1.1 Divisibility

Definition 1.1. (Divisible)
A is divisible by k if there exists an integer a, $a \in \mathbb{Z}$, such that $A = ka$. □

The notation $k \mid A$ says that A is divisible by k. Synonyms for "A is divisible by k" are

- k divides A
- "A is a multiple of k"
- "k is a factor of A"

Theorem 1.1. *Suppose A is divisible by k. Then B is divisible by k* $\iff$ $(A+B)$ *is divisible by k.*

Proof.
$\Longrightarrow$
Since A is divisible by k, $A = ka$ for some integer a. We are assuming that B is divisible by k, so $B = kb$ for some integer b. Then $(A+B) = (ka+bk) = k(a+b)$. And this means that the sum $A+B$ is divisible by k.

$\Longleftarrow$
Suppose that $(A+B)$ is divisible by k. Then $A+B = kc, c \in \mathbb{Z}$. Then

$$\begin{aligned} B &= (A+B) - A && (1.1)\\ &= kc - ka \qquad \text{(WHY?)} && (1.2)\\ &= k(c-a), && (1.3)\\ &= kp, \quad p \in \mathbb{Z} \qquad \text{(WHY?)} && (1.4)\\ & && (1.5) \end{aligned}$$

and this means B is divisible by k. □

Exercise 1

Prove each of the following.

1. A number is divisible by 3 if and only if the sum of the digits is divisible by 3.
2. A number is divisible by 9 if and only if the sum of the digits is divisible by 9.
3. A number is divisible by 11 if and only if the number formed by

   (sum of odd position digits) – (sum of even position digits)

   is a multiple (positive, negative, or zero) of 11.

1.2 Prime numbers

Definition 1.2. (Prime number) A *prime number* is a natural number greater than 1 whose only factors are one and itself. A natural number that is not a prime number is called a *composite number*. The numbers 0 and 1 are neither prime nor composite numbers. □

1.2.1 Finding prime numbers

It is a fact that no formula has been found that will produce all and only prime numbers. There is, however, a systematic way to produce a list of prime numbers. It is called the *Sieve of Eratosthenes*. Here is how it works.