then,
$3x - 7 =$ the number of oranges in John's bag,
$x + 3x - 7 = 53$. [*What are you thinking?*]
So, $x = 15$, and $3x - 7 = 38$.
$\therefore$ Alicia's bag held 15 and John's bag contained 38 oranges.

Example 5.8

Jim had 8 times as many dimes as quarters and 12 times as many nickels as dimes. If he had \$40.95, how many quarters did he have?

Solution

Let $x =$ the number of quarters.
Then,

$\$.25x =$ the *v*alue of the quarters,
$8x =$ the quantity of dimes,
$(\$.10)8x =$ the *v*alue of the dimes,
$12 \cdot 8x =$ the quantity of nickels,
$(\$.05)(12 \cdot 8x) =$ the *v*alue of nickels.

Since the *v*alue of all the coins is \$40.95,

$$\begin{aligned} \$.25x + \$(.10)8x + \$(.05)(12 \cdot 8x) &= \$40.95 \\ 25x + (10)8x + (5)(12 \cdot 8x) &= 4095 \\ 25x + 80x + 480x &= 4095 \\ 585x &= 4095 \\ x &= 7. \end{aligned}$$

$\therefore$ Jim had 7 quarters.

Remark 5.2
If you are careful to write " Let $x =$ the number of quarters." Then continue to think "the number of quarters" whenever you see or write "x", you will not mix up the *value* of the quarters with the *number* of quarters. You will avoid the mistake that beginners typically make with this problem. ■

The purpose of Exercise (5.1) is help you learn an effective process for solving word problems. To benefit from this exercise, focus on the *process*, not the answer. Your work should be similar to the preceding examples and to the solutions in the appendix. In the future, when the answers are not obvious, you might be glad you learned this process.

Exercise 5.1

1. Josh earned \$10 more than Joe. Their total earnings were \$1200. How much did each earn?
2. The capacity of a tank is 10 gallons less than the capacity of another tank. If together they hold 152 gallons, how many gallons does each tank hold?
3. A rectangle is 10 inches longer than it is wide and half its perimeter is 152 inches. How wide and long is the rectangle?
4. Mrs. Lu gave her children \$2.40 in quarters, dimes, and nickels. The number of nickels was twice the number of quarters. The number of quarters was twice the number of dimes. How many of each coin did she give out?
5. Two inches are cut off one side of a square paper and 3 inches are added to an adjacent side. The resulting rectangle has a perimeter of 54 inches. What are the dimensions of the original square?
6. Linda has a certain number of nickels and Betsy has the same number of pennies. The sum of the amounts of money possessed by the two girls is \$2.10. How much money has each girl?
7. John is three times as old as Dick, and three years ago the sum of their ages was 22 years. How old is each now?

Although no law says you have to draw a picture, sometimes it can be helpful to do so.

Example 5.9

Alex, Bill, and Carla share a sum of money in the ratio 4 : 5 : 6. If Carla receives \$80 more than Alex, find the sum of money shared by the three children?

Solution.

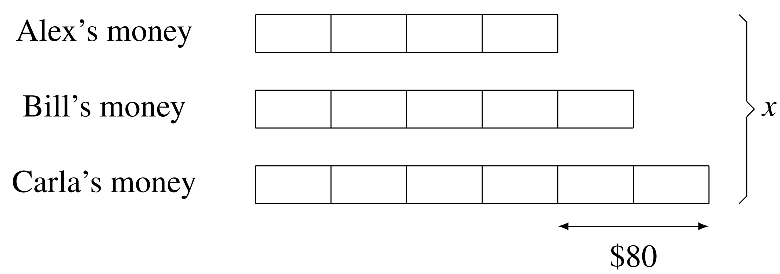


FIGURE 5.3. A very helpful picture.

The picture makes plain that each unit represents \$40 and the total, represented by "x", is 15 such units. So, the total amount of money shared by the three children is $15 \cdot \$40 = \$600$.

The solution could be written as follows.

Let $x =$ the sum of money (\$).
Then,

$$\frac{6}{15}x = \text{Carla's share (\$)},$$

$$\frac{4}{15}x = \text{Alex's share},$$

$$\frac{6}{15}x - \frac{4}{15}x = 80.$$

$$6x - 4x = 1200. \qquad x = 600.$$

$\therefore$ The sum of money shared by the three people is \$600.

Exercise 5.2

1. The area of one circle is $\frac{7}{8}$ that of a rectangle. If the sum of the areas is 105 square inches, what is the area of the rectangle?
2. Jim had \$4.40 in nickels, dimes, and quarters. If he had the same number of each denomination of coin, how many coins did he have?
3. In a collection of stamps, the number of Canadian stamps is 14 less than three times the number of United States stamps. The total number of Canadian and United States stamps is 78. How many United States stamps are there?
4. Alice had 18 times as many nickels as Bob had quarters. Together they had \$18.40. How many coins did each child have?
5. Barb is 4 times as old as Cindy. Four years ago the sum of their ages was 37 years. How old is Cindy?
6. Russel though he would be clever by paying a debt using an equal number of of nickels, dimes, and quarters? How many coins did he use to pay the debt of \$25.60?
7. One tank contains 5 gallons of water and is being filled at the rate of 12 gallons per minute. Another tank contains 61 gallons and is being drained at the rate of 4 gallons per minute. After how many minutes do the tanks contain the same volume of water?
8. Abigail has \$142 and Barbara has \$20. How much money should Abigail give to Barbara so that Abigal would have five times as much money as Barbara?
9. A farmer used $\frac{7}{8}$ as much fuel in July as he used in June. If he used 310 gallons of fuel less in July than in June, find the quantity of fuel he used in June?
10. Sue had three lame horses. The veterinarian's charge for the second horse was $\frac{2}{3}$ the charge for the first horse and the charge for the third was $1\frac{3}{4}$ the charge for the second. If the total charge was \$408, how much was the charge for the first horse?
11. Arthur and Jill had equal amounts of money. After Arthur spent \$30 and Jill spent \$15, the ratio of Arthur's money to Jill's money was 4 : 5. How much did each person have originally?
12. In January, Ace had twice as much money as Sam. In February, Ace increased his money by 40% and Sam increased his by 30%. At the end of February, Ace had \$12 more than Sam. How much money did each boy have in January?
13. In a school, 70% of the teachers are female. If there are 36 more female than male teachers, how many teachers does the school have?

14. Alice had $\frac{3}{5}$ as many marbles as Bridget. After Bridget gave $\frac{1}{4}$ of her marbles to Alice, Alice had 170 marbles. How many marbles did Alice have originally?
15. Roger spent 30% of his money on a book and $\frac{3}{5}$ of the remainder on a pen. He had \$28 left. How much money did he have at first?

5.3. Units of measure

Practical applications of mathematics *always* include units of measure such as minutes, hours, feet, meters, kilograms, pounds, hogsheads and many others. Knowing the distance from here to there is 612, is not especially helpful. If it is 612 feet, you will be there in a jiffy, but let it be 612 miles and you will need to pack several lunches unless you go by jet plane.

Many of the units with which you will work will be in the form of a fraction. An example of this is miles per hour which we write as $\frac{m}{h}$ or $\frac{mi}{h}$. We could write this as $\frac{1 \cdot m}{1 \cdot h}$. We usually do not bother writing the "1". When you see $\frac{m}{h}$, it is helpful to know that the numerator is a product of two factors, "1" and "m" and the denominator is a product of two factors, "1" and h". Similarly, the numerator of $\frac{60m}{h}$ is the product of factors 60 and m. Just as always, factors common to the numerator and denominator may be canceled.

Example 5.10

$$\frac{60m}{m} = \frac{60\cancel{m}}{\cancel{m}} = 60.$$

Example 5.11

$$\frac{\frac{60m}{h}}{m} = \frac{60m}{h} \cdot \frac{1}{m} = \frac{60\cancel{m}}{h} \cdot \frac{1}{\cancel{m}} = \frac{60}{h}.$$

Example 5.12

$$\frac{20m}{\frac{10m}{h}} = 20m \cdot \frac{h}{10m} = \cancel{20}\cancel{m} \cdot \frac{h}{\cancel{10}\cancel{m}} = 2h.$$

Exercise 5.3

Simplify the following.

1. $\dfrac{42m}{\frac{6m}{h}}$

2. $\dfrac{12\,\text{gallon}}{\frac{6\,\text{gallon}}{\text{minute}}}$

3. $\dfrac{15\,\text{hogshead}}{\frac{3\,\text{hogshead}}{h}}$

4. Divide 180 miles by 30 miles per hour.

5. Divide 15 miles by 10 miles per hour.

6. Divide 24 liters by 5 liters per minute.

5.4. Conversion of units

To rewrite a of measure, we can use the fact that multiplying by the number 1 changes the name but not the quantity. Along with this, it is helpful to know facts such as the following.

$$\begin{aligned} 1 \text{ minute} &= 60 \text{ seconds}, & 1 \text{ hour} &= 60 \text{ minutes} \\ 1 \text{ foot} &= 12 \text{ inches}, & 1 \text{ mile} &= 5280 \text{ feet} \\ 1 \text{ meter} &= 100 \text{ centimeters}, & 1 \text{ kilometer} &= 1000 \text{ meters} \\ 1 \text{ gram} &= 1000 \text{ milligrams}, & 1 \text{ kilogram} &= 1000 \text{ grams} \end{aligned}$$

Does $\frac{60 \text{ minutes}}{\text{hour}} = 1$? Yes, because,

$$60 \text{ minutes} = 1 \text{ hour} \iff \frac{60 \text{ minutes}}{\text{hour}} = 1.$$

Example 5.13

Rewrite 30 miles per hour as miles per minute.

Solution

$$\left(\frac{30 \text{ mile}}{\cancel{\text{hour}}}\right) \cdot \left(\frac{1 \cancel{\text{hour}}}{60 \text{ minute}}\right) = 0.5 \frac{\text{mile}}{\text{minute}}$$

Example 5.14

Rewrite 4 meters per second as kilometers per hour.

Solution

$$\left(\frac{4 \cancel{\text{m}}}{\cancel{\text{s}}}\right) \cdot \left(\frac{60 \cancel{\text{s}}}{1 \cancel{\text{min}}}\right) \cdot \left(\frac{60 \cancel{\text{min}}}{1 \text{ h}}\right) \cdot \left(\frac{1 \text{ km}}{1000 \cancel{\text{m}}}\right) = 14.4 \frac{\text{km}}{\text{h}}$$

Exercise 5.4

Rewrite each expression using juxtaposition instead of the "×" symbol.

1. Rewrite 2 miles per minute as miles per hour.
2. Rewrite 1200 meters per minute as kilometers per hour.
3. Rewrite 2200 centimeters per second as meters per minute.
4. Rewrite 1200 feet per second as miles per hour.
5. Rewrite 33000 feet per minute as miles per hour.

5.5. Distance, rate, and time

It is a fact of physics that the distance one travels is directly proportional to the constant rate at which one travels and to the duration of time one travels at that rate. This is not hard to believe, because your own experience tells you it is true.

You know that if you travel at a certain rate for two hours you will travel twice as far as you would in one hour; in four hours, four times as far as in one hour, in six hours, six times as far as in one hour. This is summed up by saying that "distance traveled is *directly proportional to the time* of travel".

Similarly, you know that if you travel for a three hours at twice the rate of your friend who also travels three hours, you will travel twice as far as she. Travel at four times her rate and you will cover four times the distance she does in a given period of time. And this means "distance traveled is *directly proportional to the rate* of travel".

The fact that distance traveled, at a constant rate, is directly proportional to time of travel at that rate and to the rate itself is captured in the following equation

$$d = rt, \tag{5.1}$$

where d = distance traveled, r = constant rate of travel, and t = time of travel.

Equation (5.1) is humble in appearance, but important precisely because of its simplicity.

The phrase "constant rate" has now appeared several times. It means that the rate is unvarying. A car that travels at a constant rate of speed, neither speeds up nor slows down. When the driver presses the accelerator, the car speeds up and for that time its speed is *not* constant. From your experience, you know that you can feel the change in speed.

Another phrase that means constant rate is "uniform rate". Instead of saying "rate of travel", we usually just say "speed". It is also a fact that when traveling at a constant rate, the average rate of travel equals the rate of travel.

We are careful to note that Equation (5.1) holds only for cases of constant rate.

Equation (5.1) can be rewritten in two other forms

$$r = \frac{d}{t} \tag{5.2a}$$

$$t = \frac{d}{r} \tag{5.2b}$$

Equation (5.2a) tells us that in all cases of travel at a constant rate, the ratio of distance to time is constant.

Example 5.15

The following table provides information about a one hour journey by automobile. Assume the speed was constant in each time interval. Was the speed constant throughout this journey?

Distance traveled for several time intervals				
time (minutes)	0 – 15	15 – 30	30 – 45	45 – 60
distance (miles)	25	25	30	25

TABLE 5.1. A journey by automobile

The speed was not constant for the whole trip. From 30 to 45 minutes the ratio $\frac{\text{distance}}{\text{time}}$ was $\frac{30 \text{ mile}}{15 \text{ minute}} = 2$ miles per minute, but this differs from the speed in the other time intervals which was $\frac{25 \text{ mile}}{15 \text{ minute}}$, about 1.6 miles per minute.

Example 5.16

Sally traveled by car at a constant speed of $50\frac{m}{h}$. If she traveled 450 miles, for what duration of time did she travel?

Solution

$$d = rt$$
$$t = \frac{d}{r}$$
$$d = 450\,m$$
$$r = 50\,\frac{m}{h}$$
$$t = \frac{450\cancel{m}}{50\frac{\cancel{m}}{h}}$$
$$t = 9\,h$$

Indenting the two lines where d and t are determined organizes the work clearly. We feel that determining d and t are "tasks" within the overall "project" of finding t.

$\therefore$ Sally traveled by car for 9 hours.

Example 5.17

Ali took 3 hours to travel from Place A to Place B at an average speed of 80 km/h. On the way back, he drove at an average speed of 60 km/h. How long did he take to travel from Place B to Place A?

Solution

Begin by stating the idea you will use to solve the problem. Here the idea is that distance is directly proportional to the time taken and to the rate of travel; that is, $d = rt$.

Let $d = \text{distance}_{B\to A}$, $r = \text{speed}_{B\to A}$, $t = \text{time}_{B\to A}$.

$$d = rt$$
$$t = \frac{d}{r}$$
$$r = 60\,\frac{km}{h}$$
$$d = \frac{80 \text{ km}}{h} \cdot 3h = 240 \text{ km}$$
$$t = \frac{240\cancel{km}}{60\frac{\cancel{km}}{h}}$$
$$t = 4\,h.$$

$\therefore$ Ali took 4 hours to return to Place A from Place B.

Example 5.18

Emily took 8 hours to travel from Place A to Place B at an average speed of 40 km/h. Dylan took only 5 hours for the trip. What was Dylan's average speed for the trip from A to B?

Solution

Let d = distance A→B, r = Dylan's speed A→B, t = Dylan's time A→B.

$$d = rt$$

$$r = \frac{d}{t}$$

$$t = 5h$$

$$d = \frac{40\ \text{km}}{h} \cdot 8h = 320\ \text{km}$$

$$r = \frac{320\ km}{5\ h}$$

$$r = 64\,\frac{km}{h}.$$

∴ Dylan's average speed from A to B was 64 km/h.

Example 5.19

Becky drove for 1 hour at 70 mph. Then she drove for 6 hours at 56 mph. What was her average speed for the entire journey?

"mph" is a common way to write "miles per hour".

Solution

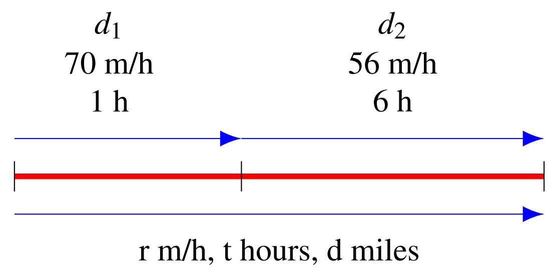


FIGURE 5.4

Let d = total distance, r = average speed, t = total time.

$$d = rt$$

$$r = \frac{d}{t}$$

$$t = 1h + 6h = 7h$$

$$d = d_1 + d_2$$

$$= \frac{70\,m}{h} \cdot 1h + \frac{56\,m}{h} \cdot 6h = 406 \text{ m}$$

$$r = \frac{406\,m}{7\,h}$$

$$r = 58\,\frac{m}{h}.$$

$\therefore$ Becky's average speed for the entire journey was 58 mph.

Example 5.20

Tom traveled from Town A to Town B. He drove the first 30 miles at 50 mph. He drove the remaining 80 miles at 75 mph. What was Tom's average speed for the trip from Town A to Town B?

Solution

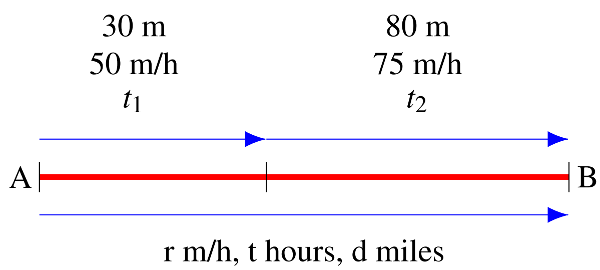


FIGURE 5.5

Let $d =$ total distance, $r =$ average speed, $t =$ total time.

$$d = rt$$

$$r = \frac{d}{t}$$

$$d = 30\,m + 80\,m = 110\,m$$

$$t = t_1 + t_2$$

$$= \frac{30\,m}{\frac{50\,m}{h}} + \frac{80\,m}{\frac{75\,m}{h}} = \frac{5}{3}\text{ h}$$

$$r = \frac{110\,m}{\frac{5}{3}h}$$

$$r = 66\,\frac{m}{h}.$$

$\therefore$ Tom's average speed driving from Town A to Town B was 66 mph.

Example 5.21

Al traveled 2/3 of the way from Town A to Town B in 4 hours. He traveled the remaining 90 km in 2 hours. What was his average speed for the whole trip?

Solution

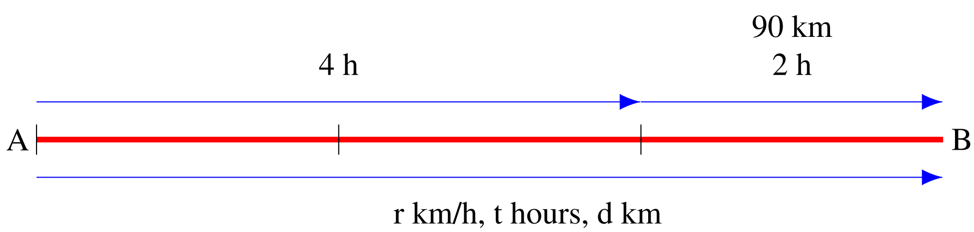


FIGURE 5.6

Let $d =$ total distance, $r =$ average speed, $t =$ total time.

$$d = rt$$

$$r = \frac{d}{t}$$

$$d = 3 \cdot 90\ km = 270\ km$$

$$t = 2\ h + 4\ h$$

$$r = \frac{270\ km}{6\ h}$$

$$r = 45\,\frac{km}{h}.$$

$\therefore$ Al's average speed driving from Town A to Town B was 45 km/h.

Remark 5.3
Each of the examples (5.16)-(5.21) required the key idea that $d = rt$. Every problem you work that concerns distance, rate, and time will rely on this key idea. You might as well begin, as in the examples, by stating this idea. As soon as you have done so, your focus shifts to two well defined tasks. Specifically, you must find values for the two quantities in the equation $d = rt$ that determine the quantity you wish to know. Devote your effort to those two tasks.

Remark 5.4
The pictures in examples (5.18)-(5.21) were useful. It will often be advantageous to draw such pictures when working problems involving distance, rate, and time.

Exercise 5.5

1. Jack spent 4 hours driving from Town A to Town B. His average speed was 65 mph. Find the distance of Town B from Town A.

2. A train traveled at a constant speed of 80 mph. How long did it take to travel 280 miles?

3. A boy ran at a constant speed of 8 mph for 15 minutes. Then he walked at 3 mph for 90 minutes. How far did the boy travel?

4. Janice walked 24 km. For the first 8 km, she walked at a rate of 6 km/h. Then she walked the rest of the way at 4 km/h. Find the time it took Janice to complete her walk.

5. Mr. Curtis bicycled 4 hours covering $\frac{5}{6}$ of the total distance to home. The last 30 km of his journey took 3 hours. What was his average speed for the journey home? (Round answer to nearest km/h)

6. Mrs. Jones drove from her house to her office at an average speed of 40 mph. It took her 3/4 hour. On the way home, her average speed was 50 mph. How many minutes did it take her to drive home?

7. Jack and Jill each drove from Place A to Place B, a distance of 160 miles. Jack's average speed was 40 mph. Jill left Place A 60 minutes later than Jack, but she arrived at Place B 30 minutes earlier than Jack. What was Jill's average speed?

8. Sally drove a round trip from Place A to Place B. The distance from A to B was 30 miles. She drove from Place A to Place B at an average speed of 60 mph. She drove back from Place B to Place A at an average speed of 30 mph. What was her average speed for the entire round trip?

9. A train traveled 3/5 of the way from Station A to Station B at an average speed of 60 mph. If the distance from Station A to Station B was 150 miles and the total time to travel from A to B was $3\frac{1}{2}$ hours, what was the average speed of the train for the remaining portion of the trip?

10. A train traveled 1/6 of the way from Station A to Station B in 1/2 hour at an average speed of 80 mph. The average speed of the train for the whole journey was 60 mph. How long did it take the train to travel the remaining portion of the journey?

11. Trains A and B traveling in the same direction on parallel tracks at 90 mph and 60 mph respectively pass the same station at the same time. How many minutes later will the distance between the trains be 10 miles?

12. Alice and Bill walk toward each other on path. Alice walks at a constant speed of 4 feet/second. Bill walks at a constant speed of 6 feet/second. If they are 3600 feet apart at noon, at what time will they meet?

13. Fred, who is walking at a constant speed of 300 feet/minute, passes point A at 1:00 PM. Gladys, running at a constant speed of 900 feet/minute to catch Fred, passes point A at 1:10 PM. At what time does Gladys catch Fred?

14. Trains A and B are 264 miles apart and traveling toward each other on parallel tracks at 72 mph and 60 mph respectively. How far apart will the two trains 12 hours later?

5.6. Additional word problems

Exercise 5.6

Solve the following.

1. If each child is given 3 marbles, there will be 8 marbles left over. But if each child were given 2 additional marbles, there would be 4 marbles too few. How many children are there?
2. The perimeter of a rectangle is 48 inches. The rectangle is 7 times longer than it is wide. How wide is the rectangle?
3. The sum of three consecutive integers is 456. Find these three integers.
4. A child is currently 14 years old and her mother is 45 years old. In how many years will the mother be twice as old as the daughter?
5. A retail store bought a number of motors at \$225 each. Twelve of the motors were damaged when moved by a fork lift truck. The retail store sold the undamaged motors for \$250 each and made a profit of \$5500. How many motors did store purchase?
6. The profit of a small company was 6% more this year than last year. If the company's profit this year was \$212000, what was its profit last year?
7. An jet airliner always arrives on time in Boston if it departs at 2:00 PM and flies at 450 mph. Today it left 25 minutes late, flew at 500 mph, and arrived 5 minutes early. At what time did the airliner arrive in Boston today?
8. Tom can paint the fence in 6 hours and Huck can paint the same fence in 9 hours. Tom begins, paints for a while, then Huck takes over. When Huck has painted 1 hour longer than Tom painted, the fence painting is completed. For how long did Tom paint the fence?
9. A truck passed a weigh station at 65 mph. Thirty minutes later, the truck slowed down to 40 mph because of heavy traffic and continued the trip at that speed. It reached its destination 30 minutes late. How far is the trucks destination from the weigh station?
10. A police officer was pursuing a robber. The officer drove at a uniform speed of 100 mph, the robber at a uniform speed of 95 mph. When observed at 1:00 PM, the robber was 1/2 mile ahead of the police officer. At what time does the police officer catch up to the robber?

Chapter 6

Linear Function

6.1. Introduction

We begin recalling what is by now a familiar object, the linear equation. An equation is linear if it can be written in the form

$$ax + b = c$$

where x, a, b and c are any numbers and $a \neq 0$.

As you know, when a, b and c are given, there is exactly one value for the unknown x that makes a linear equation true. For example: if $3x + 7 = 5$, then only $x = \frac{-2}{3}$ makes the equation a true statement.

We now wish to investigate the equation

$$ax + b = y$$

where a and b are any constants with $a \neq 0$ and x and y are variables. The meanings of "constants" and "variables" will become clear to you in this chapter. Here is an example of such an equation:

$$y = 2x + 7, \tag{6.1}$$

where x and y are any numbers that make the equation true.

Instead of discussing Equation (6.1), let us take the simpler equation

$$y = 2x \tag{6.2}$$

Equation (6.2) is unusual compared to equations you have seen, because it appears to contain two unknowns instead of just one. Equation (6.2) means that $2x$ and y are names for the same number. Maybe you are wondering "How do I solve it?"

Substituting $x = 3$ and $y = 6$, the RHS is 6 and the LHS is 6. Since LHS = RHS, the pair of numbers $(x = 3, y = 6)$ make Equation (6.2) a true statement. Therefore, the pair of numbers $(x = 3, y = 6)$ is a solution of Equation (6.2).

But that is not all! The pairs

$$(x = 2, y = 4),$$
$$(x = 5, y = 10),$$
$$(x = 6, y = 12),$$
$$(x = \frac{1}{2}, y = 1),$$
$$(x = \frac{2}{3}, y = \frac{4}{3}),$$
$$(x = \frac{3}{7}, y = \frac{6}{7}),$$

are all solutions too.

Perhaps by now you are thinking that there is no end to the pairs of numbers that make Equation (6.2) true. Correct! There are infinitely many pairs of numbers that solve Equation (6.2).

One cannot help but notice how easy it is to to find solutions to Equation (6.2). Just substitute a particular number for x, then multiply by 2 and the corresponding value for y appears.

6.1.1. Seriously boring

The author is the first to admit that solving Equation (6.2) is not interesting. Worse, the task is unending. Find a million solutions for Equation (6.2) and there will remain infinitely more solutions for you to find.

6.1.2. So what is interesting?

Equation (6.2) shows how two numbers are related and it shows how to find all and only those pairs of numbers that have that relationship.

6.2. The idea of a function: it's about the relationship!

Equation (6.2) is interesting when our attention shifts from the pairs of numbers that satisfy it to the relationship of each pair's numbers. The relationship, when one looks at Equation (6.2) is crystal clear: the value of y is double the value or x.

It is easy to find pairs of numbers that satisfy Equation (6.2), because the recipe for making them is right in front of our eyes. Just substitute a particular number for x and multiply by 2.

Finding pairs of numbers that make $y = 2x$ true just amounts to picking any old number for x, then following the rule *multiply by 2*. Now, you know how your calculator feels!

For any value of x, the rule produces the one value of y that makes $y = 2x$ true. To express this idea, we say $y = 2x$ gives "y as a *function* of x." We call

$y = 2x$ a *function*; we call y the *dependent variable* and x the *independent variable*. The "2" is called a constant, because value of "2" does not vary.

For the time being, this is a good enough idea of what a function is that we use it to define the idea of a function.

Definition 6.1 (Function)
A **function** is a rule that shows how the value of one variable, called the **dependent variable**, is uniquely determined by the value of another variable, called the **independent variable**.

Remark 6.1
The independent variable is often called the "argument of the function" and the dependent variable is often called the "value of the function".

Remark 6.2
Not every equation in which two variables appear is a function. Only equations in which the dependent variable is *uniquely* determined by the independent variable are of the special kind called functions. For example, the equation $x + y^2 = 5$ does *not* determine a unique value of y for every value of x. When $x = 1$, y can equal either 2 or -2 and equation $x + y^2 = 5$ will be true. So in $x + y^2 = 5$, y is not a function of x.

Definition 6.2 (Linear function)
A function that can be written in the form

$$y = ax + b, a \neq 0,$$

is a **linear function.** ■

6.3. Ordered pair

On page 147, we listed several pairs of values for x and y that made the equation $y = 2x$ true. We wrote, for example, $(x = 2,\ y = 4)$ and $(x = 5,\ y = 10)$. If we agree that within each parenthesis, the value of the independent variable will be first and the value of the dependent variable second, then there is no need to keep writing "$x = \ ,\ y =$ ". Writing "$(2,4)$" is enough, because the order indicates that the value of the dependent variable is equal to 4 when the value of the independent variable is 2. We call pairs such as $(2,4)$ "ordered" pairs.

Definition 6.3 (Ordered pair)
A pair of numbers within parenthesis such as (a,b) is called an **ordered pair**. In (a,b), a is called the **first element** of the pair and b is called the **second element** of the pair. The number b is the value of the dependent variable corresponding to the value a of the independent variable. ■

Remark 6.3
The order of the elements in an ordered pair matters. The ordered pairs $(2,7)$ and $(7,2)$ are not the same. The ordered pair $(2,7)$ satisfies $3x+1=y$, but the ordered pair $(7,2)$ does not.

Example 6.1

Suppose y is a function of x such that $y=2x$. Write any five ordered pairs whose elements satisfy $y=2x$.

Solution

Five such pairs are $(-2,-4),(0,0),(3,6),(\frac{9}{5},\frac{18}{5}),(100,200)$.

Example 6.2

Suppose t is a function of s such that $t=2+s$. Write any five ordered pairs whose elements satisfy $t=2+s$.

Solution

Five such pairs are $(-5,-3),(-1,1),(8,10),(\frac{1}{2},\frac{5}{2}),(\frac{3}{4},\frac{11}{4})$.

Example 6.3

Suppose y is a function of x such that $y=5+x$. Write the pairing of x and y as ordered pairs for $x=-2,-1,0,\frac{2}{5},4$.

Solution

$(-2,3),(-1,4),(0,5),(\frac{2}{5},\frac{27}{5}),(4,9)$.

Example 6.4

Suppose the values of x and y are paired by a certain function as follows: $(1,3),(2,6),(3,9),(4,12)$. Write y as a function of x.

Solution

Notice that the second element (value of dependent variable) of each pair is 3 times the first element (value of independent variable). So the function is $y=3x$.

Example 6.5

Given the function $t = 5s$, which of the following ordered pairs is not (are not) produced by this function? The candidates are: $(-3,-15),(-1,-5)$, $(0,0),(2,11),(4,20),(\frac{3}{7},\frac{15}{7}),(6,32)$.

Solution

The function $t = 5s$ requires that the second element be 5 times the first element. But, $11 \neq 2\cdot 5$ and $32 \neq 6\cdot 5$. So (2,11) and (6,32) are not produced by the function $t = 5s$.

Exercise 6.1

1. Suppose y is a function of x such that $y = 4x$. Write the pairing of x and y as ordered pairs for $x = -3,-2,0,1,\frac{3}{8},5$.
2. Suppose t is a function of s such that $t = s+3$. Write the pairing of s and t as ordered pairs for $s = 0,1,2,3,\frac{10}{3}$.
3. Suppose z is a function of w such that $z = 3w$. Write the pairing of w and z as ordered pairs for $w = \frac{-2}{3},-1,0,1,2$
4. Could the ordered pair $(2,9)$ be produced by the function $y = 3x$?
5. Could the ordered pair $(-1,5)$ be produced by the function $y = x-2$?
6. Suppose the values of x and y are paired by a certain function as follows: $(-2,-10),(0,0),(2,10),(3,15)$. Write y as a function of x.
7. Suppose the values of x and y are paired by a certain function as follows: $(1,5),(2,6),(3,7),(4,8)$. Write y as a function of x.
8. Suppose the values of s and t are paired by a certain function as follows: $(-2,10),(-1,5),(0,0),(1,-5),(2,-10)$. Write t as a function of s.
9. Show that the ordered pairs $(2,6)$ and $(6,2)$ are not the same.

6.4. Domain and Range

There are many ideas associated with functions. Two of these ideas are "domain" and "range".

Definition 6.4 (Domain)
The set of numbers whose members serve as the arguments of the function is called the **domain** of the function.

Definition 6.5 (Range)
The set of numbers that the function takes as values is called the **range** of the function.

Remark 6.4
The domain of the function is not always stated. If it is not, then the domain is taken to be the largest set of numbers for which the function makes mathematical sense. If a number would cause a division by zero, that number is excluded from the domain.

Example 6.6

Suppose $y = 2x$ and the domain of the function is $\{0, 1, 3, 5, 7\}$. What is the range of the function?

Solution

When $x = 0, y = 0$. When $x = 1, y = 2$. When $x = 3, y = 6$. When $x = 5, y = 10$. When $x = 7, y = 14$. So the range is $\{0, 2, 6, 10, 14\}$.

Example 6.7

Suppose $y = 2x$ and the domain of the function is $\{0, 2, 4, 6, 8\}$. What is the range of the function?

Solution

When $x = 0, y = 0$. When $x = 2, y = 4$. When $x = 4, y = 8$. When $x = 6, y = 12$. When $x = 8, y = 16$. So the range is $\{0, 4, 8, 12, 16\}$.

Example 6.8

Suppose $y = 2x$ and the domain of the function is all numbers between 0 and 5. What is the range of the function?

Solution

When $x = 0, y = 0$. When $x = 5, y = 10$. Since x takes all values between 0 and 5, y takes all values between 0 and 10. So, the range is all numbers between 0 and 10 including 0 and 10. ■

If we know the ordered pairs produced by a function, we know the domain and the range. Remember, the domain is the set of values the argument takes and these will be the first elements of the ordered pairs. The range is the set of values the function produces and these will be the second elements of the ordered pairs.

Example 6.9

Suppose a function produces these, and only these, ordered pairs:

$$(-1,-3),(0,0),(2,6),(10,30).$$

What is the domain and what is the range of this function?

Solution

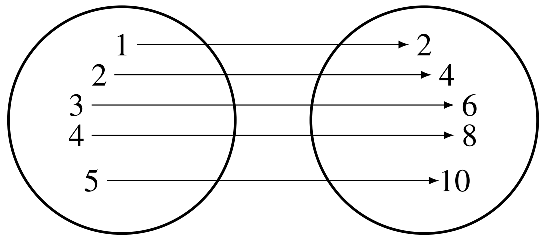
The domain is $\{-1,0,2,10\}$ and the range is $\{-3,0,6,30\}$.

Exercise 6.2

1. Suppose that $y = x - 5$ domain $\{0,4,8,11\}$. What is the range of $y = x - 5$?
2. If $z = 10w$ domain $\{-3,-1,0,2,9\}$, what is the range?
3. If $t = s + 5$ domain $\{0, \frac{1}{3}, \frac{1}{2}, 1, 3\}$, what is the range?
4. What is the range of the function $y = 3x + 1$ when the domain is $\{-2,-1,0,1,2\}$?
5. What is the range of the function $y = 3x + 1$ when the domain is $\{3,4,5,6\}$?
6. What is the range of the function $y = 3x + 1$ when the domain is $\{7,8,9,10,11\}$?
7. A function produces all and only the following pairs
$$(1,\frac{2}{3}),(2,\frac{3}{4}),(3,7),(5,11),(12,27).$$
What is the domain and range of the function?

6.5. Idea of function as correspondence

We have said that a function shows how a number in a set called "the range of the function" is determined by another number in a set called "the domain of the function". The correspondence of numbers in the domain with numbers in the range is shown in Figure (6.1) for the function $y = 2x$. The picture is drawn for only a few of the numbers in each set.



Domain $= \{1, 2, 3, \cdots\}$ Range $= \{2, 4, 6, \cdots\}$

FIGURE 6.1. Domain and Range of $y = 2x$

Mathematicians often find it useful to emphasize that a function shows the correspondence of members of sets. You too will find this idea useful in subsequent mathematics courses. To write "y corresponds to x", we use an arrow:

$$x \longrightarrow y.$$

We can write the correspondences shown in Figure (6.1) like this:

$$1 \longrightarrow 2$$
$$2 \longrightarrow 4$$
$$3 \longrightarrow 6$$
$$4 \longrightarrow 8$$
$$5 \longrightarrow 10$$

When we write, for example, $1 \longrightarrow 2$, we should realize that the function *pairs* 1 with 2, 2 with 4, and so on. Writing $(1,2)$ expresses the same idea as writing $1 \longrightarrow 2$. Therefore,

$1 \longrightarrow 2$ is equivalent to $(1,2)$

$2 \longrightarrow 4$ is equivalent to $(2,4)$

$3 \longrightarrow 6$ is equivalent to $(3,6)$

$4 \longrightarrow 8$ is equivalent to $(4,8)$

$5 \longrightarrow 10$ is equivalent to $(5,10)$

Remark 6.5
The idea of function was refined over a period of decades as people acquired deeper and deeper insight into what a function is. As a consequence of that history, the language used in discussing functions is rich (read "confusing to a beginner"). You might as well know now that "$x \longrightarrow y$" is also read, "the function sends x to y" or "the function maps x to y". For example, the function $y = 2x$ maps 1 to 2. Or, equally correct, the function $y = 2x$ sends 1 to 2.

Example 6.10

For the function $y = 3x + 4$ domain $\mathbb{Z}$, use the "$\longrightarrow$" notation to write $x \longrightarrow y$ for $x = -8, -5, -1, 0, 1, 4$.

Solution

$$\begin{aligned} -8 &\longrightarrow -20 \\ -5 &\longrightarrow -11 \\ -1 &\longrightarrow 1 \\ 0 &\longrightarrow 4 \\ 1 &\longrightarrow 7 \\ 4 &\longrightarrow 16 \end{aligned}$$

Example 6.11

Suppose that the domain of a function is $\{-3, -2, -1, 0, 2, 5\}$ and that the function maps -3 to -12, -2 to -8, -1 to -4, 0 to 0, 2 to 8, 5 to 20. Write the function and state its range.

Solution

We are given that

$$\begin{aligned} -3 &\longrightarrow -12 \\ -2 &\longrightarrow -8 \\ -1 &\longrightarrow -4 \\ 0 &\longrightarrow 0 \\ 2 &\longrightarrow 8 \\ 5 &\longrightarrow 20 \end{aligned}$$

Noticing that -12 is 4 times -3, -8 is 4 times -2 and so on, the function is $y = 4x$. The range is $\{-12, -8, -4, 0, 8, 20\}$.

Exercise 6.3

1. For the function $y = 5x$ with domain $\{1, 2, 3, \cdots\}$, draw a picture like Figure (6.1) for elements $\{1, 2, 3, 4\}$ of the domain.
2. For the function $y = 5 - x$ with domain $\{\cdots, -3 - 2, -1, 0, 1, 2, 3, \cdots\}$, draw a picture like Figure (6.1) for elements $\{-5, -1, 0, 1, 7\}$ of the domain.
3. Use the "$\longrightarrow$" notation to express the idea that 2 corresponds to 7, 3 corresponds to 10, and 5 corresponds to 16.
4. For the function $y = 2x + 3$ domain $\mathbb{Z}$, use the "$\longrightarrow$" notation to write $x \longrightarrow y$ for $x = -7, -2, -1, 0, 1, 3$. [See Example (6.10).]
5. Suppose a function has domain $\{1, 2, 3, 4, 5\}$ and maps 1 to 10, 2 to 20, 3 to 30, 4 to 40, and 5 to 50. Write the function and state its range. [See Example (6.11).]

6.6. Romeo and Juliet

Suppose Romeo is exactly two years younger than Juliet. You would have no trouble telling Romeo how old Juliet will be when he is 13 years old. If Romeo wanted Juliet's age when he is 14, you would say she will be 16. And if he asks, "How old will Juliet be when I am 15?", you would say, "17". If Romeo asks for her age when he is 16, you might begin to get fed up and make a chart for him so that he could just look up Juliet's age without bothering you. You provide the handy pocket chart shown below.

If you object that in fact Romeo was older than Juliet, you are thinking of the wrong Romeo and Juliet. The Romeo and Juliet referred to here live in the Bronx.

Romeo's age in years	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Juliet's age in years	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**

TABLE 6.1. Romeo and Juliet Ages

All is well with Romeo, until one night you get a call because he wishes to know how old Juliet will be when he is 31 years old. Instead of continuing the table, you might simply tell him a rule for finding Juliet's age. "Just add 2 to your age in years to get Juliet's age in years."

Rule for finding Juliet's age, j (years), given Romeo's age, r (years):

(6.3) $$j = r + 2.$$

Table (6.1), Equation (6.3), and the instruction “Just add 2 to Romeo’s age in years to get Juliet’s age in years,” all express the same idea. Each provides Juliet’s age, given Romeo’s age. In mathematics, we say each tells Juliet’s age *as a function of* Romeo’s age.

6.6.1. Cute story, but what’s the point?

First point. The author was hoping that you would look at Table (6.6) and realize it is just another way of expressing correspondence or pairing. If you did not think of Table (6.6) that way, look at it again and think

$$\begin{array}{lcc} 10 \longrightarrow 12 & \ldots \text{or} \ldots & (10,12) \\ 11 \longrightarrow 13 & \ldots \text{or} \ldots & (11,13) \\ 12 \longrightarrow 14 & \ldots \text{or} \ldots & (12,14) \end{array}$$

and so on up to

$$20 \longrightarrow 22 \quad \ldots \text{or} \ldots \quad (20,22).$$

Second point. We often use a function to accomplish a practical end. Just as Romeo used a function to know Juliet’s age, given his age, so also do engineers use functions to predict the deflection of a bridge as a train passes over it, and resource biologists to predict the population of wolves in a region over a period of time. Those are somewhat complicated functions that you will get to know in subsequent mathematics courses.

Example 6.12

Suppose Jane leaves work at 5:00 PM and drives home at a constant speed of 40 mph arriving home at 5:30 PM. Write the distance Jane covers as a function of time. Also, state the domain and range of the function.

Solution

Let d represent distance in miles and t represent time in hours. Then $d = 40t$. The domain is all numbers between 0 and $\frac{1}{2}$. When $t = 0$, $d = 0$ and when $t = \frac{1}{2}$, $d = 20$, so the range is all numbers between 0 and 20.

Example 6.13

John drives at a constant speed of 60 mph. Write the distance d miles he covers as a function of time t hours.

Solution

$d = 60t$.

Remark 6.6
The author knows from experience that the phrases "distance d miles" and "time t hours" confuse beginning algebra students. All that "distance d miles" means is "when you write the function, use d to represent distance and the units are miles". All that "time t hours" means is "use the letter t to represent time and the units are hours". In other words, you are just being told what letters and units to use. That's all there is to it.

Example 6.14

A tank, initially empty, is being filled with water at the rate of 5 gallons per minute. Write the volume V gallons of water in the tank as a function of time t minutes.

Solution

$V = 5t$.

Example 6.15

A tank that initially contained 800 gallons of water is being drained at the rate of 4 gallons per minute. Write the volume V gallons of water in the tank as a function of time t minutes. Also, state the domain and range of the function.

Solution

$V = 800 - 4t$. At time $t = 0$, the tank contains 800 gallons. It will be empty when $4t = 800$. So, the tank will be empty when $t = \frac{800}{4}$ minutes which is 200 minutes. Therefore the domain is all numbers between 0 and 200. The range is all numbers between 0 and 800.

Example 6.16

Make a table that shows the quantity of water in the tank from Example (6.15) at 20 minute intervals.

Solution

t (min)	0	20	40	60	80	100	120	140	160	180	200
V (gal)	800	720	640	560	480	400	320	240	160	80	0

Example 6.17

Barb saves \$400 per month. Write her total saved T dollars as a function of n months.

Solution

$T = 400n.$

Example 6.18

A construction supervisor wishes to know how many lineal feet of sidewalk a crew paves as a function of time. The crew paved 36 feet in 12 hours, 45 feet in 15 hours, and 93 in 31 hours.

Solution

Noting that

$$\begin{aligned} 12 &\longrightarrow 36 \\ 15 &\longrightarrow 45 \\ 31 &\longrightarrow 93 \end{aligned}$$

The supervisor realizes that in general

$$t \longrightarrow 3t.$$

The supervisor concludes that the function is $L = 3t$ where L represents lineal feet of sidewalk and t represents time in hours.

Example 6.19

The supervisor from Example (6.18) must tell her boss how long the crew will take to pave 306 lineal feet of sidewalk. What should she tell her boss?

Solution

She knows that the quantity of sidewalk paved as a function of time is $L = 3t$ where L is in lineal feet and t is in hours. Substituting 306 for L she obtains the equation $306 = 3t$. After she solves this for t, she tells her boss it will take 102 hours to complete 306 lineal feet of sidewalk.

Exercise 6.4

1. Write the function that gives d miles as a function of t hours, if the constant speed is 60 $^{\text{mi}}/_{\text{hr}}$.

2. A pail that initially contains 2 gallons of water is filled at a rate of 3 $^{\text{gal}}/_{\text{min}}$.
   a) Write the volume V gallons as a function of time t minutes.
   b) Make a table that shows the volume at increments of 2 minutes from 0 minutes up to 12 minutes.
3. Gary's combine harvests soybeans at the rate of 7 $^{\text{acres}}/_{\text{hr}}$.
   a) Write the number of acres A harvested as a function of time t hours.
   b) Find the value of this function at $t = 7$.
4. Jill walks to school at a constant rate of 4 $^{\text{mi}}/_{\text{hr}}$. The distance from her house to school is 0.75 mi.
   a) Write the time remaining, t, in Jill's walk to school as a function of the distance, d, she has walked.
   b) State the domain of the function.
   c) State the range of the function.
5. Suppose that the correspondence between two variables, x and y, is as shown below.

$$x \longrightarrow y$$
$$0 \longrightarrow 0$$
$$1 \longrightarrow 5$$
$$2 \longrightarrow 10$$
$$3 \longrightarrow 15$$
$$4 \longrightarrow 20$$

   a) Write the function as a rule that shows how the value of y is determined by the value of x.
   b) State the domain of the function.
   c) State the range of the function.
   d) Find the value of this function when x is 70.

6.7. Graphs

One of the best ways to acquire an understanding of a particular function is to create a picture of it. That picture is referred to as the graph of the function. We will return to our discussion of functions in a little while. But for now, the topic is graphs.

6.7.1. The coordinate plane

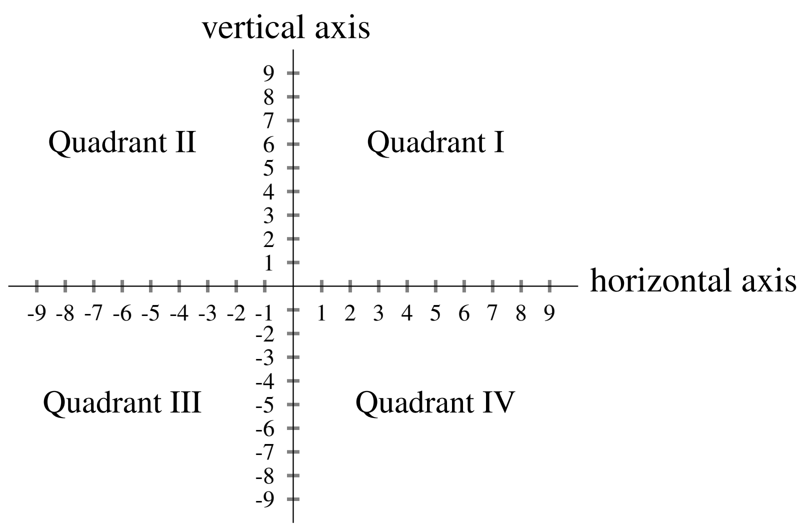


FIGURE 6.2. The coordinate plane

Figure (6.2) shows the coordinate plane. The point at which the axes intersect is called the *origin*. The axes intersect at a right angle. It is understood that each axis continues in both directions in spite of the fact that in this book we do not put arrows on the ends of the axes. The two axes divide the plane into four regions called quadrants. The quadrants are named with roman numerals as in the figure.

Other names for the coordinate plane are the “Cartesian plane”, the “Cartesian coordinate system”, or the “rectangular coordinate system”.

Yes, there are many coordinate systems that are not rectangular.

The exact location of any point in the plane can be given by an ordered pair of numbers that are always written in parentheses. These numbers are called the *coordinates* of the point. The first number of the pair is the horizontal coordinate of the point, the second number the vertical coordinate.

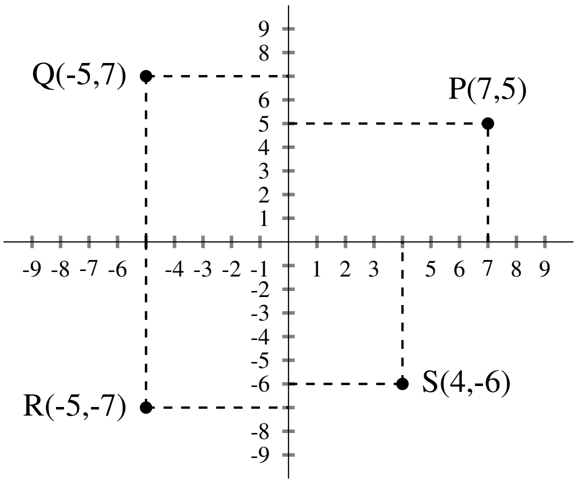


FIGURE 6.3. The coordinate plane

Figure (6.3) shows several points in the plane labeled with their coordinates. The dashed lines show how the coordinates of the points are obtained.

It is worth noting that although we rarely write the numbers on the axes as ordered pairs, we could do so. Figure (6.4) shows how this would look.

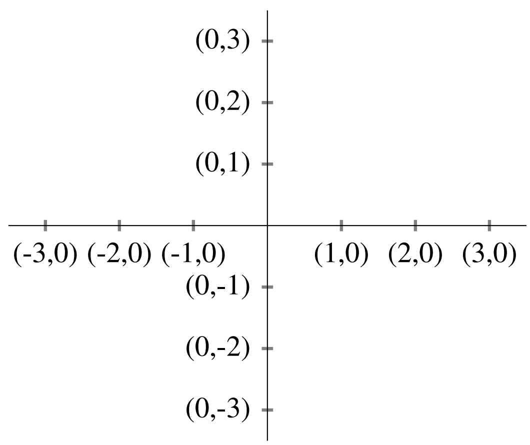


FIGURE 6.4. The coordinates of points on the axes.

The first coordinate of every point on the vertical axis is 0. The second coordinate of every point on the horizontal axis is 0.

Exercise 6.5

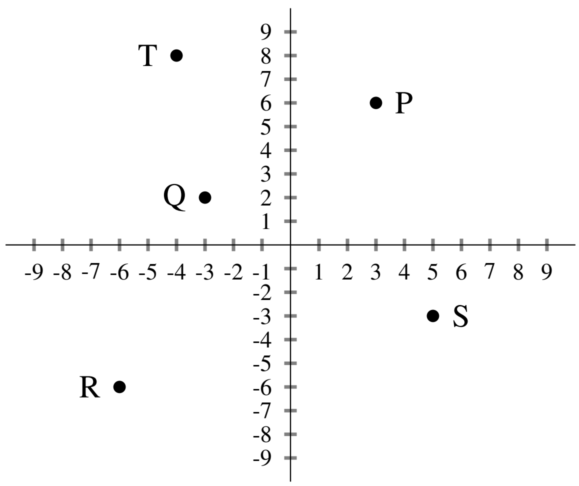


FIGURE 6.5.

1. Based on the appearance of Figure (6.5), state the coordinates of each point. (The first question is answered as an example.)
   a) T. Answer: $(-4,8)$.
   b) P.
   c) Q.
   d) R.
   e) S.
2. In which quadrant does each point in Figure (6.5) lie?
   a) T.
   b) P.
   c) Q.
   d) R.
   e) S.
3. Plot each of the following points.
   a) $A(3,5)$.
   b) $B(3,-5)$.
   c) $C(-3,5)$.
   d) $D(-3,-5)$.
4. Draw the axes. Label the points $-3,-2,-1,1,2,3$ on the horizontal axis and $-3,-2,-1,1,2,3$ on the vertical axis, using ordered pairs.

6.8. The graph of *y* = 2*x*

Figure (6.6), shows the function

$$y = 2x, \quad \text{domain} = \{-3, -2, -1, 0, 1, 2, 3, 4\}$$

as a correspondence and as a graph.

$$\begin{array}{c} x \longrightarrow y \\ -3 \longrightarrow -6 \\ -2 \longrightarrow -4 \\ -1 \longrightarrow -2 \\ 0 \longrightarrow 0 \\ 1 \longrightarrow 2 \\ 2 \longrightarrow 4 \\ 3 \longrightarrow 6 \\ 4 \longrightarrow 8 \end{array}$$

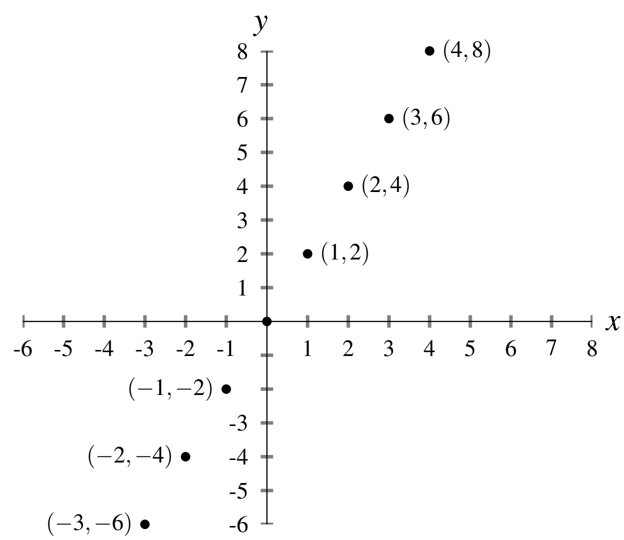


FIGURE 6.6. $y = 2x$, domain $\{-3, -2, -1, 0, 1, 2, 3, 4\}$

Who can help but think, "The points make a straight line!"?

So it seems. But, looks can deceive.

We might try to get a finer view of how the function behaves at values of x between, for example, 0 and 1. We consider the same rule, but with a different domain.

$$y = 2x, \text{ domain} = \{-0.5, -0.375, -0.25, \cdots, 1\}.$$

Figure (6.7) shows the graph. Note that the horizontal (x-axis) is marked every 0.125, the vertical (y-axis) every 0.25.

The evidence that graph of the function $y = 2x$ is a straight line of points is somewhat convincing. But we can do better. We can be certain. But to achieve such certainty, we need a few more ideas. For now we will leave open the question *Do the points produced by the function $y = 2x$ all lie on the same straight line?*

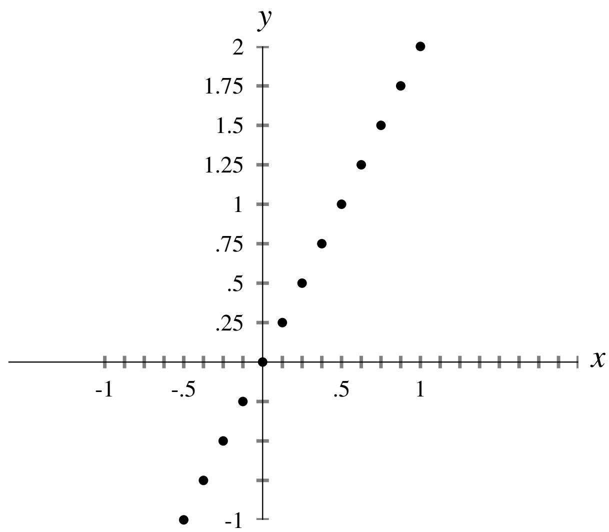


FIGURE 6.7. $y = 2x$, domain $= \{-0.5, -0.375, \cdots, 1\}$

6.8.1. Rise and run

If you study Figure (6.8), you will learn what is meant when we say "rise" and "run".

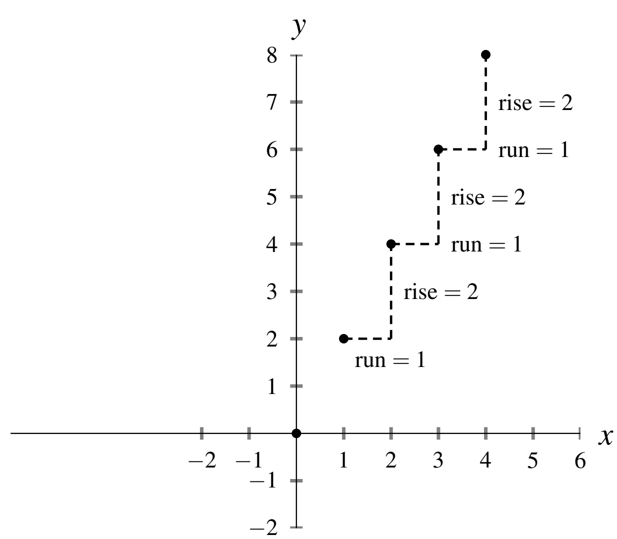


FIGURE 6.8. Graph of $y = 2x$, domain $= \{0, 1, 2, 3, 4\}$

If you have the idea that the run is "how far to the right" and the rise is "how far up", then you are on the correct track. There is an additional detail that you will discover when you study Figure (6.9).

The additional idea is that rise can be "how far down" in which case it is a negative number. The ratio of rise to run, $\frac{\text{rise}}{\text{run}}$, is a number that plays an

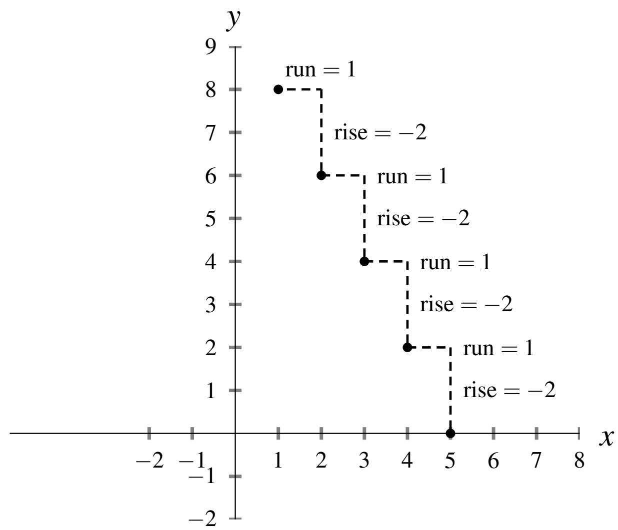


FIGURE 6.9. Graph of $y = 2x$, domain $= \{1,2,3,4,5\}$

important part in the study of functions. If you examine Figure (6.8), you will see that the $\frac{\text{rise}}{\text{run}} = \frac{2}{1} = 2$. In Figure (6.9), $\frac{\text{rise}}{\text{run}} = \frac{-2}{1} = -2$.

The phrases "How far up" and "How far right" are valuable, because they express our sense of what "rise" and "run" should mean. Let us see if we can say more precisely, though, what our ideas of rise and run are.

Let P and Q be two points whose coordinates are $P(x_1, y_2)$ and $Q(x_2, y_2)$ as shown in Figure (6.10).

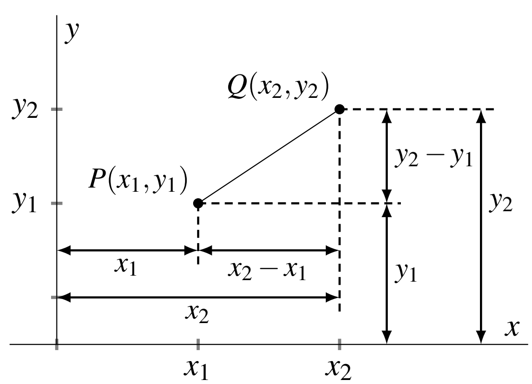


FIGURE 6.10. Computing $\frac{\text{rise}}{\text{run}}$ ratio.

Guided by Figure (6.10), the answer to "How far is Q up from P?" is $y_2 - y_1$. And the answer to "How far right is Q from P?" is $x_2 - x_1$.

Not only have we made our ideas about rise and run more precise, we have even found a way to measure "How far up" and "How far right". The computations are shown in Equation (6.4) and Equation (6.5).

$$\text{rise} = y_2 - y_1 \tag{6.4}$$

and

$$\text{run} = x_2 - x_1. \tag{6.5}$$

Example 6.20

Find the rise, run, and $\frac{\text{rise}}{\text{run}}$ ratio for the points $P(2,1), Q(7,5)$.

Solution

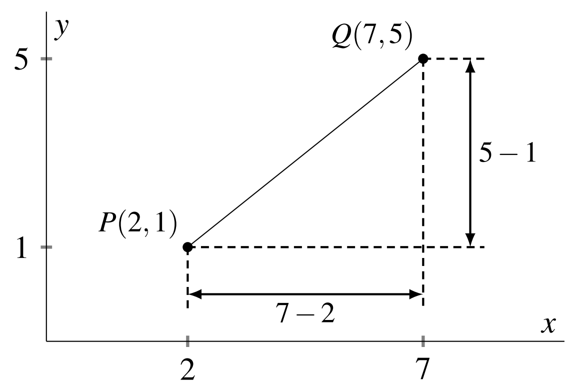


FIGURE 6.11. Example (6.20)

The rise is 4, the run is 5, and $\frac{\text{rise}}{\text{run}} = \frac{4}{5}$. ■

Another way to think through Example (6.20) would be like this: From point P, point Q is 5 right then 4 up, so the run is 5, the rise is 4, and $\frac{\text{rise}}{\text{run}} = \frac{4}{5}$. Figure (6.12) illustrates this.

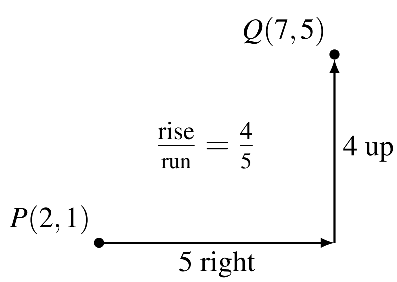


FIGURE 6.12. Another good way to think about Example (6.20)

Example 6.21

Find the ratio $\frac{\text{rise}}{\text{run}}$ for each of the following pairs of points.

(a) $P(-3,-5)$ and $Q(9,2)$
(b) $P(-2,7)$ and $Q(4,-5)$
(c) $P(3,-1)$ and $Q(-4,7)$

Solution

Both ways of thinking will be used for each problem.

(a) Q is 12 right and 7 up from P, so $\frac{\text{rise}}{\text{run}} = \frac{7}{12}$.
(a) $y_2 - y_1 = 2-(-5) = 7$ and $x_2 - x_1 = 9-(-3) = 12$. So, $\frac{\text{rise}}{\text{run}} = \frac{7}{12}$.

(b) Q is 6 right and 12 down from P, so $\frac{\text{rise}}{\text{run}} = \frac{-12}{6} = -2$.
(b) $y_2 - y_1 = -5-7 = -12$ and $x_2 - x_1 = 4-(-2) = 6$. So, $\frac{\text{rise}}{\text{run}} = \frac{-12}{6} = -2$.

(c) Q is 7 left and 8 up from P, so $\frac{\text{rise}}{\text{run}} = \frac{8}{-7} = \frac{-8}{7}$.
(c) $y_2 - y_1 = 7-(-1) = 8$ and $x_2 - x_1 = -4-3 = -7$. So, $\frac{\text{rise}}{\text{run}} = \frac{8}{-7} = \frac{-8}{7}$.

Exercise 6.6

[Part 1] *Find the ratio $\frac{rise}{run}$ for each pair of points.*

1. $P(2,5)$ and $Q(6,8)$
2. $P(1,4)$ and $Q(7,2)$
3. $P(-5,8)$ and $Q(3,10)$
4. $P(-2,-5)$ and $Q(2,3)$
5. $P(-1,-7)$ and $Q(-3,-11)$
6. $P(-4,-3)$ and $Q(-2,6)$
7. $P(1,7)$ and $Q(-3,-8)$
8. $P(0,0)$ and $Q(-3,4)$

[Part 2]

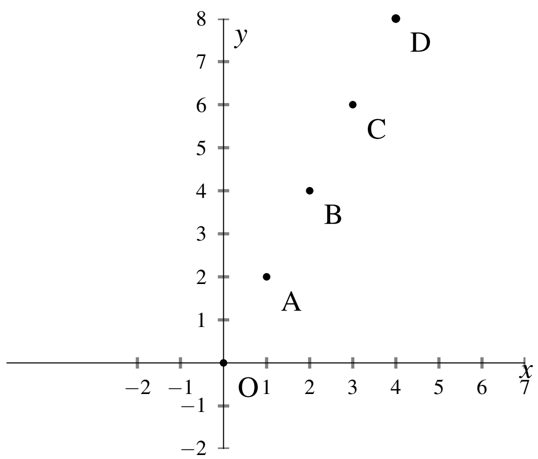


FIGURE 6.13. Graph of $y = 2x$, domain $= \{0,1,2,3,4\}$

1. Based on the appearance of Figure (6.13), complete the following table of the ratio $\frac{\text{rise}}{\text{run}}$ for each pair of points. (The first row is completed as an example.)

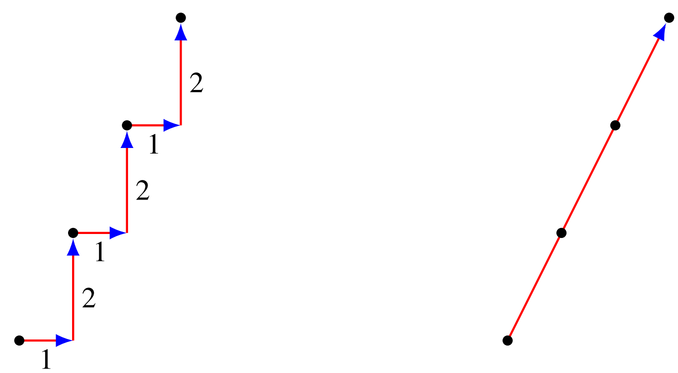
	0	**A**	**B**	**C**	**D**
0	undefined	$\frac{2}{1} = 2$	$\frac{4}{2} = 2$	$\frac{6}{3} = 2$	$\frac{8}{4} = 2$
A		undefined			
B			undefined		
C				undefined	
D					undefined

2. What is notable about the $\frac{\text{rise}}{\text{run}}$ that you filled in?
3. Why does the word "undefined" appear in certain cells of the table?

6.8.2. Collinear points

Think of the map to buried treasure so popular among the pirates of old.

In Figure (6.14), the path from one point to the next point is always the same, "1 right, 2 up". We can say more.The path from *every point* to *every other point* is in the *ratio* $\frac{2 \text{ up}}{1 \text{ right}}$. The consequence is that the four points lie on the same line. This idea is general and we state it in Definition (6.6).



(a) Direction constant. (b) Straight path.

FIGURE 6.14. $\frac{rise}{run}$ shows direction.

Definition 6.6 (Collinear points)
For a given collection of points, if the ratio $\frac{rise}{run}$ is the same for every pair of points in the collection, then all the points of the collection are **collinear** (lie on the same straight line). ■

6.8.3. The graph of *y = 2x*

On page 164 we wondered *Will the points produced by the function $y = 2x$ always lie on the same straight line?* The answer is "Yes". And we can be certain of this. All we need to show is that for *every* pair of points produced by $y = 2x$, the ratio $\frac{\text{rise}}{\text{run}}$ is identical. Here goes.

Let points $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points produced by $y = 2x$ where $x_1 \neq x_2$. Then $y_1 = 2x_1$ and $y_2 = 2x_2$. The rise is $2x_2 - 2x_1$. The run is $x_2 - x_1$. So

$$\begin{aligned}
\frac{\text{rise}}{\text{run}} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{2x_2 - 2x_1}{x_2 - x_1} \\
&= 2\left(\frac{x_2 - x_1}{x_2 - x_1}\right) \\
&= 2.
\end{aligned}$$

Since $\frac{\text{rise}}{\text{run}}$ is identical for *every* pair of points produced by $y = 2x$, the points produced by $y = 2x$ are collinear.

6.8.4. The function *y* = *ax* unbounded

We have just discovered in Section (6.8.2) that the graph of $y = 2x$ is a set of collinear points for every possible domain. If we allow the domain to be all the numbers, then the points of the graph fill in to produce a solid continuous straight line. The graph is in Figure (6.15). It is understood that the graph, like the axes, continues in both directions.

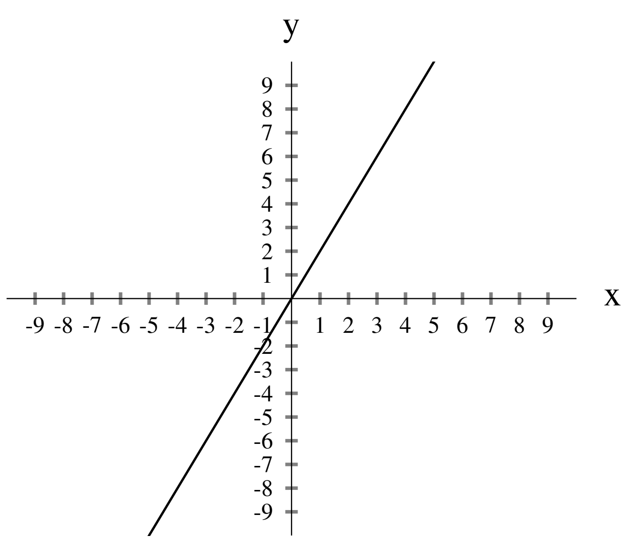


FIGURE 6.15. Graph of $y = 2x$, domain all numbers.

Remark 6.7

There is a difference between plotting points and graphing. We might plot points in science class to make a chart of, for example, temperature and number of cricket chips per hour. The chart would consist of accurately plotted points on carefully laid out axes. When we graph a function in mathematics, we plot only a few points. The process by which we arrived at the graph of $y = 2x$ in Figure (6.15) is a good example of how we graph in mathematics. We knew to draw a straight line, because we understood that the constant $\frac{\text{rise}}{\text{run}}$ guaranteed the graph would be a straight line.

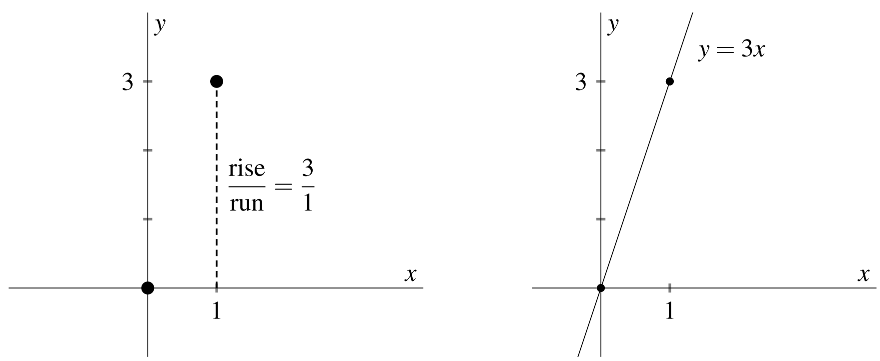
6.9. Graphing the function *y* = *ax*

The function $y = 2x$ is but one example of $y = ax$ where a is a constant. By now we know that the number a is the $\frac{\text{rise}}{\text{run}}$ ratio. We could write $y = \left(\frac{\text{rise}}{\text{run}}\right)x$, but letting "$a$" represent $\frac{\text{rise}}{\text{run}}$ makes a better looking equation. In fact, since ratio $\frac{\text{rise}}{\text{run}}$ turns up so often, it is convenient to give it a name. We agree to call this ratio the **slope**.

Example 6.22

Graph the function $y = 3x$, domain all numbers.

Solution



(a) Plot $(0,0)$. Then use slope. (b) Draw the line.

FIGURE 6.16. Graphing $y = 3x$

Figure (6.16) shows the steps. (a) The point $(0,0)$ is on the graph, because $0 = 3 \cdot 0$. The slope is $3 = \frac{3}{1}$ and we use it to locate another point on the line thinking "1 right, 3 up,". (b) Then draw the straight line through points $(0,0)$ and $(1,3)$. ■

Figure (6.16)(a), shows what we thought as we made Figure (6.16)(b). You would not show Part (a) when graphing.

The several examples that follow should answer some of your questions about graphing.

Example 6.23

Graph $y = x$.

Solution

The graph goes through the origin. The slope is 1, think "$y = 1 \cdot x$". So another point on the line is $(1,1)$.

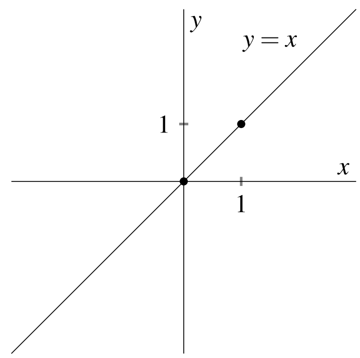


FIGURE 6.17. Example (6.23)

Example 6.24

Graph $y = \dfrac{2}{3}x$.

Solution

The graph goes through the origin. The slope is $\frac{2}{3}$, think "three right, two up". So another point on the line is $(3,2)$.

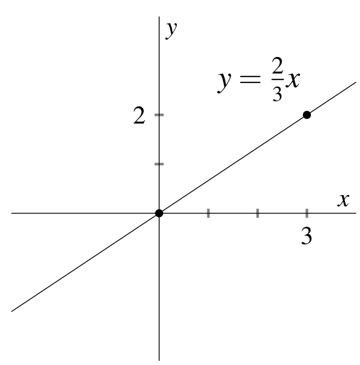


FIGURE 6.18. Example (6.24)

Example 6.25

Graph $y = \dfrac{-2}{3}x$.

Solution

The graph goes through the origin. The slope is $\frac{-2}{3}$, think "three right, two down". So another point on the line is $(3,-2)$.

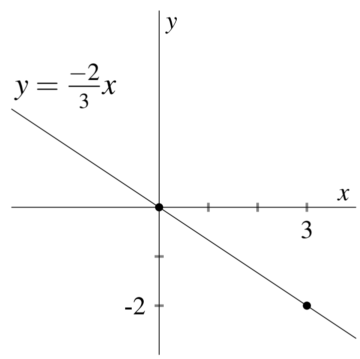


FIGURE 6.19. Example (6.25)

Example 6.26

Graph $y = \dfrac{-1}{4}x$.

Solution

The graph goes through the origin. The slope is $\frac{-1}{4}$, think "four right, one down". So another point on the line is $(-1,4)$.

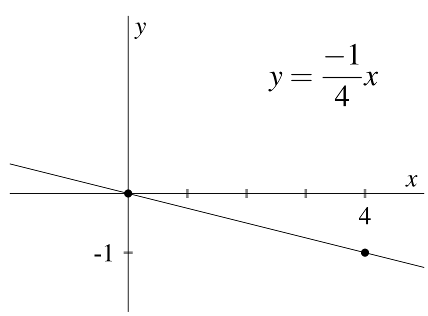


FIGURE 6.20. Example (6.26)

Example 6.27

Graph $y = 3x$.

Solution

The graph goes through the origin. The slope is 3, so another point on the line is $(1,3)$.

In Figure (6.21), we write the coordinates of the point $(1,3)$ beside the point, since there are several numbers shown on the axes.

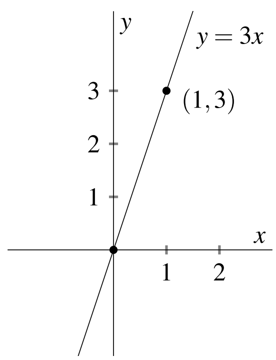


FIGURE 6.21. Example (6.27)

Exercise 6.7

1. Graph the function $y = 4x$.
2. Graph the function $y = \frac{1}{4}x$.
3. Graph the function $y = -2x$.
4. Graph the function $y = -3x$.
5. Graph the function $y = \dfrac{-2}{3}x$.
6. Graph the function $y = 8x$.

6.9.1. Graphs to equations

Sometimes we wish to write the function, given its graph. For example, consider the graph of Figure (6.22) which is of a straight line through the origin $(0,0)$.

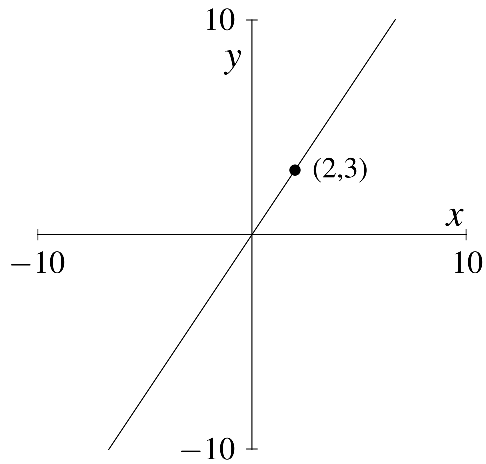


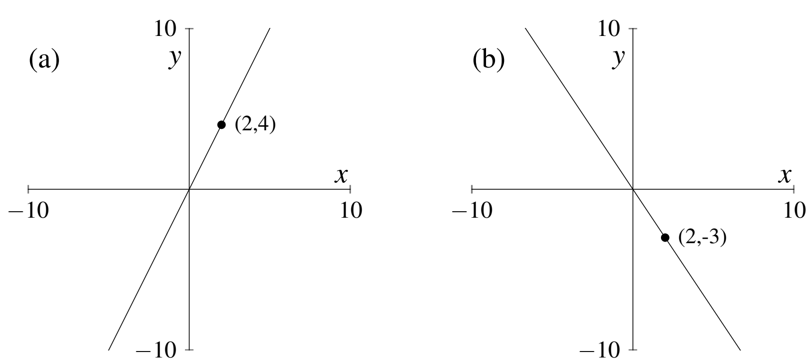
FIGURE 6.22. A straight line through points $(0,0)$ and $(2,3)$.

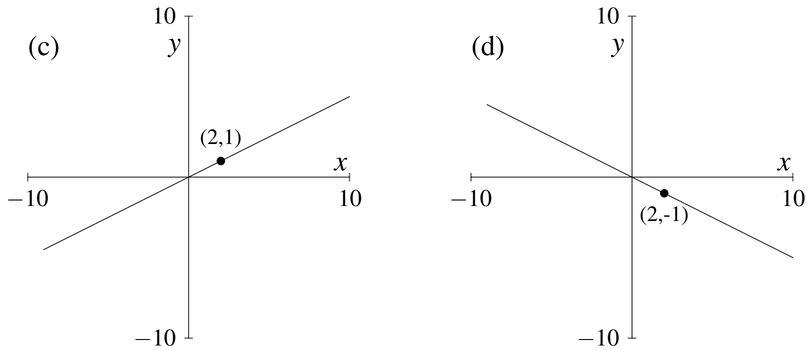
Since the graph is a straight line through the origin, it must have an equation of the form $y = ax$ where a is the slope of the line. The point $(2,3)$ on the line is 2 right of the $(0,0)$ and 3 up from $(0,0)$, so the slope equals $\frac{3}{2}$. Therefore the function must be $y = \frac{3}{2}x$.

Exercise 6.8

For each of the following graphs, write the function.

1.





6.10. Slope

We have been using the words “rise” and “run” quite a bit. But, it is unlikely that you will hear them used in more advanced mathematics courses. What we have been calling the “rise” is the change in the value of the dependent variable. Similarly, the run is the change in the value of the independent variable. So,

$$\frac{\text{rise}}{\text{run}} = \frac{\text{change in the value of the dependent variable}}{\text{change in the value of the independent variable}}. \tag{6.6}$$

Nobody really wants to say “the change in the value of the dependent variable *over* the change in the value of the independent variable”. Usually we do not have to, because usually we are talking about a particular function. If, for example, $y = ax$, then we just say “the change in y over the change in x”.

Improved, but admittedly, not so much. We can do better yet. In mathematics, the uppercase Greek letter Δ means “change”. To express the idea *change in y*, we just write Δy. Similarly for the change in x, just write Δx. We can express the idea of Equation (6.6) by writing Equation (6.7)

Δ is pronounced “delta”.

$$\frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}. \tag{6.7}$$

We now define “slope”.

Definition 6.7 (Slope)
If $y = ax$, then the **slope** a is $\dfrac{\Delta y}{\Delta x}$. ■

Remark 6.8
If the function were written $t = as$, we would write the slope as $\dfrac{\Delta t}{\Delta s}$. If the function were written $z = aw$, we would write the slope as $\dfrac{\Delta z}{\Delta w}$.

Example 6.28

Suppose $z = \frac{7}{6}w$. Using the Δ notation, write the rise, the run, and the slope.

Solution

$$\Delta z = 7, \quad \Delta w = 6, \quad \frac{\Delta z}{\Delta w} = \frac{7}{6}.$$

Example 6.29

For the graph in Figure (6.23), use the Δ notation to write the rise, the run, and the slope.

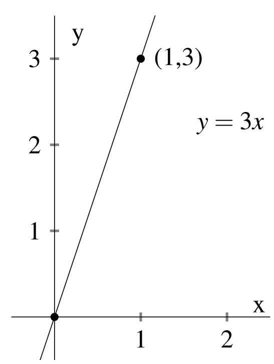


FIGURE 6.23. Example (6.29)

Solution

$$\Delta y = 3, \quad \Delta x = 1, \quad \frac{\Delta y}{\Delta x} = \frac{3}{1} = 3.$$

Example 6.30

For the graph in Figure (6.24), use the Δ notation to write the rise, the run, and the slope.

Solution

$$\Delta y = -3, \quad \Delta x = 2, \quad \frac{\Delta y}{\Delta x} = \frac{-3}{2}. \quad \blacksquare$$

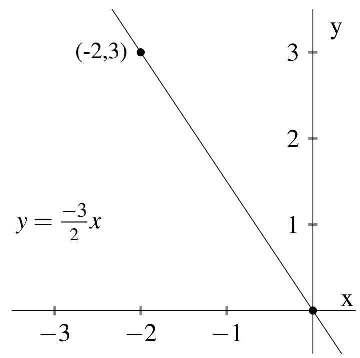


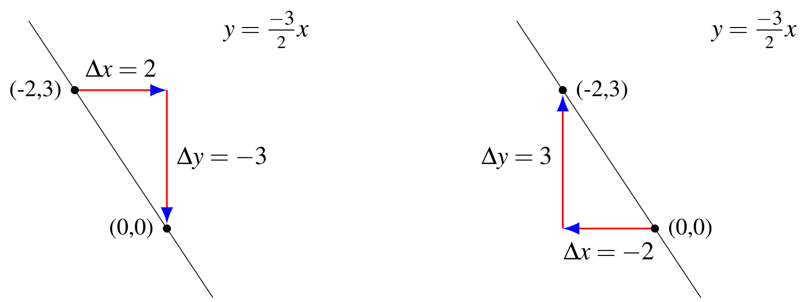
FIGURE 6.24. Example (6.30)

Someone is bound to point out that Example (6.30) could also be worked like this:

Alternate solution to Example (6.30)

$$\begin{aligned}\Delta y &= 3,\\ \Delta x &= -2,\\ \frac{\Delta y}{\Delta x} &= \frac{3}{-2}\\ &= \frac{-3}{2}. \quad \blacksquare\end{aligned}$$

Both the solution and the alternate solution provide the correct answer. So which solution is better? Figure (6.25) shows a picture for each solution.



(a) Solution. (b) Alternate solution.

FIGURE 6.25. Alternate solutions for Example (6.30)

When studying the solution (a) shown in Figure (6.25), you should think:

> As the values of x increase from -2 to 0, the values of y decrease from 3 to 0.

When studying the solution (b) shown in Figure (6.25), you should think:

As the values of x decrease from 0 to -2, the values of y increase from 0 to 3.

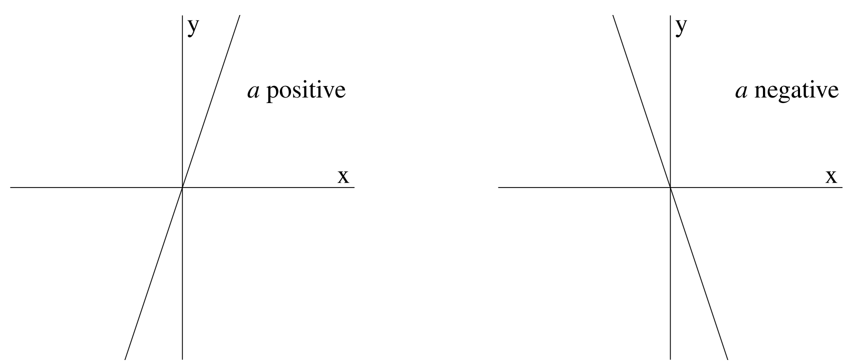
"Which correct solution is better?" The original solution which is shown in Figure (6.25)(a) is better. The thinking that accompanies it, "As the values of x increase from -2 to 0, the values of y decrease from 3 to 0", is well suited to some mathematics you will use a few years from now. We prefer to think about what happens to y as x *increases*, and "increases" means to the right on the x-axis.

Since we have the ideas "increasing" and "decreasing", we may as well define two ideas that you will use in subsequent courses.

Definition 6.8 (Increasing (decreasing) function)
Let y be a function of x. If the value of y increases as the value of x increases, we say y is an **increasing function** of x. Alternatively, if the value of y decreases as the value of x increases, we say y is a **decreasing function** of x.

Theorem 6.1
Let y be a function of x of the form $y = ax$. If a is positive, y is an increasing function of x. If a is negative, y is a decreasing function of x. ■

Figure (6.26) illustrates these ideas.



(a) y is an increasing function of x. (b) y is a decreasing function of x.

FIGURE 6.26. Graphing $y = ax$

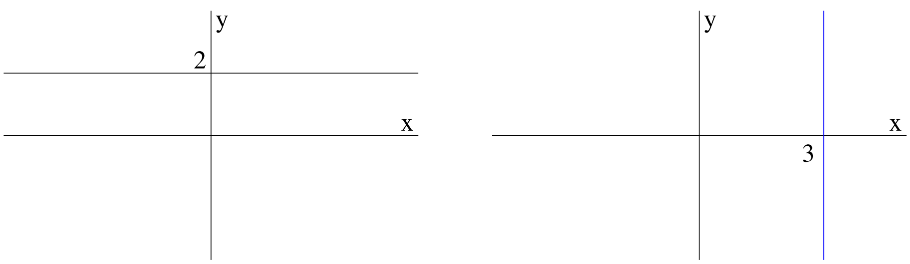
6.10.1. Slope - extreme cases

Horizontal line

What is the slope of a horizontal line? Since the rise, Δy, of a horizontal line is 0, the slope is $\frac{0}{\Delta x} = 0$. Therefore, the slope of a horizontal line is 0.

Vertical line What is the slope of a vertical line?
Since the run, Δx, of a vertical line is 0, the slope would be $\frac{\Delta y}{0}$. But this is undefined. Therefore, the slope of a vertical line is undefined.

Figure (6.27) illustrates these ideas.



(a) Slope is 0.　　(b) Slope is undefined.

FIGURE 6.27. Slope: extreme cases

You might be surprised to know that in each of these cases, we can still write an equation for the line. In Figure (6.27) (a), the equation is $y = 2$. In (b), the equation is $x = 3$.

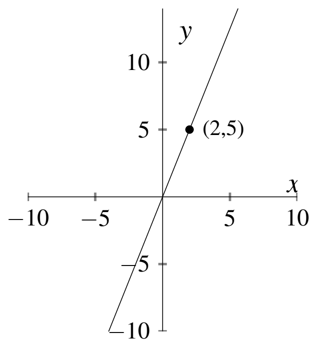
Exercise 6.9

[Part 1] *Use the Δ notation to write the rise, run, and slope. State whether the function is increasing or decreasing.*

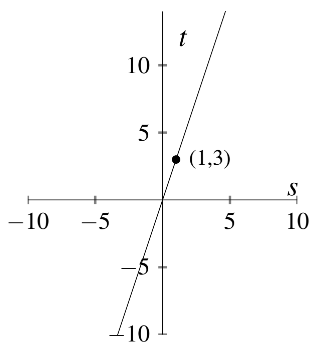
1. $y = \frac{2}{5}x$
2. $z = \frac{3}{11}w$
3. $t = \frac{-1}{4}s$
4. $y = 3x$
5. $y = \frac{5}{2}x$
6. $t = -9s$

[Part 2] *Use the Δ notation to write the rise, run, and slope. State whether the function is increasing or decreasing.*

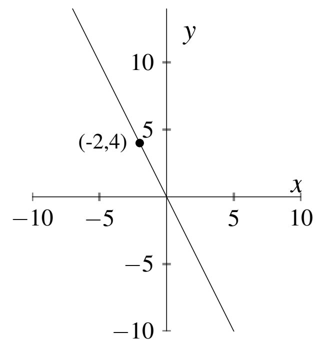
1.



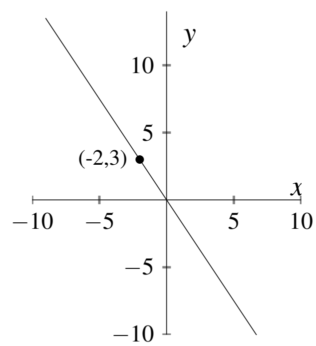
2.



3.



4.



6.11. Linear functions and rate of change

Suppose that

$$y = 2x. \tag{6.8}$$

Equation (6.8) says that whatever number x is, y is double that number. It is not quite so obvious that Equation (6.8) also says that whatever the change in x, the change in y is twice that. Table (6.2) provides some evidence for this claim. The first row of Table (6.2) shows that when $x = 1, y = 2$. These values are called "old x and old y". When $x = 3, y = 6$. These are called "new x and new y". The change in x, Δx, is the old value minus the new value which is $3 - 1 = 2$. The change in y, Δy, is "new y" subtract "old y" which is $6 - 2 = 4$. The last column of the table points out that the change in y is 2 times the change in x, $\Delta y = 2\Delta x$; in this example, the computation is $4 = 2 \cdot 2$.

We can do better than merely provide evidence that the change in y is double the change in x when $y = 2x$. We can prove that this must be so.

old x	new x	Δx	old y	new y	Δy	result
1	3	2	2	6	4	4 is 2 times 2
3	9	6	6	18	12	12 is 2 times 6
2	13	11	4	26	22	22 is 2 times 11
9	109	100	18	218	200	200 is 2 times 100
1/2	11/16	3/16	1	11/8	3/8	3/8 is 2 times 3/16

TABLE 6.2. Several computations showing effect on y of increasing x when $y = 2x$.

Proof. Let $y = 2x$. Suppose that x changes by an amount Δx and y by an amount Δy. Then

$$\begin{aligned} y+\Delta y &= 2(x+\Delta x) \\ y+\Delta y &= 2x+2\Delta x \\ y+\Delta y &= y+2\Delta x, \qquad \text{[WHY?]} \\ \Delta y &= 2\Delta x \end{aligned}$$

Therefore, the change in y is 2 times the change in x. ■

Another way to express the idea that the change in y is 2 times the change in x is to say that "the rate of change in y with respect to x is 2". This is true for all non-zero values of a. Theorem (6.2) says just that.

Theorem 6.2
If $y = ax, a \neq 0$, then the number a is the rate of change of y with respect to x. ■

It may be reassuring to know that Theorem (6.2) says what we have known, though perhaps not so generally, since 6th grade or earlier.

For example, you know that distance traveled equals the speed times the time of traveling at that speed. That is,

$$\text{distance} = (\text{speed}) \times (\text{time}), \text{ provided the speed is constant.} \tag{6.9}$$

Now, Equation (6.9) has the form $y = ax$, with speed playing the role of a. Theorem (6.2) tells us that a must be the rate of change in y with respect to x. And speed is exactly that: *the rate of change in distance with respect to time.*

Example 6.31

At what rate does the circumference of a circle change with respect to its diameter?

Solution

The circumference, C, of a circle is a function of its diameter, D. The function is $C = \pi D$. This equation has the form $y = ax$ with a constant, C and D variables. Therefore, the rate of change in circumference with respect to diameter, $\frac{\Delta C}{\Delta D}$, is π.

Exercise 6.10

1. A tank is being filled with water. The volume, V liters, of water in the tank is a function of time, t minutes. In fact, $V = 23\frac{\text{L}}{\text{min}}t$. Find $\frac{\Delta V}{\Delta t}$.
2. For a triangle whose base is 10 feet, at what rate is the triangle's area changing with respect to the triangle's height in feet?
3. Sally runs at a rate $1\frac{1}{2}$ that of Peter. If Peter covers 800 feet in a certain period of time, how many feet will Sally cover in that same period of time?
4. Tank A is being filled according to the function $V = 40t$ while tank B is being filled according to the function $V = 60t$, where V is in liters and t is in minutes. How much has the volume in tank B increased when the volume of tank A has increased from 30L to 60L?
5. * Tank A is being filled according to the function $V_A = 40t$ while tank B is being filled according to the function $V_B = 60t$, where V_A and V_B are in liters and t is in minutes. How much has the volume in tank B increased when the volume of tank A has increased an amount ΔV_A liters? Answer in terms of ΔV_A.

6.12. Graphs and rate of change

Consider the graph in Figure (6.28).

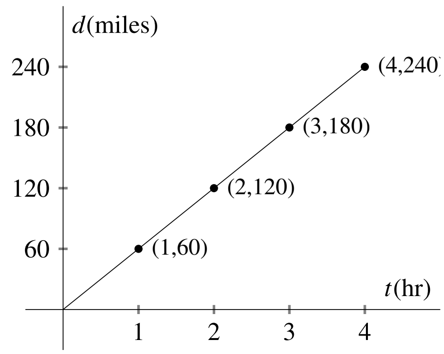


FIGURE 6.28. A four hour trip at a constant speed.

Time is shown on the horizontal axis. Distance is shown on the vertical axis. What feature of the graph shows the speed?

Well, speed is the rate at which d changes with respect to t. And, that is the slope of the line! Or, what is equivalent, it is the ratio $\frac{\text{rise}}{\text{run}}$. Or, what is equivalent, the ratio $\frac{\Delta d}{\Delta t}$.

To find the speed, think as illustrated in Figure (6.29).

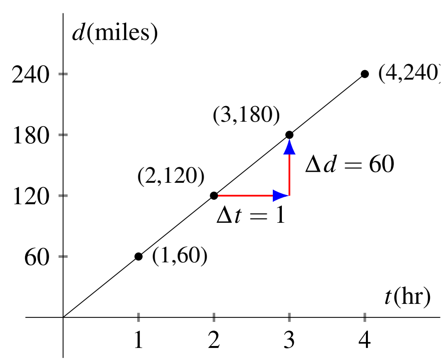


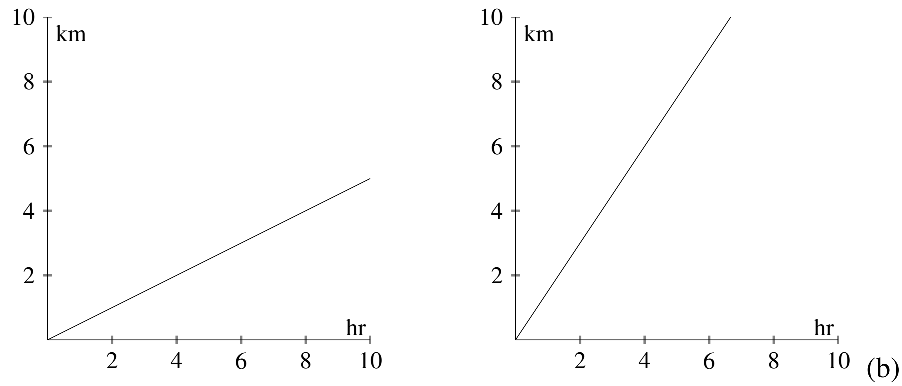
FIGURE 6.29. Same four hour trip.

Since speed is rate of change in distance with respect to time,

$$\text{the speed } = \frac{\Delta d}{\Delta t} = \frac{60 \text{ mile}}{1 \text{ hour}} = 60\frac{\text{mi}}{\text{hr}}.$$

Example 6.32

Figure (6.30) shows the distance Tom traveled as a function of time and the distance Sue traveled as a function of time. Which person's speed was greater?



(a) Tom's journey. Sue's journey.

FIGURE 6.30. Tom and Sue, Example (6.32)

Solution

The slope is greater for Sue than for Tom. The slope of the line is $\frac{\Delta\text{distance}}{\Delta\text{time}}$. So, the rate of change in distance with respect to time is greater for Sue than for Tom. Sue traveled at the greater speed.

Exercise 6.11

1. Alice drove at a constant speed of 40 mph from Town A to Town B. She completed the trip in 3 hours. Write the distance she covered, d miles, as a function of time t hours. Be sure to state the domain of the function.
2. A bucket of ice, temperature $0\,^{\circ}C$, was heated on a stove. The water boiled ($100\,^{\circ}C$) after 20 minutes of heating. Assuming that the rate of increase of water temperature was constant, write the temperature $T\,^{\circ}C$, as a function of time t minutes.

3. The graph showing distance and time of a journey by automobile is shown in Figure (6.31).

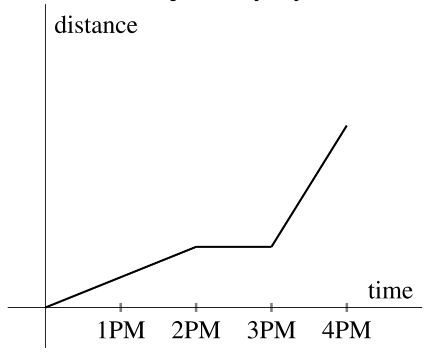


FIGURE 6.31. Four hour car ride.

a) During what time did the car travel the fastest?
b) The driver stopped for lunch. When was that?
c) How long did the lunch stop take?

6.13. $y = ax + b$

We have discussed the function $y = ax$, a constant. It is the special case of $y = ax + b$ when $b = 0$. Now we consider $y = ax + b, a \neq 0, b \neq 0$.

6.13.1. The role of b

To appreciate the effect of b in $y = ax + b$, think how you would compute $y = \frac{1}{2}x + 3$ when, for example, $x = 2$. First you would determine the value of $\frac{1}{2}x$. That is $\frac{1}{2} \cdot 2 = 1$. Then you would add 3. When $x = 4$, you would compute $\frac{1}{2} \cdot 4 = 2$, then you would add 3. The "+3" tacked onto "$\frac{1}{2}x$" just causes y to be 3 greater than $\frac{1}{2}x$ for every value of x. Figure (6.32) illustrates this idea.

The lines $y = ax$ and $y = ax + b$ must be parallel, because they have identical slope. As Figure (6.32) shows, the line $y = 2x + 3$ is the line $y = 2x$ shifted 3 units in the vertical direction. When a line is shifted without any rotation, as in Figure (6.32), we say the line is "translated". Every point on the line $y = ax + b$ is a point on $y = ax$ translated b units vertically.

The graph of $y = ax + b$ where b is a negative number, is the graph of $y = ax$ translated b units in the negative vertical direction (otherwise known as "down"). Figure (6.33) shows the graphs of $y = \frac{1}{2}x$ and $y = \frac{1}{2}x - 3$.

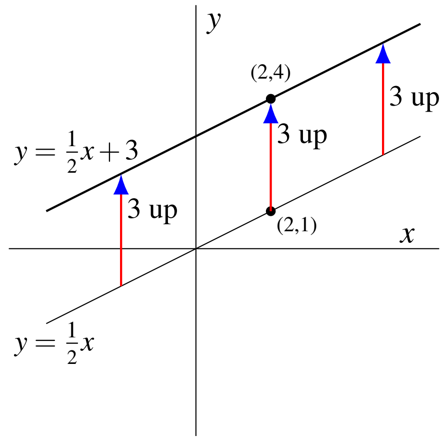


FIGURE 6.32. $y = 2x$ compared to $y = 2x + 3$.

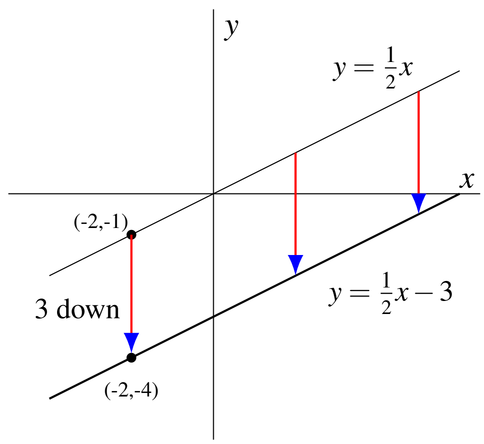


FIGURE 6.33. $y = 2x$ compared to $y = 2x - 3$.

6.13.2. Intercepts

The points at which a graph crosses the axes are landmarks. They are called intercepts.

Definition 6.9 (Vertical intercept)
The second coordinate of the point at which a graph crosses the vertical axis is called the **vertical intercept**.

Definition 6.10 (Horizontal intercept)
The first coordinate of the point at which a graph crosses the vertical axis is called the **horizontal intercept**.

Remark 6.9
If the function is written $y = ax + b$, we often call the vertical axis the "y axis" and speak of the "y intercept". If the function is written $t = as + b$,

the vertical axis may be called the "t axis" and the vertical intercept the "t intercept".

Example 6.33

State the intercepts of the function $y = 2x + 6$ shown in Figure (6.34).

Solution

The line $y = 2x + 6$ crosses the y axis at the point $(0,6)$ where the second coordinate is 6. It crosses the x axis at $(-3,0)$.Therefore, the y intercept is 6 and the x intercept is -3.

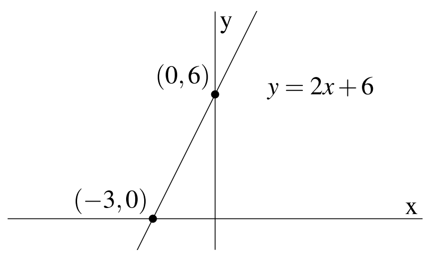


FIGURE 6.34

■

Was it necessary to make a graph? No. Recall from page 162 that the first coordinate of every point on the vertical axis is 0 and that the second coordinate of every point on the horizontal axis is zero. These facts are used in Example (6.34).

Example 6.34

Find the intercepts of $y = 8x + 5$.

Solution

Find the y intercept by setting $x = 0$. Then $y = 8(0) + 5$. So the y intercept is 5.
Find the x intercept by setting $y = 0$. $0 = 8x + 5 \implies x = \frac{-5}{8}$. So the x intercept is $\frac{-5}{8}$.

6.13.3. Solving problems just by looking at them

When a linear function is written $y = ax + b$, all one needs to do is look! The coefficient of x must be the slope, the constant b must be the y intercept. When we answer by merely looking at $y = ax + b$, we say that we found the slope and y intercept "by inspection".

Example 6.35

Determine the slope and y intercept of $y = 19x + 100$.

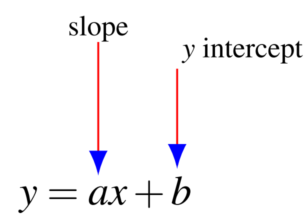
Solution

By inspection, the slope is 19 and the y intercept is 100.

6.13.4. Slope-intercept form

We call the form $y = ax + b$ the “slope-intercept” form of a linear function.

Slope-intercept form



Example 6.36

Determine the slope and t intercept of $t = \frac{2}{3}x - 5$.

Solution

The function written in slope-intercept form is $t = \frac{2}{3}x + (-5)$. By inspection, the slope is $\frac{2}{3}$ and the t intercept is -5.

Example 6.37

Determine the slope and x intercept of $3x + 5y - 7 = 9$.

Solution

Begin by writing $3x + 5y - 7 = 9$ in slope-intercept form.

$$3x + 5y - 7 = 9$$

$$5y = -3x + 16.$$

$$y = \frac{-3}{5}x + \frac{16}{5} \tag{6.10}$$

Equation (6.10) is in slope-intercept form. By inspection, the slope is $\frac{-3}{5}$ and the y intercept is $\frac{16}{5}$.

Exercise 6.12

Write the slope and both intercepts for each of the following.

1. $y = 3x + 7$
2. $t = 5s + \frac{1}{2}$
3. $t = \frac{5}{8}s + \frac{3}{2}$
4. $y = \frac{-5}{9} + 6$
5. $y = \frac{-3}{4} - 1$
6. $2y + x = 6$
7. $3x - y + 2 = 0$
8. $2(3x - y) - 7 = 10$
9. $12 - (x + 7) = 5y$
10. $\frac{x}{3} + \frac{y}{2} = 1$

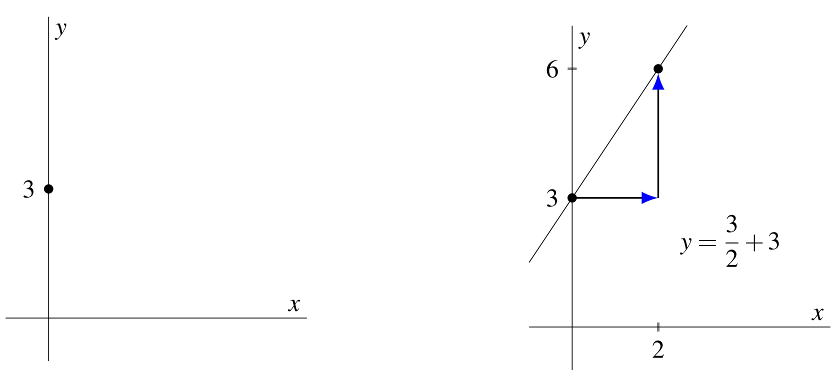
Example 6.38

Graph the function $y = \frac{3}{2}x + 3$.

Solution

The y intercept is 3, so the point $(0, 3)$ is on the line.

Since the slope is $\frac{3}{2}$, 2 units right then 3 units up produces another point on the line.



(a) Plot $(0, 3)$.

(b) Draw the line.

FIGURE 6.35. $y = \frac{3}{2} + 3$

An alternative method for graphing a function is to find the intercepts, then draw a line though them.

Example 6.39

Graph the function $y = \frac{2}{5}x + 2$.

Solution

The y intercept is 2, so the point $(0,2)$ is on the line. The x intercept is -5, so the point $(-5,0)$ is on the line.

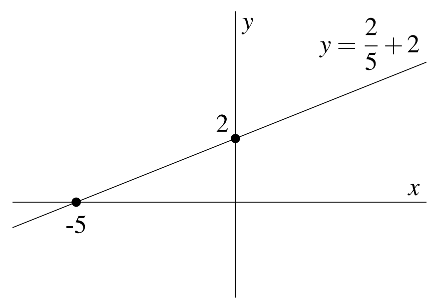


FIGURE 6.36. $y = \dfrac{2}{5}x + 2$

Exercise 6.13

Graph each of the following functions. Label the intercepts.

1. $y = \frac{1}{2}x + 3$
2. $y = x + 3$
3. $y = \frac{2}{3}x + 2$
4. $y = \frac{-3}{5}x + 3$
5. $t = -6s + 6$
6. $y = \frac{3}{4}x + 3$
7. $y = \frac{2}{7}x - 2$
8. $z = \frac{1}{3}w - 1$

Example 6.40

Places A and B are 1000 meters apart. Sue walks 200 meters per minute from place A to Place B.

(a) Write Sue's distance from Place A, d meters, as a function of time t minutes. State the domain of the function. Graph the function.

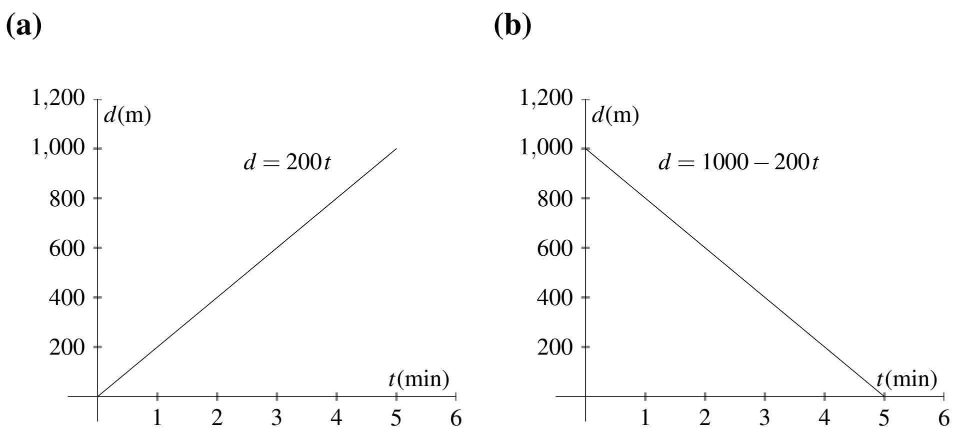
(b) Write Sue's distance from Place B, d meters, as a function of time t minutes. State the domain of the function. Graph the function.

Solution

(a) The function is $d = 200t$. Sue will complete her walk when d is 1000 m. Solving $1000 = 200t$, for t, we find that $t = 5$ when $d = 1000$. So the domain of the function is all numbers between 0 and 5.

(b) The function is $d = 1000 - 200t$. Sue will complete her walk when d is 0 m. Solving $0 = 1000 - 200t$, for t, we find that $t = 5$ when $d = 0$. So the domain of the function is all numbers between 0 and 5.

The graphs of both functions are shown side by side in Figure (6.37).



(a) Distance from Place A.

(b) Distance to Place B.

FIGURE 6.37. Sue's walk, Example (6.40)

Remark 6.10
Example (6.40), notice that Sue's distance from Place A is an increasing function of time and her distance from Place B is a decreasing function of time.

Example 6.41

A tank of capacity 1600 gallons initially contains 200 gallons of water. Then water is added to the tank at the rate of 50 gallons per minute until the tank is full.

(a) Write the volume of water in the tank, V gallons, as a function of time t minutes. Graph the function.

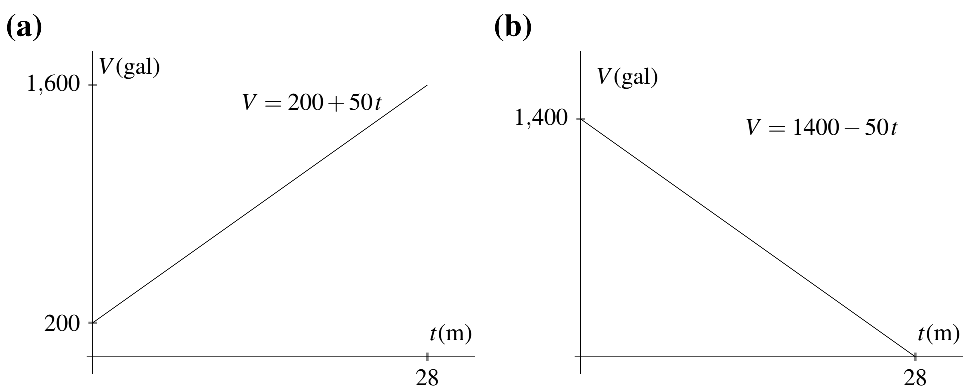
(b) Write the volume of water that must be added to fill the tank, V gallons, as a function of time t minutes. Graph the function.

Solution

(a) The function is $V = 200 + 50t$. The tank will be full when V is 1600 gal. Solving $1600 = 200 + 50t$, for t, we find that $t = 28$ when $V = 1600$. So the domain of the function is all numbers between 0 and 28.

(b) The function is $V = 1400 - 50t$. The tank will be full when V is 0 gal. Solving $0 = 1400 - 50t$, for t, we find that $t = 28$ when $V = 0$. So the domain of the function is all numbers between 0 and 28.

The graphs of both functions are shown side by side in Figure (6.38).



(a) Volume water in tank. (b) Volume water to add.

FIGURE 6.38. Tank filling, Example (6.41)

■

Part (a) of Example (6.41) suggests a helpful way to think of a linear function.

Slope-intercept form

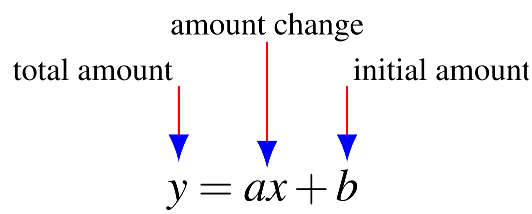


Figure (6.39) illustrates this idea.

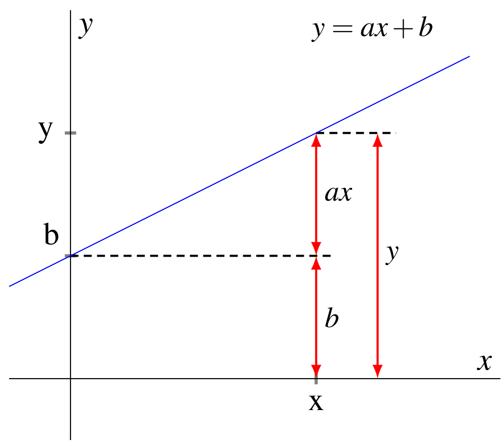


FIGURE 6.39. A helpful way to think about $y = ax + b$

Exercise 6.14

1. Station B is 900 miles east of Station A. Station C is 300 miles east of Station A. A train travels at 60 mph from Station C to Station B.
   a) Write the distance of the train from Station A, y miles, as a function of time t hours. Graph the function. State whether the distance is an increasing or decreasing function of time.
   b) Write the distance of the train from Station B, y miles, as a function of time t hours. Graph the function. State whether the distance is an increasing or decreasing function of time.

2. Alex walks at 1.5 meters per second (m/s) and Barb runs at 4 m/s. Al and Barb each start at the same time, but Al has a 10 meter head start. How long does it take for Barb to catch Alex? Write Alex's distance, y_A meters (m) as a function of time t seconds (s) and and Barb's distance y_B m as a function of time t s. Then graph both functions on the same axes and estimate the time at which Barb catches Alex.
3. Find the exact answer to Question (2).
4. On page 183, we showed that if $y = 2x$, then the change in y is 2 times the change in x. Is it also true that if $y = 2x + 5$, then the change in y is 2 times the change in x? Provide the reasoning for your answer.
5. Draw a graph like the one in Figure (6.39) for the function $y = 2x + 5$.

Appendices

Appendix A

Answers to Exercises

Answers to Exercise 1.1

(**1**) $0, 1, 2, \cdots, 1000$. (**2**) $7, 8, 9, \cdots, 93$. (**3**) $5, 6, 7, \cdots$.
(**4**) Yes. Exhibit a 1-1 correspondence. (**5**) Yes. Exhibit a 1-1 correspondence. (**6**) Yes. Exhibit a 1-1 correspondence.

Answers to Exercise 1.2

(**1**) $a = 1 \times a$. (**2**) $\frac{a}{a} = 1$. (**3**) $0 + b = 0$. (**4**) $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$
(**5**) $a \times b = b \times a$ (**6**) $2 \times y$, where $y = 1, 2, 3, \cdots$.

Answers to Exercise 1.3

(**1**) $101 + a$. (**2**) $a + 3 = b + 5$. (**3**) Symmetric. (**4**) The Principle of Substitution
(**5**) (1) Adding the same number to each side of an equation preserves the equality. (2) Multiplying both sides of an equation by the same number preserves the equality.
(**6**) Proof
Let a, b and c be any numbers. Suppose that $a = b$.

$$\begin{aligned} a \times c &= a \times c, && \text{equality is reflexive} \\ a \times c &= b \times c, && \text{substitution, we supposed } a = b. \end{aligned}$$

Therefore, if $a = b$ then $a \times c = b \times c$. ■

Answers to Exercise 1.4

(**1**) 11 (**2**) 36 (**3**) 15 (**4**) 17

Answers to Exercise 1.5

[Part 1] (**1**) True. (**2**) False. (**3**) False. (**4**) True. (**5**) False.
(**6**) True. (**7**) True. (**8**) False. (**9**) True. (**10**) True.

[Part 2] (**1**) $\in$. (**2**) $\in$. (**3**) $\notin$. (**4**) $\in$.
[Part 3] (**1**) The set of even numbers. (**2**) The set of multiples of 5 that are no less than 10.

Answers to Exercise 1.6

(**1**)

$$\begin{aligned}(2+7)+13 &= 2+(7+13) && \text{associative}\\ &= 2+(13+7) && \text{commutative}\\ &= (2+13)+7 && \text{commutative}\end{aligned}$$

(**2**)

$$\begin{aligned}(9\times 11)\times 6 &= 9\times(11\times 6) && \text{associative}\\ &= 9\times(6\times 11) && \text{commutative}\\ &= (9\times 6)\times 11 && \text{associative}\end{aligned}$$

(**3**)

$$\begin{aligned}1+4+(7+12) &= 1+(7+12)+4 && \text{commutative}\\ &= (1+7)+12+4 && \text{associative}\\ &= 12+(1+7)+4 && \text{commutative}\end{aligned}$$

(**4**)

$$\begin{aligned}3+(7+12)+(4\times 7) &= (3+7)+12+(4\times 7) && \text{associative}\\ &= (7+3)+12+(4\times 7) && \text{commutative}\\ &= (7+3)+(4\times 7)+12 && \text{commutative}\\ &= (4\times 7)+(7+3)+12 && \text{commutative}\\ &= (4\times 7)+12+(7+3) && \text{commutative}\end{aligned}$$

(**5**) $2, 1001, 6$ (**6**) Yes. Because $\frac{12}{3}=4$.

Answers to Exercise 2.1

[Part 1] (**1**) $7+(-7)=0$. (**2**) $4+(-4)=0$. (**3**) $2090+(-2090)=0$. (**4**) $-33+33=0$. (**5**) $-51+51=0$. (**6**) $-8+8=0$. (**7**) $x+(-x)=0$. (**8**) $-x+(-(-x))=0$ or $-x+x=0$.
[Part 2] (**1**) 1. (**2**) 2. (**3**) Yes. (**4**) Yes.
[Part 3] (**1**) a. (**2**) c. (**3**) Yes. (**4**) Yes.

Answers to Exercise 2.2

(**1**) -3. (**2**) -6. (**3**) -1. (**4**) -2. (**5**) -7. (**6**) -16. (**7**) -6. (**8**) -9 (**9**) -5 (**10**) -16

Answers to Exercise 2.3

(**1**) Let a and b be positive numbers. Since the integers are closed under subtraction, there is a number, call it c, such that
$-a-b=c$. This means that $-a=c+b$.

$$\begin{aligned} -a+(-b) &= (c+b)+(-b) && \text{substitution, } -a=c+b \\ &= c+(b+(-b)) && \text{associative} \\ &= c+0 && \text{inverse elements} \\ &= c && \text{identity element} \\ &= -a-b && \text{substitution, } c=a-b \end{aligned}$$

Therefore.

$$-a+(-b)=-a-b. \quad \blacksquare \tag{2.7}$$

(**2**) This is the subtraction of the integer $(-b)$. But, according to equation (2.7), that is accomplished by adding the additive inverse of $-b$. Since the additive inverse of $-b$ is b, $-a-(-b)=-a+b$. ■

Answers to Exercise 2.4

(**1**) 5 (**2**) -4 (**3**) 2 (**4**) 7 (**5**) 3 (**6**) 5 (**7**) 3 (**8**) 7 (**9**) 6
(**10**) -5 (**11**) -10 (**12**) 12 (**13**) 2 (**14**) 3 (**15**) -2 (**16**) -5
(**17**) -2 (**18**) -8 (**19**) -6 (**20**) 3 (**21**) -2 (**22**) 1 (**23**) -6
(**24**) -3 (**25**) 6 (**26**) -1 (**27**) 8 (**28**) 6 (**29**) -3 (**30**) 4

Answers to Exercise 2.5

[Part 1] (**1**) 7 (**2**) 113 (**3**) 9 (**4**) $15+3=18$
[Part 2] (**1**) No. We no longer need the separate cases $a+(-b)$ and $a+b$. When b is positive, case $a+b$. When b is negative, case $a+(-b)$. (**2**) 21 (**3**) 133

Answers to Exercise 2.6

[Part 1] (**1**) $7b$ (**2**) $9x$ (**3**) ad (**4**) $2ab$ (**5**) $5(x+2)$ (**6**) $9a(3y+4)+2$ (**7**) $5(a+b)$ (**8**) $7(a+1)$ (**9**) $7(2a+d)$ (**10**) $2(3a+4)$ (**11**) $6(2x-y)$ (**12**) $-9(5x-2y)$
[Part 2] (**1**) $3\times(2\times a+5)$. (**2**) $4\times a\times b\times c$ (**3**) $x\times y$
(**4**) $x\times y\times(2\times y+5\times x)$ (**5**) $3\times x(5\times x+3\times y+7)$

Answers to Exercise 2.7

[Part 1] (**1**) $ab+ac$ (**2**) $3a+6$ (**3**) $4x+4y$ (**4**) $2x+10$ (**5**) $7a+21b$
(**6**) $12+6x$ (**7**) $10a+15b$ (**8**) $11x+22$

[Part 2]

(**1**)

Proof.

$$\begin{aligned} a(b+c+d) &= ab+a(c+d) \\ &= ab+ac+ad \end{aligned}$$

$\therefore a(b+c+d) = ab+ac+ad$ ■

(**2**)

Proof.

$$a(b+c+d+e) = a(b+c+d)+ae$$

Using the previous result

$$= ab+ac+ad+ae$$

$\therefore a(b+c+d+e) = ab+ac+ad+ae$ ■

Answers to Exercise 2.8

(**1**)

Proof. Second part, $0 \cdot a = 0$.
Let a be any number.

$a \cdot 0 = 0$	Theorem (2.2)
$0 \cdot a = 0$	multiplication is commutative

$\therefore\ 0 \cdot a = 0$

■

(**2**)

Proof. Second part, $(-a)\cdot(b) = -(ab)$
Let a and b be any numbers other than 0,

$$\begin{aligned} a+(-a) &= 0 \\ b(a+(-a)) &= 0 \\ ba+b\cdot(-a) &= 0 && \text{Distribution} \\ -(ba)+ba+b\cdot(-a) &= -(ba) \\ b\cdot(-a) &= -(ba) \\ (-a)\cdot(b) &= -(ab) \end{aligned}$$

■

Answers to Exercise 2.9

(**1**) $25n+25$ (**2**) $-4-20a$ (**3**) $-5-10k$ (**4**) $-2a+10$
(**5**) $-20x-20$ (**6**) $-5n-20$ (**7**) $5+5k$ (**8**) $-6p+10$ (**9**) $10x-10$
(**10**) $20-20n$ (**11**) $-20m-12$ (**12**) $12-15r$ (**13**) $5x+20$
(**14**) $4+4n$ (**15**) $9-12b$ (**16**) $2-8r$ (**17**) $5+10x$ (**18**) $4x-4$
(**19**) $-12-12a$ (**20**) $-4p-16$ (**21**) $8-12x$ (**22**) $25n-15$
(**23**) $3m+6$ (**24**) $-5r-15$ (**25**) $-20x-15$ (**26**) $20n+15$
(**27**) $2-4v$ (**28**) $-15x+6$ (**29**) $9n+6$ (**30**) $8a+4$ (**31**) $10k+2$
(**32**) $25+10x$ (**33**) $2x+4$ (**34**) $-4+4n$ (**35**) $-15+3m$
(**36**) $-8v-10$ (**37**) $15n-20$ (**38**) $-2+10n$ (**39**) $-15+10m$
(**40**) $-8r-2$

Answers to Exercise 2.10

(**1**) $-2-2n$ (**2**) $8b-8$ (**3**) $-8-8n$ (**4**) $-2x-6$ (**5**) $-4x-4$
(**6**) $-15-3a$ (**7**) $-20k-4$ (**8**) $-4p-2$ (**9**) $4x-4$ (**10**) $-5n+20$
(**11**) $-3-9m$ (**12**) $-12r+8$ (**13**) $-5+3x$ (**14**) $-10+2n$
(**15**) $-4b-8$ (**16**) $-2+2v$ (**17**) $20x+20$ (**18**) $-15+5n$
(**19**) $-12a+3$ (**20**) $-25k+5$ (**21**) $-4-20x$ (**22**) $3+6n$
(**23**) $-3k+3$ (**24**) $-12+6p$ (**25**) $-3-3x$ (**26**) $-15+12x$
(**27**) $-4r+3$ (**28**) $4x+12$ (**29**) $8n+6$ (**30**) $-3v-6$ (**31**) $-20x+12$
(**32**) $-6x+6$ (**33**) $-16a+16$ (**34**) $-5+15p$ (**35**) $-25x+20$
(**36**) $-2+8n$ (**37**) $5+20m$ (**38**) $-9r+12$ (**39**) $-5x-15$
(**40**) $-3n+15$

Answers to Exercise 2.11

(**1**) $2a+5$ (**2**) $15a+7b$ (**3**) $8x+2y+2$ (**4**) $7a-b-4c+9$ (**5**) $5a+7b+3c-13$ (**6**) $5x+9y+8z-9$ (**7**) $-10x+10y+10z-8$ (**8**) $2a-8c-8$ (**9**) Simplified. (**10**) Simplified.

Answers to Exercise 2.12

(**1**) $8a^5$ (**2**) $16x^3y$ (**3**) Already simplified. (**4**) $6a^3b^3c^2$ (**5**) Already simplified. (**6**) Already simplified.

Answers to Exercise 2.13

(**1**) $10a+3$ (**2**) $8a+5b+9$ (**3**) $18a+5b+1$- (**4**) $7x+1$ (**5**) $-2x+7$ (**6**) $32-10y$ (**7**) $-16x-1$ (**8**) $16a+9b-15$ (**9**) $8a^2+8a+51$ (**10**) $2a^2+4ab-a+2b+b^2$ (**11**) - $4a^3b+9a^2b+7b+3a^3-7$

Answers to Exercise 2.14

(**1**) $10m-92$ (**2**) $10-47r$ (**3**) $-5-20x$ (**4**) $-15n+30$ (**5**) $-21+72b$ (**6**) -63 (**7**) $76x+8$ (**8**) $-4a+18$ (**9**) $-13-12r$ (**10**) $13+36k$ (**11**) $-3-9x$ (**12**) $8v+47$ (**13**) $50+7x$ (**14**) $-12m-7$ (**15**) $-5+80v$ (**16**) $-81n+32$ (**17**) $-5-14b$ (**18**) -16 (**19**) $44-25k$ (**20**) $-48+40x$ (**21**) $-5b-36$ (**22**) $92a+10$ (**23**) $-9x-4$ (**24**) $-6n-4$ (**25**) $-6-11x$ (**26**) $-20n+3$ (**27**) $3-6k$ (**28**) $10p+40$ (**29**) $-56x-18$ (**30**) $25-8n$ (**31**) $-10+15m$ (**32**) $19-63r$ (**33**) $2k+11$ (**34**) $-3n$ (**35**) $-10b-49$ (**36**) $-39v+40$ (**37**) $-60-70x$ (**38**) $10n+75$ (**39**) $19a+100$ (**40**) $-68k-14$ (**41**) $80p+50$ (**42**) $-x-10$ (**43**) $-28n-2$ (**44**) $34-22m$ (**45**) $-14r+35$ (**46**) $38-4v$ (**47**) $-39n+84$ (**48**) $-11+25b$ (**49**) $48x+112$ (**50**) $-92m+38$ (**51**) $-59n-44$ (**52**) $-44n+29$ (**53**) $37x+74$ (**54**) $-28x+77$ (**55**) $-13v+2$ (**56**) $-23+20n$ (**57**) $8-38k$ (**58**) $14a+18$ (**59**) $25b+23$ (**60**) $-77n-76$

Answers to Exercise 2.15

(**1**) $10-3a$ (**2**) $6-5y$ (**3**) $8-3x$ (**4**) $60-8x$ (**5**) $-3x-10$ (**6**) $-2x-15$ (**7**) x (**8**) $-x-2$ (**9**) $18-10a$ (**10**) $2x-11$ (**11**) $x-7$ (**12**) $3x-31$ (**13**) $5b-13$ (**14**) $2x-2$ (**15**) $4-x$ (**16**) $180-10x$ (**17**) $-x$ (**18**) $42-2x$

Answers to Exercise 2.16

(**1**) $19n+4$ (**2**) $42a+48$ (**3**) $-48k-55$ (**4**) $-24x-9$ (**5**) 18 (**6**) $-46n-8$ (**7**) $26-42m$ (**8**) $54-43p$ (**9**) $x-3$ (**10**) $5n-42$ (**11**) $2m-1$ (**12**) $4r-6$ (**13**) $27-87x$ (**14**) $15n-49$ (**15**) $19-36b$

(**16**) $90v-5$ (**17**) $6x-32$ (**18**) $4x-2$ (**19**) $-18a-9$ (**20**) $-92k-90$
(**21**) $-64p-57$ (**22**) $15x-49$ (**23**) $15n-24$ (**24**) $43m-70$
(**25**) $8r-61$ (**26**) $-35x-7$ (**27**) 8 (**28**) $3b-12$ (**29**) $16-30v$
(**30**) $8-8x$ (**31**) $11n-5$ (**32**) $36-9a$ (**33**) $18k-26$ (**34**) $17x-40$
(**35**) $4-3x$ (**36**) $-45n-85$ (**37**) $-60m-45$ (**38**) $45p-6$
(**39**) $-18x-90$ (**40**) $56n-51$

Answers to Exercise 2.17

(**1**) $20-72m$ (**2**) $15-8r$ (**3**) $-43x-4$ (**4**) $-33n-75$
(**5**) $-39b-2$ (**6**) $3-30v$ (**7**) $-12x-42$ (**8**) $61-26x$ (**9**) $36-38a$
(**10**) $112k-24$ (**11**) $6-20p$ (**12**) $30x-24$ (**13**) $16n-18$
(**14**) $61-45m$ (**15**) $19r-10$ (**16**) $32-40x$ (**17**) $34-60n$
(**18**) $18-31b$ (**19**) $6v+32$ (**20**) $-44x-20$ (**21**) $88-11n$
(**22**) $10-28a$ (**23**) $28k-13$ (**24**) $-15x-39$ (**25**) $-37x-8$
(**26**) $-19n-27$ (**27**) $-39m-2$ (**28**) $15p+5$ (**29**) $7-35x$
(**30**) $-9n-21$ (**31**) $-10b-16$ (**32**) $31r-53$ (**33**) $-47x-24$
(**34**) $-47n-83$ (**35**) $25b+14$ (**36**) $22-43v$ (**37**) $-54x+46$
(**38**) $-63x-16$

Answers to Exercise 3.1

[Part 1]

(**1**)

$$27\div 3=x$$

means that

$$3\cdot x=27.$$

Multiply both sides $\frac{1}{3}$,

$$\frac{1}{3}\cdot 3\cdot x=\frac{1}{3}\cdot 27.$$

Since 3 and $\frac{1}{3}$ are multiplicative inverses, and 1 is the identity element for multiplication,

$$1\cdot x=\frac{1}{3}\cdot 27$$

$$x=\frac{1}{3}\cdot 27$$

$$x=9$$

$$\therefore\quad 27\div 3=9.\quad\blacksquare$$

(**2**)

$$31 \div 6 = x$$

means that

$$6 \cdot x = 31$$

$$\frac{1}{6} \cdot 6 \cdot x = \frac{1}{6} \cdot 31$$

$$1 \cdot x = \frac{1}{6} \cdot 31$$

$$x = \frac{31}{6}.$$

[Part 2] (**1**) $9 \div 3 = 9 \cdot \frac{1}{3} = 3$ (**2**) $17 \div 8 = 17 \cdot \frac{1}{8} = \frac{17}{8}$
(**3**) $52 \div 13 = 52 \cdot \frac{1}{13} = \frac{52}{13} = 4$ (**4**) $5 \div 12 = 5 \cdot \frac{1}{12} = \frac{5}{12}$

Answers to Exercise 3.2

(**1**) $\frac{-15}{28}$ (**2**) $\frac{36}{55}$ (**3**) $\frac{7}{22}$ (**4**) $\frac{-2}{3}$ (**5**) 4 (**6**) $\frac{28}{3}$ (**7**) $\frac{-13}{3}$

(**8**) $\frac{-3}{7}$ (**9**) $\frac{3}{8}$ (**10**) $\frac{2}{15}$ (**11**) $\frac{3}{11}$ (**12**) $\frac{-4}{15}$ (**13**) $\frac{3}{16}$ (**14**) $\frac{1}{18}$

(**15**) $\frac{-2}{5}$ (**16**) $\frac{1}{20}$ (**17**) $\frac{ab}{6}$ (**18**) $\frac{9a}{5}$ (**19**) $\frac{-12x}{5y}$

(**20**) $4x$ (**21**) $\frac{-7}{2}$ (**22**) $\frac{-a}{12b}$ (**23**) $\frac{2}{5}$ (**24**) $\frac{-a}{bc}$ (**25**) $\frac{-7a}{bc}$

Answers to Exercise 3.3

(**1**)

Proof. Let a be any number other than 0.

$$\begin{aligned} \text{LHS} &= \frac{a}{a} \\ &= a \cdot \frac{1}{a}, && \text{definition division} \\ &= 1 && \text{inverse element} \\ &= \text{RHS} \end{aligned}$$

$$\therefore \frac{a}{a} = 1, a \neq 0$$

∎

(**2**)

Proof. Let a be any number.

$$\begin{aligned} \text{LHS} &= \frac{a}{1} \\ &= a \cdot \frac{1}{1}, && \text{definition division} \\ &= a \cdot 1, && \text{Theorem (3.1) with } a = 1 \\ &= a, && \text{idemtity element} \\ &= \text{RHS} \end{aligned}$$

$$\therefore \frac{a}{1} = a$$

∎

(**3**)

Proof.

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= \frac{a}{b} \cdot 1 + \frac{c}{d} \cdot 1 && \text{identity element} \\ &= \frac{a}{b} \cdot \frac{d}{d} + \frac{c}{d} \cdot \frac{b}{b} && \text{Theorem (3.1)} \\ &= \frac{ad}{bd} + \frac{cb}{bd} && \text{Theorem (3.4)} \\ &= ad \cdot \frac{1}{bd} + cb \cdot \frac{1}{bd} && \text{Theorem (3.4)} \\ &= \frac{1}{bd}(ad + bc) && \text{distribution} \\ &= \frac{ad + bc}{bd} && \text{definition division} \end{aligned}$$

■

(4)

Proof.

$$
\begin{aligned}
\frac{a}{b}+\frac{c}{b} &= a\cdot\frac{1}{b}+c\cdot\frac{1}{b} && \text{definition division}\\
\frac{a}{b}+\frac{c}{b} &= \frac{1}{b}(a+c) && \text{distribution}\\
&= \frac{a+c}{b} && \text{definition division}
\end{aligned}
$$

■

Answers to Exercise 3.4

(1) $\frac{2}{5}$ **(2)** $\frac{8}{7}$ **(3)** 22 **(4)** $\frac{6}{7}$ **(5)** 2 **(6)** 5 **(7)** $\frac{2a-2b}{a+b}$ **(8)** $\frac{10x}{27}$

(9) $\frac{5}{4}$ **(10)** $\frac{2a-5}{13}$ **(11)** $\frac{8a-11}{15}$ **(12)** $\frac{8x-13}{4}$ **(13)** $\frac{28a+4}{15}$

(14) $\frac{7x-2}{3}$

Answers to Exercise 3.5

(1) $\frac{-a-9}{10}$ **(2)** $\frac{2b-31}{6}$ **(3)** $\frac{5y+2}{6}$ **(4)** $\frac{2x+9}{24}$ **(5)** $\frac{7a+1}{4}$

(6) $\frac{17a+2}{9}$ **(7)** $\frac{4x-38}{15}$ **(8)** $\frac{2x-23}{28}$ **(9)** $\frac{x+4}{18}$ **(10)** $\frac{-8x-13}{3}$

(11) $\frac{29x+2}{21}$ **(12)** $\frac{-7x-11}{12}$ **(13)** $\frac{9x-2}{12}$ **(14)** $\frac{-11a+24}{9}$

(15) $\frac{-b+5}{12}$ **(16)** $\frac{5b-84}{22}$ **(17)** $\frac{-5b-3}{8}$ **(18)** $\frac{10a+2b-11}{24}$

(19) $\frac{9b+4}{12}$ **(20)** $\frac{-10x-23}{12}$ **(21)** $\frac{x+2}{6}$ **(22)** $\frac{-7a+24}{15}$ **(23)** $\frac{35x+5}{36}$

(24) $\frac{8a+23}{12}$ **(25)** $\frac{3x-10}{6}$ **(26)** $\frac{-5x+10}{6}$

Answers to Exercise 3.6

(1) $6x-6$ **(2)** $-6n-6$ **(3)** $2m-6$ **(4)** $3r+6$ **(5)** $\frac{-3n}{2}$
(6) $\frac{15x-11}{6}$ **(7)** $-2v$ **(8)** $\frac{5-3b}{2}$ **(9)** $\frac{1}{2}$ **(10)** $\frac{-14n-1}{6}$ **(11)** $-3a$
(12) $\frac{3k+2}{2}$ **(13)** $\frac{12p-20}{3}$ **(14)** $\frac{3-x}{9}$ **(15)** $\frac{12n+16}{9}$ **(16)** $\frac{4m+15}{18}$
(17) $\frac{-r-4}{2}$ **(18)** $\frac{-3x-1}{2}$ **(19)** $\frac{99-88n}{18}$ **(20)** $\frac{8-4b}{3}$ **(21)** $\frac{5-5v}{3}$
(22) $\frac{43x-15}{2}$ **(23)** $\frac{-8n-75}{18}$ **(24)** $\frac{12-5a}{6}$ **(25)** $\frac{23-9v}{6}$ **(26)** $\frac{-22x-43}{6}$
(27) $\frac{-3a-2}{4}$ **(28)** $\frac{26-3n}{3}$ **(29)** $\frac{4k+1}{2}$ **(30)** $\frac{-9p-1}{3}$ **(31)** $\frac{13x+6}{9}$
(32) $\frac{9k+4}{3}$ **(33)** $\frac{-2m-10}{3}$ **(34)** $\frac{31x+2}{12}$ **(35)** $\frac{5-18x}{6}$ **(36)** $\frac{2-n}{2}$

Answers to Exercise 3.7

(1) $\frac{27p-22}{18}$ **(2)** $\frac{39-44n}{18}$ **(3)** $\frac{70-33x}{18}$ **(4)** $\frac{13x-12}{9}$ **(5)** $\frac{-8r-9}{9}$
(6) $\frac{-18x-110}{9}$ **(7)** $\frac{22n-27}{12}$ **(8)** $\frac{-2b-15}{3}$ **(9)** $\frac{31v-35}{6}$ **(10)** $\frac{1-4x}{9}$
(11) $\frac{25x-98}{18}$ **(12)** $\frac{7-43a}{6}$ **(13)** $\frac{-13k-18}{12}$ **(14)** $\frac{10p-21}{3}$ **(15)** $\frac{-75x-8}{18}$
(16) $\frac{13x+22}{6}$ **(17)** $\frac{-16m-36}{9}$ **(18)** $\frac{-4r-3}{2}$ **(19)** $\frac{32x+53}{12}$ **(20)** $\frac{14n-5}{18}$
(21) $\frac{16-24b}{9}$ **(22)** $\frac{-48v-13}{12}$ **(23)** $\frac{12-35n}{12}$ **(24)** $\frac{4-6x}{3}$ **(25)** $\frac{-20a-18}{3}$
(26) $\frac{40-42k}{9}$ **(27)** $\frac{30x+34}{9}$ **(28)** $\frac{23x+7}{12}$ **(29)** $\frac{35-9x}{3}$ **(30)** $\frac{15-4m}{2}$

Answers to Exercise 4.1

(1) $5(3)+8=23, \quad \therefore x=3$ is a solution. **(2)** $7(5)-9=26, \quad \therefore x=5$ is a solution. **(3)** $5(6)-4=26,\ 26\neq 20, \quad \therefore a=6$ is not a solution.

(4) $\dfrac{2\left(\dfrac{15}{4}\right)}{5}+\dfrac{3}{2}=3, \quad \therefore x=\dfrac{15}{4}$ is a solution.

Answers to Exercise 4.2

(1) 1 **(2)** 3 **(3)** 0 **(4)** 3 **(5)** 2 **(6)** -2 **(7)** -1 **(8)** -2 **(9)** -1
(10) 3 **(11)** -2 **(12)** 2 **(13)** 0 **(14)** 1 **(15)** 1 **(16)** 0 **(17)** -3
(18) 0 **(19)** 2 **(20)** 3 **(21)** 3 **(22)** -2 **(23)** -3 **(24)** 0 **(25)** 1
(26) -2 **(27)** -2 **(28)** -2 **(29)** 3 **(30)** -3

Answers to Exercise 4.3

(**1**) $x = 6$ (**2**) $x = \frac{2}{3}$. (**3**) $x = 150$. (**4**) $x = 135$. (**5**) $x = \frac{6}{7}$.
(**6**) $x = \frac{3}{10}$. (**7**) $x = 42$. (**8**) $x = 112$. (**9**) $x = \frac{60}{77}$. (**10**) $x = \frac{1}{4}$.
(**11**) $x = \frac{1}{23}$. (**12**) $x = \frac{1331}{4440}$.

Answers to Exercise 4.4

(**1**) $25x = 72$ (**2**) $801x = 400$ (**3**) $131x = 60.2$- (**4**) $1191x = 130$
(**5**) $x = 500$

Answers to Exercise 4.5

(**1**) -4 (**2**) 4 (**3**) 3 (**4**) 4 (**5**) 6 (**6**) -2 (**7**) -3 (**8**) 4 (**9**) -5
(**10**) 5 (**11**) -6 (**12**) 5 (**13**) 3 (**14**) -4 (**15**) 6 (**16**) 5 (**17**) -1
(**18**) 2 (**19**) -1 (**20**) -1 (**21**) -1 (**22**) 2 (**23**) 3 (**24**) -1
(**25**) 5 (**26**) -5 (**27**) -2 (**28**) 0 (**29**) -3 (**30**) 0

Answers to Exercise 4.6

(**1**) 3 (**2**) 4 (**3**) -5 (**4**) 5 (**5**) -5 (**6**) -4 (**7**) -4 (**8**) -4
(**9**) -6 (**10**) -5 (**11**) -5 (**12**) 4 (**13**) -6 (**14**) 3 (**15**) 6
(**16**) -4 (**17**) -1 (**18**) 0 (**19**) -1 (**20**) 2 (**21**) 1 (**22**) 1
(**23**) -2 (**24**) -2 (**25**) 4 (**26**) 0 (**27**) 3 (**28**) -1 (**29**) -3
(**30**) 5

Answers to Exercise 4.7

(**1**) 3 (**2**) -2 (**3**) -3 (**4**) -2 (**5**) 9 (**6**) 2 (**7**) -1 (**8**) 8
(**9**) -11 (**10**) -2 (**11**) 0 (**12**) 2 (**13**) 1 (**14**) -1 (**15**) 15
(**16**) 4 (**17**) -1 (**18**) 1 (**19**) -1 (**20**) -15 (**21**) -9 (**22**) 0
(**23**) 14 (**24**) -4 (**25**) 8 (**26**) 7 (**27**) -3 (**28**) -9 (**29**) 4
(**30**) 1 (**31**) -11 (**32**) 8 (**33**) 1 (**34**) 4 (**35**) -2 (**36**) -2
(**37**) -1 (**38**) -11 (**39**) 4 (**40**) -4

Answers to Exercise 4.8

(**1**) $\frac{1}{2}$ (**2**) $\frac{5}{3}$ (**3**) -2 (**4**) $\frac{-5}{3}$ (**5**) $-\frac{7}{2}$ (**6**) 1 (**7**) $\frac{3}{2}$ (**8**) 0 (**9**) 2
(**10**) $\frac{1}{2}$ (**11**) $\frac{5}{2}$ (**12**) $\frac{3}{2}$ (**13**) $\frac{5}{2}$ (**14**) $\frac{4}{3}$ (**15**) $\frac{-5}{2}$ (**16**) $\frac{-7}{3}$
(**17**) $\frac{-4}{3}$ (**18**) $\frac{-2}{3}$ (**19**) $\frac{2}{3}$ (**20**) $\frac{1}{2}$ (**21**) $\frac{1}{2}$ (**22**) -1 (**23**) 1
(**24**) 1

Answers to Exercise 4.9

(**1**) 0 (**2**) $\frac{7}{3}$ (**3**) -1 (**4**) -3 (**5**) $\frac{-1}{3}$ (**6**) 2 (**7**) $\frac{-10}{3}$ (**8**) $\frac{8}{3}$
(**9**) $\frac{-1}{2}$ (**10**) $\frac{-4}{3}$ (**11**) -2 (**12**) $\frac{-5}{3}$ (**13**) $\frac{4}{3}$ (**14**) $\frac{-3}{2}$ (**15**) $\frac{1}{3}$
(**16**) $\frac{-4}{3}$ (**17**) $\frac{-5}{3}$ (**18**) $\frac{7}{3}$ (**19**) $\frac{2}{3}$ (**20**) $\frac{-5}{3}$ (**21**) $\frac{-4}{3}$ (**22**) -1
(**23**) $\frac{-11}{3}$ (**24**) $\frac{3}{2}$

Answers to Exercise 4.10

[Part 1]

(**1**) Let $a = 2, b = 3$, then

$$\begin{aligned}\frac{1}{a}(a+b) &= \frac{1}{2}(2+3)\\ &= \frac{5}{2}\end{aligned}$$

but,

$$\begin{aligned}\frac{1}{\not{a}}(\not{a}+b) &= \frac{1}{\not{2}}(\not{2}+3)\\ &= 1(1+3)\\ &= 4\end{aligned}$$

(**2**) Let a, b be any numbers provided that $a \neq 0$.

$$\begin{aligned}\text{LHS} &= \frac{1}{a}(ab+ad)\\ &= \frac{1}{a}(a)(b+d)\\ &= b+d.\end{aligned}$$

$$\text{RHS} = \frac{1}{\not{a}}(\not{a}b+\not{a}d)$$

$$\text{RHS} = a+b$$

Since, LHS = RHS, the equation is true.

[Part 2]

(**1**) $\frac{4a+1}{2}$ (**2**) $a+1$ (**3**) $\frac{40-6a}{5}$ (**4**) $4-14a$ (**5**) Simplified.
(**6**) Simplified. (**7**) $2+a$ (**8**) $2+2a$ (**9**) $1+2a$ (**10**) $1+b$
(**11**) $1+b$ (**12**) Simplified. (**13**) $9+3b$ (**14**) $27-3b$

Answers to Exercise 4.11

(**1**) $\frac{-7}{2}$ (**2**) -3 (**3**) -3 (**4**) $\frac{-7}{2}$ (**5**) $\frac{-7}{2}$ (**6**) -2 (**7**) $\frac{-11}{3}$ (**8**) $\frac{-7}{2}$ (**9**) 18 (**10**) $\frac{-83}{26}$ (**11**) $\frac{2}{21}$ (**12**) $\frac{-4}{3}$ (**13**) $\frac{-3}{29}$ (**14**) $\frac{-4}{3}$ (**15**) $\frac{23}{18}$ (**16**) 3

Answers to Exercise 4.12

(**1**) $\frac{11}{3}$ (**2**) -3 (**3**) $\frac{5}{2}$ (**4**) $\frac{-7}{2}$ (**5**) -3 (**6**) $\frac{-11}{3}$ (**7**) $\frac{-7}{2}$ (**8**) 3
(**9**) $\frac{24}{17}$ (**10**) $\frac{-4}{7}$ (**11**) $\frac{-16}{17}$ (**12**) $\frac{-34}{7}$ (**13**) 12 (**14**) $\frac{-5}{2}$ (**15**) $\frac{-24}{37}$
(**16**) $\frac{-13}{87}$

Answers to Exercise 4.13

(**1**) $\frac{-2}{17}$ (**2**) $\frac{-8}{27}$ (**3**) $\frac{2}{13}$ (**4**) $\frac{14}{3}$ (**5**) $\frac{23}{15}$ (**6**) $\frac{-16}{15}$ (**7**) $\frac{-8}{3}$ (**8**) $\frac{45}{17}$
(**9**) $\frac{-16}{29}$ (**10**) $\frac{-32}{27}$ (**11**) -37 (**12**) $\frac{-5}{7}$ (**13**) $\frac{-95}{218}$ (**14**) $\frac{-29}{13}$
(**15**) $\frac{-49}{54}$ (**16**) $\frac{-19}{80}$ (**17**) $\frac{13}{23}$ (**18**) $\frac{-5}{6}$ (**19**) $\frac{3}{2}$ (**20**) $\frac{-7}{6}$ (**21**) $\frac{1}{44}$

Answers to Exercise 4.14

(**1**) -1 (**2**) -1 (**3**) -1 (**4**) -4 (**5**) $\frac{7}{3}$ (**6**) -4 (**7**) 9

Answers to Exercise 4.15

[Part 1]
(**1**) The equation $\frac{3}{7}x+2=0$ as written does not appear to be a linear equation, because the coefficient of x, $\frac{3}{7}$ is not an integer.
(**2**)

$$\begin{aligned}\frac{2}{5}x+\frac{1}{2} &= 3x-6\\ 10\left(\frac{2}{5}x+\frac{1}{2}\right) &= 10(3x-6)\\ 4x+5 &= 30x-60\\ -24x+65 &= 0\end{aligned}$$

which satisfies the requirements of definition (4.2) and is therefore a linear equation.

(3)

$$\begin{aligned} 3(2x-7)+5=2(3x-11)+12 &\Longleftrightarrow 6x-16=6x-10 \\ &\Longleftrightarrow 0=6 \end{aligned}$$

(a) Apparently the equation $3(2x-7)+5=2(3x-11)+12$ is equivalent to the statement "$0=6$". Since the statement "$0=6$" is always false, the equation $3(2x-7)+5=2(3x-11)+12$ is likewise always false. We conclude that there is no value of x that makes $3(2x-7)+5=2(3x-11)+12$ true. We say that $3(2x-7)+5=2(3x-11)+12$ has no solution.

(b) Theorem (4.1) says that every linear equation has a solution. This result appears to contradict that theorem. Fear not. The principles of mathematics have note gone "Bonkers", because the equation "$3(2x-7)+5=2(3x-11)+12$" is *not* a linear equation. It is equivalent to "$0=6$" which is not of the form $ax+b=0,$ where $a\neq 0$.

(4)

$$\begin{aligned} 3(2x-7)-9=2(3x-11)-8 &\Longleftrightarrow 6x-30=6x-30 \\ &\Longleftrightarrow 0=0 \end{aligned}$$

(a) Apparently the equation $3(2x-7)-9=2(3x-11)-8$ is equivalent to the statement "$0=0$" which is always true. This means that every number when substituted for x makes $3(2x-7)-9=2(3x-11)-8$ true.

(b) This appears to contradict the portion of theorem (4.1) which says the solution of a linear equation is *unique*. But the equation "$3(2x-7)-9=2(3x-11)-8$" is not a linear equation, because it is equivalent to "$0=0$". And, "$0=0$" is not of the form $ax+b=0,$ where $a\neq 0$

(5) Many possible answers.
(6) Proof of Theorem (4.1).

Proof. Suppose an equation is a linear equation. Then, by definition, it can be written in the form $ax+b=0$, where a,b integers with $a\neq 0$.

Since every integer has an inverse under addition, there exists a number $-b$ that we add to both sides.

$$ax+b=0 \Longleftrightarrow ax=-b.$$

Since every integer, by virtue of being a rational number, has an inverse under multiplication, there exists a number $\frac{1}{a}$ (remember, the equation is linear, so $a \neq 0$) by which we multiply both sides.

$$\iff x = \frac{-b}{a}.$$

The number $\frac{-b}{a}$ exists, because the rational numbers are closed under division. Therefore, every linear equation has a solution in the rational numbers. ■

(**7**) Proof of Theorem (4.1).

Proof. Suppose a, b integers, where $a \neq 0$. Then the $ax + b = 0$ is a linear equation. Further suppose that there are two different solutions to this equation. Call them x_1 and x_2 where $x_1 \neq x_2$. This implies the following two following equations.

$$ax_1 + b = 0 \tag{4.22}$$

$$ax_2 + b = 0. \tag{4.23}$$

Since equality is transitive,

$$ax_1 + b = ax_2 + b$$

Subtract b from both sides and divide by a which is not 0. Then,

$$x_1 = x_2.$$

This contradicts the supposition that $x_1 \neq x_2$. Therefore, every linear equation has a unique solution in the rational numbers. ■

(**8**) Every rational number except 0 has a unique inverse under multiplication.

Proof. Every rational number other than 0 has an inverse under multiplication. We need only show that no rational number has two or more inverses under multiplication. Suppose that a rational number $a \neq 0$ has two distinct inverses for multiplication. Call the two distinct inverses b and c. This means that $ab = 1$ and $ac = 1$. So, $ab = ac$. But this implies $b = c$. That contradicts the supposition that b and c are distinct. Therefore, every rational number other than 0 has a distinct inverse under multiplication. ■

[Part 2] (**1**) $x = 6$ (**2**) $x = 4$ (**3**) $x = 5$ (**4**) $x = 5$ (**5**) $x = 2$

Answers to Exercise 5.1

(**1**)
Let $x =$ amount Joe earned (\$).
Then,
$x+10 =$ amount Josh earned,
$x+x+10 = 1200.$
$x = 595.$
$\therefore$ Joe earned \$595 and Josh earned \$605.

(**2**)
Let $x =$ the capacity of one tank (gallons).
Then,
$x-10 =$ the capacity of the other tank,
$x+x-10 = 152.$
$x = 81.$
$\therefore$ one tank holds 71 gallons and the other holds 81 gallons.

(**3**)
Let $x =$ the the width of the rectangle (inches).
Then,
$x+10 =$ the length of the rectangle,
$2x+2(x+10)$ = perimeter,
$x+(x+10) =$ half of perimeter,
$x+(x+10) = 152.$
$x = 71.$
$\therefore$ The width of the rectangle is 71 inches and the length is 81 inches.

(**4**)
Let $x =$the number of dimes.
Then,
$2x =$ the number of quarters,
$4x =$ the number of nickels,
$10x =$ the value of the dimes (cents),
$50x =$ the value of the quarters (cents),
$20x =$ the value of the nickels (cents),
$10x+50x+20x = 240.$
$x+5x+2x = 24.$
$x = 3.$
$\therefore$ She had 3 dimes, 6 quarters, and 12 nickels.

(**5**)
Let $x =$length of side of original square paper (inches).
Then,

$x-2=$ the width of rectangular piece,
$x+3=$ the length of rectangular piece,
$2(x-2)+2(x+3)=54.$
$x=13.$

$\therefore$ The original paper was 13 inches by 13 inches.

(**6**)
Let $x=$ the number of Linda's nickels.
Then,

$x=$ the number of Betsy's pennies (cents),
$5x=$ the value of Linda's nickels (cents),
$x=$ the value of Betsy's pennies.
$x+5x=210.$
$x=35.$

$\therefore$ Betsy has \$0.35 and Linda has \$1.75.

(**7**)
Let $x=$ Dick's age now (years).
Then,

$3x=$ John's age now,
$x-3=$ Dick's age three years ago,
$3x-3=$ John's age three years ago.
$x-3+(3x-3)=22.$
$x=7.$

$\therefore$ Dick is 7 years old now. John is 21 years old now.

Answers to Exercise 5.2

(**1**) The area of the rectangle is 56 square inches. (**2**) Jim had 11 of each denomination of coin. (**3**) There are 23 U.S. stamps. (**4**) Bob had 16 quarters and Alice had 288 nickels. (**5**) Cindy is 9 years old. (**6**) Russel used 192 coins. (**7**) The tanks contain the same volume of water after $3\frac{1}{2}$ minutes. (**8**) Abigail should give \$7 to Barbara. (**9**) He used 2480 gallons of fuel in June. (**10**) The veterinarian charged \$144 for the first lame horse. (**11**) Each person originally had \$90. (**12**) In January, Ace had \$16 and Sam had \$8. (**13**) The school has 90 teachers. (**14**) Alice had 120 marbles to start with. (**15**) He had \$100 at first.

Answers to Exercise 5.3

(**1**) $7h$. (**2**) 2 minute. (**3**) $5h$. (**4**) 6 hours. (**5**) 1.5 hours. (**6**) 4.8 minutes.

Answers to Exercise 5.4

(**1**) $120\frac{m}{h}$ (**2**) $72\frac{km}{h}$ (**3**) $1320\frac{m}{min}$ (**4**) $818\frac{m}{h}$ (**5**) $375\frac{m}{h}$

Answers to Exercise 5.5

(**1**) 260 miles. (**2**) $3\frac{1}{2}$h or 3h 30m. (**3**) 6.5 miles. (**4**) $5\frac{1}{3}$h or 5 h 20 m (**5**) 26 km/h. (**6**) 72 (**7**) 64 mph. (**8**) 40 mph. (**9**) 30 mph. (**10**) $2\frac{1}{2}$ hours. (**11**) 20. (**12**) 12:09 PM. (**13**) 1:15 PM. (**14**) 1320 miles.

Answers to Exercise 5.6

(**1**) 6 (**2**) 3 inches (**3**) 151, 152, 153 (**4**) 17 (**5**) 100 (**6**) \$200000 (**7**) 4:00 PM (**8**) $3\frac{1}{5}$ hours or 3h 12m (**9**) 136.5 miles (**10**) 1:06 PM

Answers to Exercise 6.1

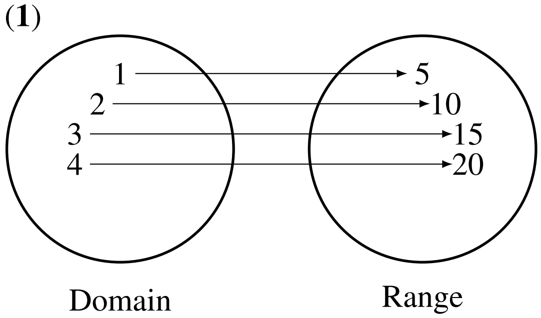
(**1**) $(-4,-12),(-2,-8),(0,0),(1,-4),(\frac{3}{8},\frac{3}{2}),(5,20)$.
(**2**) $(0,3),(1,4),(2,5),(3,15),(\frac{10}{3},\frac{19}{3})$. (**3**) $(\frac{-1}{3},-2),(-1,-3),(0,0),(1,3),(2,6)$.
(**4**) No, because $3\cdot 2 = 6 \neq 9$. (**5**) No, because $-1-2=-3 \neq 5$. (**6**) $y=5x$. (**7**) $y=x+4$. (**8**) $t=-5s$. (**9**) To see that $(2,6)$ and $(6,2)$ are not the same, note that $(2,6)$ is a solution of $y=3x$ but $(6,2)$ is not.

Answers to Exercise 6.2

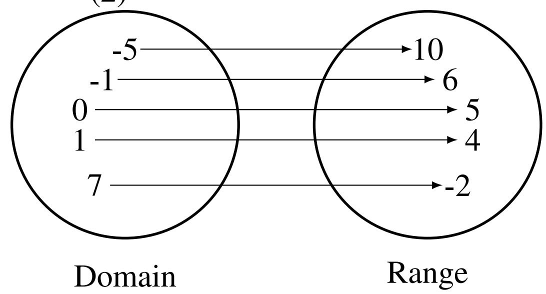
(**1**) $\{-5,-1,3,6\}$ (**2**) $\{-30,-10,0,20,90\}$ (**3**) $\{5,\frac{16}{3},\frac{11}{2},6,8\}$
(**4**) $\{-5,-2,1,4,7\}$ (**5**) $\{10,13,16,19\}$ (**6**) $\{22,25,28,31,34\}$
(**7**) Domain $=\{1,2,3,5,12\}$. Range $=\{\frac{2}{3},\frac{3}{4},7,11,27\}$.

Answers to Exercise 6.3

(**1**)



(**2**)



(**3**) $2 \longrightarrow 7, 3 \longrightarrow 10, 5 \longrightarrow 16$.
(**4**) $-7 \longrightarrow -11, -2 \longrightarrow -1, -1 \longrightarrow 1, 0 \longrightarrow 3, 1 \longrightarrow 5, 3 \longrightarrow 9$.
(**5**) $y = 10x$, range $= \{10, 20, 30, 40, 50\}$.

Answers to Exercise 6.4

(**1**) $d = 60t$. (**2**) (**a**) $V = 2 + 3t$.

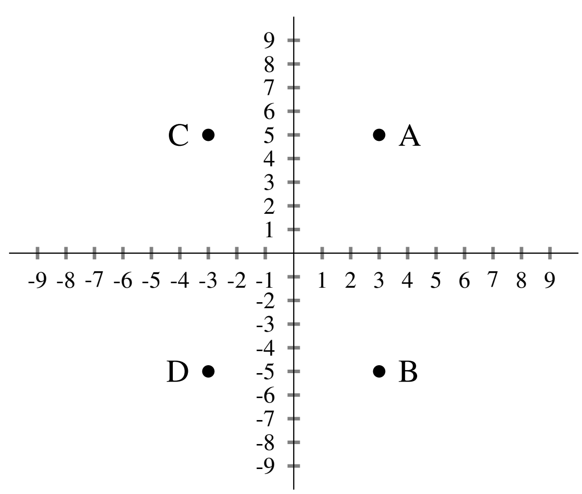
(**b**)

t (min)	0	2	4	6	8	10	12
V (gal)	2	8	14	20	26	32	38

(**3**) (**a**) $A = 7t$. (**b**) 49. (**4**) (**a**) $t = \frac{0.75-d}{4}$. (**b**) All numbers between 0 and 0.75. (**c**) All numbers between 0 and 0.1875.
(**5**) (**a**) $y = 5x$. (**b**) $\{0, 1, 2, 3, 4\}$. (**c**) $\{0, 5, 10, 15, 20\}$. (**d**) 350.

Answers to Exercise 6.5

(**1**) (**a**) Done as example. (**b**) $(3, 6)$. (**c**) $(-3, 2)$. (**d**) $(-6, -6)$. (**e**) $(5, -3)$.
(**2**) (**a**) II. (**b**) I. (**c**) II. (**d**) III. (**e**) IV.
(**3**)



(**4**) Answer should look the same as Figure (6.4) page 162.

Answers to Exercise 6.6

[Part 1] (**1**) $\frac{3}{4}$ (**2**) $\frac{-2}{6} = \frac{-1}{3}$ (**3**) $\frac{2}{8} = \frac{1}{4}$ (**4**) $\frac{8}{4} = 2$ (**5**) $\frac{-4}{-2} = 2$
(**6**) $\frac{9}{2}$ (**7**) $\frac{-15}{-4}$ (**8**) $\frac{4}{-3}$

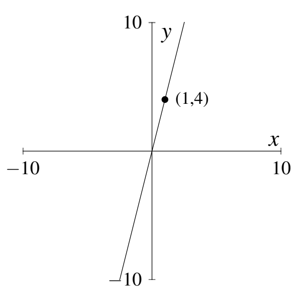
[Part 2]

(**1**)

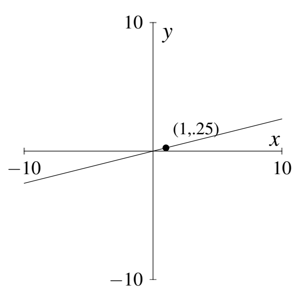
	0	**A**	**B**	**C**	**D**
0	undefined	$\frac{2}{1} = 2$	$\frac{4}{2} = 2$	$\frac{6}{3} = 2$	$\frac{8}{4} = 2$
A		undefined	$\frac{2}{1} = 2$	$\frac{4}{2} = 2$	$\frac{6}{3} = 2$
B			undefined	$\frac{2}{1} = 2$	$\frac{4}{2} = 2$
C				undefined	$\frac{2}{1} = 2$
D					undefined

(**2**) The ratio $\frac{\text{rise}}{\text{run}}$ equals 2 whenever it is defined. (**3**) The run is 0, so the ratio $\frac{\text{rise}}{\text{run}}$ is undefined because division by zero is undefined.

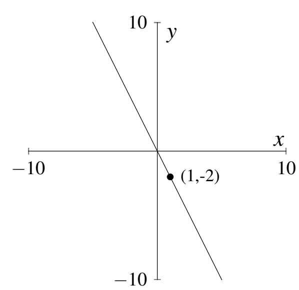
Answers to Exercise 6.7



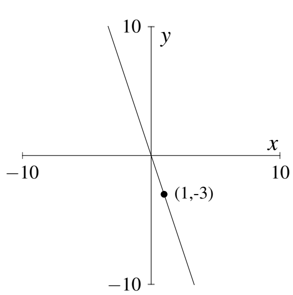
(**1**)



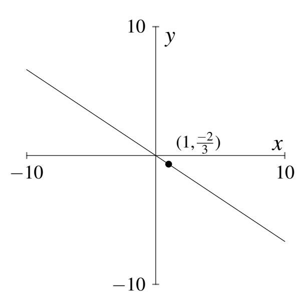
(**2**)



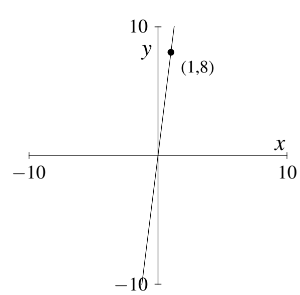
(**3**)



(**4**)



(**5**)



(**6**)

Answers to Exercise 6.8

(**1**) (a) $y = 2x$. (b) $y = \dfrac{-3}{2}x$. (c) $y = \dfrac{1}{2}x$. (d) $y = \dfrac{-1}{2}x$.

Answers to Exercise 6.9

[Part 1] (**1**) $\Delta y = 2, \Delta x = 5, \frac{\Delta y}{\Delta x} = \frac{2}{5}$, increasing (**2**) $\Delta z = 3, \Delta w = 11, \frac{\Delta z}{\Delta w} = \frac{3}{11}$, increasing (**3**) $\Delta t = -1, \Delta s = 4, \frac{\Delta t}{\Delta s} = \frac{-1}{4}$, decreasing
(**4**) $\Delta y = 3, \Delta x = 1, \frac{\Delta y}{\Delta x} = \frac{3}{1} = 3$, increasing (**5**) $\Delta y = 5, \Delta x = 2, \frac{\Delta y}{\Delta x} = \frac{5}{2}$, increasing (**6**) $\Delta t = -9, \Delta s = 1, \frac{\Delta t}{\Delta s} = \frac{-9}{1} = 9$, decreasing
[Part 2] (**1**) $\Delta y = 5, \Delta x = 2, \frac{\Delta y}{\Delta x} = \frac{5}{2}$, increasing (**2**) $\Delta t = 3, \Delta s = 1, \frac{\Delta t}{\Delta s} = \frac{3}{1} = 3$, increasing (**3**) $\Delta y = -4, \Delta x = 2, \frac{\Delta y}{\Delta x} = \frac{-4}{2} = -2$, decreasing
(**4**) $\Delta y = -3, \Delta x = 2, \frac{\Delta y}{\Delta x} = \frac{-3}{2}$, decreasing

Answers to Exercise 6.10

(**1**) $23 \frac{\text{L}}{\text{min}}$ (**2**) 5 feet (**3**) 1200 feet (**4**) 30 L (**5**) $\frac{3}{2}b$

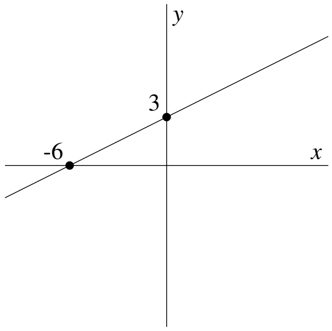
Answers to Exercise 6.11

(**1**) $d = 40t$, domain is all numbers from 0 to 3. (**2**) $T = 5t$, domain is all numbers from 0 to 20. (**3**) (**a**) 3PM-4PM (**b**) 2PM (**c**) 1 hr.

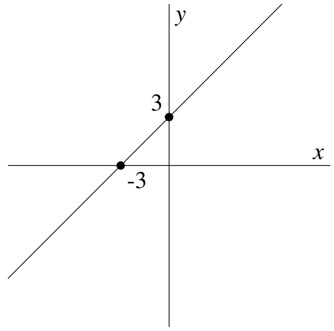
Answers to Exercise 6.12

(**1**) slope $= 3, y$ intercept $= 7, x$ intercept $= \frac{-7}{3}$
(**2**) slope $= 5, t$ intercept $= \frac{1}{2}, s$ intercept $= \frac{-1}{10}$
(**3**) slope $= \frac{5}{8}, t$ intercept $= \frac{3}{2}, s$ intercept $= \frac{-12}{5}$
(**4**) slope $= \frac{-5}{9}, y$ intercept $= 6, x$ intercept $= \frac{54}{5}$
(**5**) slope $= \frac{-3}{4}, y$ intercept $= -1, x$ intercept $= \frac{-4}{3}$
(**6**) slope $= \frac{-1}{2}, y$ intercept $= 3, x$ intercept $= 6$
(**7**) slope $= 3, y$ intercept $= 2, x$ intercept $= \frac{-2}{3}$
(**8**) slope $= 3, y$ intercept $= \frac{-17}{2}, x$ intercept $= \frac{17}{6}$
(**9**) slope $= \frac{-1}{5}, y$ intercept $= 1, x$ intercept $= 5$
(**10**) slope $= \frac{-2}{3}, y$ intercept $= 2, x$ intercept $= 3$

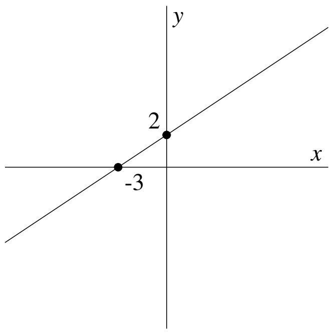
Answers to Exercise 6.13



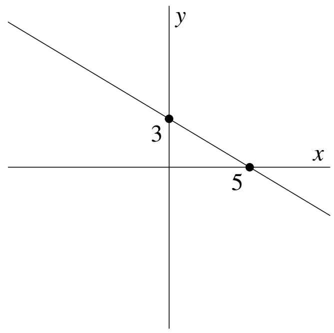
(**1**)



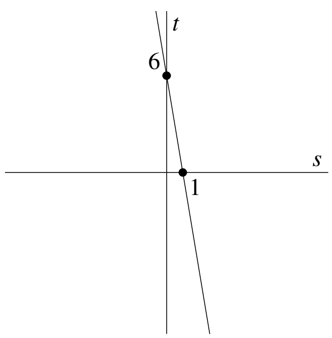
(**2**)



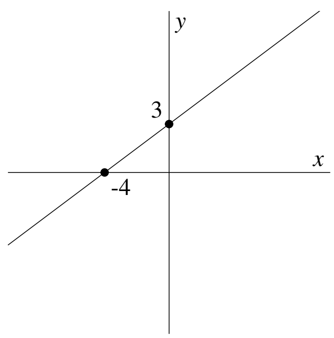
(**3**)



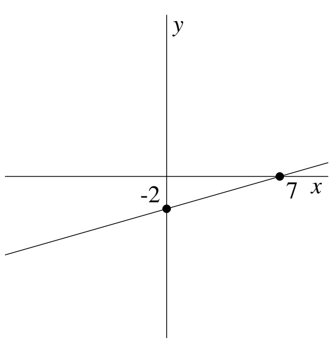
(**4**)



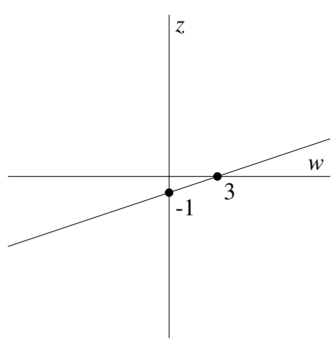
(**5**)



(**6**)



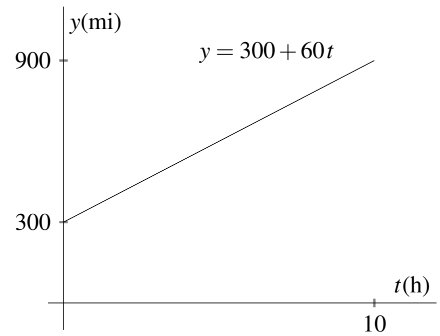
(**7**)



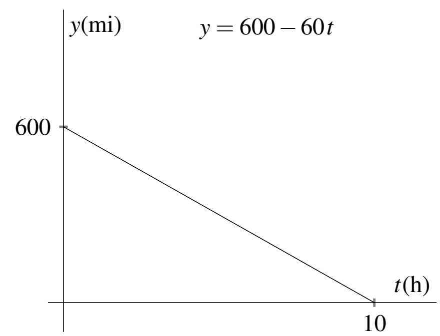
(**8**)

Answers to Exercise 6.14

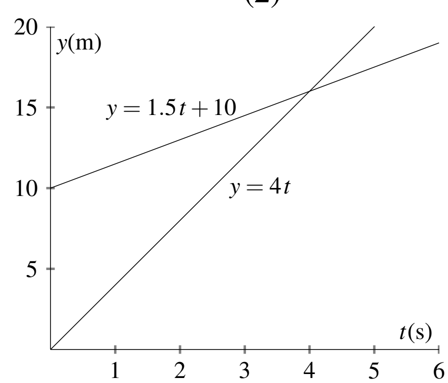
(**1**) (**a**) $y = 300 + 60t$, Domain = all numbers from 0 to 10. Increasing.



(**b**) $y = 600 - 60t$, Domain = all numbers from 0 to 10. Decreasing.



(**2**)



It appears that Barb catches Alex at about 4 seconds.

(**3**) Barb catches Alex at time 4 seconds.

(**4**) True.

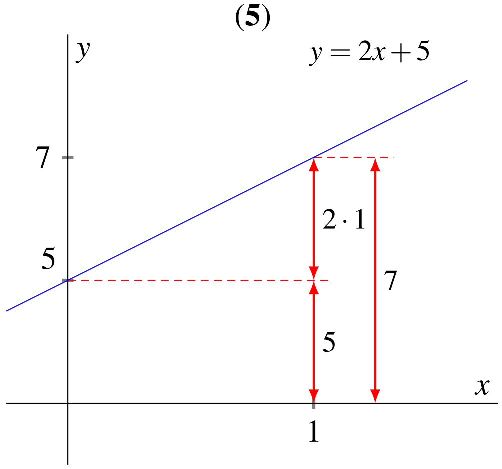
Proof. Let $y = 2x + 5$. Suppose that x changes by an amount Δx and y by an amount Δy. Then

$$\begin{aligned} y + \Delta y &= 2(x + \Delta x) + 5 \\ y + \Delta y &= 2x + 2\Delta x + 5 \\ y + \Delta y &= y + 2\Delta x \\ \Delta y &= 2\Delta x \end{aligned}$$

Therefore, the change in y is 2 times the change in x.

■

(**5**)



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List of Symbols