## Exercises 2.5



Use the **half-angle** identities developed in this section to evaluate the following expressions.

1. $\sin \frac{\pi}{12}$
2. $\sin \frac{\pi}{8}$
3. $\cos \frac{\pi}{8}$
4. $\sin \frac{\pi}{24}$
5. $\cos \frac{\pi}{24}$
6. $\cos \frac{\pi}{16}$
7. $\sin \frac{\pi}{16}$
8. $\tan \frac{\pi}{8}$
9. $\tan \frac{\pi}{12}$
10. $\tan \frac{\pi}{16}$
11. $\tan \frac{5\pi}{12}$
12. $\sin \frac{3\pi}{8}$
13. $\cos \frac{3\pi}{8}$
14. $\cos \frac{7\pi}{8}$
15. $\sin \frac{7\pi}{8}$
16. $\sin \frac{11\pi}{12}$
17. $\cos \frac{11\pi}{12}$
18. $\tan \frac{3\pi}{8}$
19. $\tan \frac{7\pi}{8}$
20. $\tan \frac{11\pi}{12}$

Verify that each of the following equations is an identity.

21. $\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \equiv \frac{1}{2} \sin \alpha.$
22. $\cos^2 \frac{\theta}{4} - \cos \frac{\theta}{2} \equiv \sin^2 \frac{\theta}{4}.$
23. $\tan \frac{\beta}{2} \equiv \frac{1 - \cos \beta}{\sin \beta}.$
24. $\tan \frac{\gamma}{2} \equiv \frac{\sin \gamma}{1 + \cos \gamma}.$
25. $\cot \frac{x}{2} \equiv \frac{1 + \cos x}{\sin x}.$
26. $\cot \frac{z}{2} \equiv \frac{\sin z}{1 - \cos z}.$
27. $\tan \frac{y}{2} \equiv \csc y - \cot y.$
28. $\sec^2 \frac{\phi}{2} \equiv \frac{2}{1 + \cos \phi}.$
29. $\sin \alpha \cot \frac{\alpha}{2} \equiv 2 \cos^2 \frac{\alpha}{2}.$
30. $\left(\sin \frac{y}{2} + \cos \frac{y}{2}\right)^2 \equiv 1 + \sin y.$
31. $\tan^2 \frac{\beta}{2} + 1 \equiv 2 \csc \beta \tan \frac{\beta}{2}.$
32. $\csc^2 \frac{x}{2} \equiv \frac{2}{1 - \cos x}.$
33. $\tan \frac{\phi}{2} \sin \phi \equiv 2 \sin^2 \frac{\phi}{2}.$
34. $\dfrac{1 + \tan \frac{z}{2}}{1 - \tan \frac{z}{2}} \equiv \sec z + \tan z.$
35. $2 \tan \frac{\alpha}{2} \csc \alpha \equiv \sec^2 \frac{\alpha}{2}.$
36. $\cot y + \csc y \equiv \cot \frac{y}{2}.$
37. $\csc (\alpha + \beta) - \cot (\alpha + \beta) \equiv \tan \left(\frac{\alpha + \beta}{2}\right).$
38. $2 \sin^2 \frac{y}{6} - \sin^2 \frac{y}{7} \equiv \cos^2 \frac{y}{7} - \cos \frac{y}{3}.$
39. $\cos^2 \frac{x}{18} \equiv \sin^2 \frac{x}{18} + \cos \frac{x}{9}.$
40. $\dfrac{1 - \tan \frac{\alpha}{2}}{1 + \tan \frac{\alpha}{2}} \equiv \frac{\cos \alpha}{1 + \sin \alpha}.$

## Exercises 2.5

1. $\dfrac{\sqrt{2-\sqrt{3}}}{2}$

3. $\dfrac{\sqrt{2+\sqrt{2}}}{2}$

5. $\dfrac{\sqrt{2+\sqrt{2+\sqrt{3}}}}{2}$

7. $\dfrac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$

9. $\sqrt{\dfrac{2-\sqrt{3}}{2+\sqrt{3}}}$ or $\sqrt{7-4\sqrt{3}}$

11. $\sqrt{\dfrac{2+\sqrt{3}}{2-\sqrt{3}}}$ or $\sqrt{7+4\sqrt{3}}$

13. $\dfrac{\sqrt{2-\sqrt{2}}}{2}$

15. $\dfrac{\sqrt{2-\sqrt{2}}}{2}$

17. $\dfrac{\sqrt{2-\sqrt{3}}}{2}$

19. $\sqrt{\dfrac{2-\sqrt{2}}{2+\sqrt{2}}}$ or $\sqrt{3-2\sqrt{2}}$

23. $$\begin{aligned}\tan^2\frac{\beta}{2} &= \frac{1-\cos\beta}{1+\cos\beta}\\ &= \frac{(1-\cos\beta)(1-\cos\beta)}{(1+\cos\beta)(1-\cos\beta)}\\ &= \frac{(1-\cos\beta)^2}{1-\cos^2\beta}\\ &= \frac{(1-\cos\beta)^2}{\sin^2\beta}\\ &= \left(\frac{1-\cos\beta}{\sin\beta}\right)^2\\ \tan\frac{\beta}{2} &= \pm\frac{1-\cos\beta}{\sin\beta}.\end{aligned}$$

if $\tan\frac{\beta}{2} > 0$, $\frac{\beta}{2}$ is in the 1st or 3rd quadrant. If $\frac{\beta}{2}$ is in the 1st quadrant, $\frac{\beta}{2}$ is between $2n\pi$ and $2n\pi + \frac{\pi}{2}$ for some $n \in J$; $\beta$ is between $4n\pi$ and $4n\pi + \pi$, which means $\sin\beta > 0$, and $\frac{1-\cos\beta}{\sin\beta} > 0$.

If $\frac{\beta}{2}$ is in the 3rd quadrant, $\frac{\beta}{2}$ is between $2n\pi + \pi$ and $2n\pi + \frac{3\pi}{2}$ for some $n \in J$; $\beta$ is between $4n\pi + 2\pi$ and $4n\pi + 3\pi$, which means $\sin\beta > 0$ and $\frac{1-\cos\beta}{\sin\beta} > 0$.

If $\tan\frac{\beta}{2} < 0$, $\frac{\beta}{2}$ is in the 2nd quadrant or the 4th quadrant. Show that in either case, $\sin\beta < 0$, so $\frac{1-\cos\beta}{\sin\beta} < 0$.

27. $$\begin{aligned}\tan\frac{y}{2} &= \frac{1-\cos y}{\sin y} \quad \text{(from No. 23)}\\ &= \frac{1}{\sin y} - \frac{\cos y}{\sin y}\\ &= \csc y - \cot y.\end{aligned}$$

34. $$\begin{aligned}\frac{1+\tan\frac{z}{2}}{1-\tan\frac{z}{2}} &= \frac{1+\dfrac{\sin z}{1+\cos z}}{1-\dfrac{\sin z}{1+\cos z}}\\ &= \frac{\left(\dfrac{1+\cos z+\sin z}{1+\cos z}\right)}{\dfrac{1+\cos z-\sin z}{1+\cos z}}\\ &= \frac{1+\cos z+\sin z}{(1+\cos z)-\sin z}\\ &= \frac{1+\cos z+\sin z}{(1+\cos z)-\sin z}\cdot\frac{1+\cos z+\sin z}{1+\cos z+\sin z}\\ &= \frac{1+\cos^2 z+\sin^2 z+2\cos z+2\sin z+2\sin z\cos z}{1+2\cos z+\cos^2 z-\sin^2 z}\\ &= \frac{2+2\cos z+2\sin z+2\sin z\cos z}{1+2\cos z+\cos^2 z-(1-\cos^2 z)}\\ &= \frac{2[1+\cos z+\sin z+\sin z\cos z]}{2\cos z+2\cos^2 z}\\ &= \frac{2(1+\cos z)(1+\sin z)}{2\cos z(1+\cos z)}\\ &= \frac{1+\sin z}{\cos z}\\ &= \frac{1}{\cos z}+\frac{\sin z}{\cos z} = \sec z+\tan z.\end{aligned}$$

38. $$\begin{aligned}2\sin^2\frac{y}{6} - \sin^2\frac{y}{7} &= 2\left(\frac{1-\cos\frac{y}{3}}{2}\right) - \left(1-\cos^2\frac{y}{7}\right)\\ &= 1-\cos\frac{y}{3} - 1 + \cos^2\frac{y}{7}\\ &= \cos^2\frac{y}{7} - \cos\frac{y}{3}\end{aligned}$$