Exercises 2.4

Using $\cos \pi/6 = \sqrt{3}/2$, $\sin \pi/6 = 1/2$, and $\tan \pi/6 = 1/\sqrt{3}$ and the double-angle formulas, evaluate the following.

1. $\sin \frac{\pi}{3}$
2. $\cos \frac{\pi}{3}$
3. $\tan \frac{\pi}{3}$

Using $\cos \pi/4 = \sqrt{2}/2 = \sin \pi/4$ and the double-angle formulas, evaluate the following.

4. $\sin \frac{\pi}{2}$
5. $\cos \frac{\pi}{2}$

Using $\sin \pi/2 = 1$ and $\cos \pi/2 = 0$, evaluate the following.

6. $\sin \pi$
7. $\cos \pi$

Verify that each of the following equations is an identity.

8. $\cos 3\theta \equiv 4 \cos^3 \theta - 3 \cos \theta.$
9. $\sin 4\phi \equiv 4 \cos \phi (\sin \phi - 2 \sin^3 \phi).$
10. $\cos 4\alpha \equiv 8 \cos^4 \alpha - 8 \cos^2 \alpha + 1.$
11. $\sin 5x \equiv 16 \sin^5 x - 20 \sin^3 x + 5 \sin x.$
12. $\cos 5y \equiv 16 \cos^5 y - 20 \cos^3 y + 5 \cos y.$
13. $\sin \frac{x}{2} \cos \frac{x}{2} \equiv \frac{1}{2} \sin x.$ $\left[\textit{Hint:} \quad x = 2\left(\frac{x}{2}\right)\right]$
14. $\cos^2 \frac{z}{2} - \sin^2 \frac{z}{2} \equiv \cos z.$
15. $\left(\cos \frac{\beta}{2} - \sin \frac{\beta}{2}\right)^2 \equiv 1 - \sin \beta.$
16. $\sin 2\alpha \equiv \frac{2 \tan \alpha}{1 + \tan^2 \alpha}.$
17. $\tan 3\phi \equiv \frac{3 \tan \phi - \tan^3 \phi}{1 - 3 \tan^2 \phi}.$
18. $\tan 2y \equiv \frac{2}{\cot y - \tan y}.$
19. $\cos 2x \equiv \cos^4 x - \sin^4 x.$
20. $1 - 2 \sin^2 \left(\frac{\pi}{4} - \theta\right) \equiv \sin 2\theta.$
21. $\cos 2z + 2 \sin^2 z \equiv 1.$
22. $\sin^2 \frac{\beta}{2} \equiv \frac{1 - \cos \beta}{2}.$
23. $\cos^2 2\gamma - 2 \cos^2 \gamma \equiv -1.$
24. $\sec 2\phi \equiv \frac{\sec^2 \phi}{1 - \tan^2 \phi}.$

25. $\sin\theta \sin 3\theta \equiv \sin^2 2\theta - \sin^2\theta.$

26. $\cot 2y \equiv \dfrac{\cot^2 y - 1}{2\cot y}.$

27. $\sec y \csc y \equiv 2\csc 2y.$

28. $\cos 4\alpha \equiv 1 - 8\sin^2\alpha\cos^2\alpha.$

29. $\cos 4z \equiv 8\sin^4 z - 8\sin^2 z + 1.$

30. $(\cos y - \sin y)^2 \equiv 1 - \sin 2y.$

31. $2\sin(\alpha + \beta)\cdot\cos(\alpha - \beta) \equiv \sin 2\alpha + \sin 2\beta.$

32. $\dfrac{\sin 3\phi}{\sin\phi} - \dfrac{\cos 3\phi}{\cos\phi} \equiv 2.$

33. $2\csc 2x \equiv \cot x + \tan x.$

34. $\cot\alpha - \cot 2\alpha \equiv \csc 2\alpha.$

35. $\csc^2 2y - \sec^2 2y \equiv 4\cot 4y \csc 4y.$

36. $\cos\theta\cos 3\theta \equiv \cos^2\theta - \sin^2 2\theta.$

37. $\tan\beta \equiv \dfrac{\sin 2\beta}{1 + \cos 2\beta}.$

38. $\cot^2\phi - \tan^2\phi \equiv \dfrac{4\cos 2\phi}{\sin^2 2\phi}.$

39. $4\sin^4\gamma \equiv 3 - 4\cos 2\gamma + \cos 4\gamma.$

40. $\dfrac{1 + \sin 2x + \cos 2x}{1 + \sin 2x - \cos 2x} \equiv \cot x.$

[1] $\sin\frac{\pi}{3} = \sin 2\cdot\frac{\pi}{6} = 2\sin\frac{\pi}{6}\cos\frac{\pi}{6} = 2\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}.$

[2] $\cos\frac{\pi}{3} = \cos 2\,\frac{\pi}{6} = \cos^2\frac{\pi}{6} - \sin^2\frac{\pi}{6} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}.$

[3] $\tan\frac{\pi}{3} = \tan 2\cdot 2\,\frac{\pi}{6} = \frac{2\tan\frac{\pi}{6}}{1-\tan^2\frac{\pi}{6}} = \frac{\frac{2}{\sqrt{3}}}{1-\frac{1}{3}} = \sqrt{3}.$

[4] $\sin\frac{\pi}{2} = \sin 2\cdot\frac{\pi}{4} = 2\sin\frac{\pi}{4}\cos\frac{\pi}{4} = 2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} = 1.$

[5] $\cos\frac{\pi}{2} = \cos 2\,\frac{\pi}{4} = \cos^2\frac{\pi}{4} - \sin^2\frac{\pi}{4} = \frac{2}{4} - \frac{2}{4} = 0.$

[6] $\sin\pi = \sin 2\cdot\frac{\pi}{2} = 2\sin\frac{\pi}{2}\cos\frac{\pi}{2} = 0.$

[7] $\cos\pi = \cos 2\,\frac{\pi}{2} = \cos^2\frac{\pi}{2} - \sin^2\frac{\pi}{2} = 0 - 1 = -1.$

11.
$$\begin{aligned}\sin 5x &= \sin(3x+2x)\\ &= \sin 3x\cos 2x + \cos 3x \sin 2x\\ &= (3\sin x - 4\sin^3 x)(1-2\sin^2 x) + (4\cos^3 x - 3\cos x)(2\sin x\cos x)\\ &= 3\sin x - 6\sin^3 x - 4\sin^3 x + 8\sin^5 x + 8\sin x\cos^4 x\\ &\quad - 6\sin x\cos^2 x\\ &= 3\sin x - 10\sin^3 x + 8\sin^5 x + 8\sin x(1-\sin^2 x)^2\\ &\quad - 6\sin x(1-\sin^2 x)\\ &= 3\sin x - 10\sin^3 x + 8\sin^5 x + 8\sin x(1 - 2\sin^2 x + \sin^4 x)\\ &\quad - 6\sin x + 6\sin^3 x\\ &= 16\sin^5 x - 20\sin^3 x + 5\sin x.\end{aligned}$$

16.
$$\begin{aligned}\frac{2\tan\alpha}{1+\tan^2\alpha} &= \frac{2\tan\alpha}{\sec^2\alpha}\\ &= \frac{2\left(\frac{\sin\alpha}{\cos\alpha}\right)}{\frac{1}{\cos^2\alpha}}\\ &= 2\left(\frac{\sin\alpha}{\cos\alpha}\right)\left(\cos^2\alpha\right)\\ &= 2\sin\alpha\cos\alpha\\ &= \sin 2\alpha.\end{aligned}$$

20.
$$\begin{aligned}\sin\left(\frac{\pi}{4}-\theta\right) &= \sin\frac{\pi}{4}\cos\theta - \cos\frac{\pi}{4}\sin\theta\\ &= \frac{\sqrt{2}}{2}(\cos\theta - \sin\theta),\end{aligned}$$

$$\begin{aligned}1 - 2\sin^2\left(\frac{\pi}{4}-\theta\right) &= 1 - 2\left(\frac{1}{2}\right)(\cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta)\\ &= 1 - [(\cos^2\theta + \sin^2\theta) - 2\sin\theta\cos\theta)]\\ &= 1 - [1 - \sin 2\theta]\\ &= \sin 2\theta.\end{aligned}$$

25.
$$\begin{aligned}\sin\theta\sin 3\theta &= \sin\theta(3\sin\theta - 4\sin^3\theta)\\ &= 3\sin^2\theta - 4\sin^4\theta\\ &= 4\sin^2\theta - \sin^2\theta - 4\sin^4\theta\\ &= 4\sin^2\theta(1-\sin^2\theta) - \sin^2\theta\\ &= 4\sin^2\theta\cos^2\theta - \sin^2\theta\\ &= (2\sin\theta\cos\theta)^2 - \sin^2\theta\\ &= \sin^2 2\theta - \sin^2\theta.\end{aligned}$$

33.
$$\begin{aligned}\cot x + \tan x &= \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}\\ &= \frac{(\cos x)(\cos x) + (\sin x)(\sin x)}{\sin x\cos x}\\ &= \frac{\sin^2 x + \cos^2 x}{\sin x\cos x}\\ &= \frac{1}{\sin x\cos x}\\ &= \frac{2}{2\sin x\cos x}\\ &= \frac{2}{\sin 2x}\\ &= 2\csc 2x.\end{aligned}$$

40.
$$\begin{aligned}\frac{1+\sin 2x + \cos 2x}{1 + \sin 2x - \cos 2x} &= \frac{1 + 2\sin x\cos x + (2\cos^2 x - 1)}{1 + 2\sin x\cos x - (1 - 2\sin^2 x)}\\ &= \frac{2\sin x\cos x + 2\cos^2 x}{2\sin x\cos x + 2\sin^2 x}\\ &= \frac{2\cos x(\sin x + \cos x)}{2\sin x(\cos x + \sin x)}\\ &= \frac{\cos x}{\sin x} = \cot x.\end{aligned}$$

33. $\cot x + \tan x = \dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x}$
$= \dfrac{(\cos x)(\cos x) + (\sin x)(\sin x)}{\sin x \cos x}$
$= \dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x}$
$= \dfrac{1}{\sin x \cos x}$
$= \dfrac{2}{2 \sin x \cos x}$
$= \dfrac{2}{\sin 2x}$
$= 2 \csc 2x.$

40. $\dfrac{1 + \sin 2x + \cos 2x}{1 + \sin 2x - \cos 2x} = \dfrac{1 + 2 \sin x \cos x + (2 \cos^2 x - 1)}{1 + 2 \sin x \cos x - (1 - 2 \sin^2 x)}$
$= \dfrac{2 \sin x \cos x + 2 \cos^2 x}{2 \sin x \cos x + 2 \sin^2 x}$
$= \dfrac{2 \cos x (\sin x + \cos x)}{2 \sin x (\cos x + \sin x)}$
$= \dfrac{\cos x}{\sin x} = \cot x.$

Exercises 2.5

1. $\dfrac{\sqrt{2 - \sqrt{3}}}{2}$
3. $\dfrac{\sqrt{2 + \sqrt{2}}}{2}$
5. $\dfrac{\sqrt{2 + \sqrt{2 + \sqrt{3}}}}{2}$
7. $\dfrac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2}$
9. $\sqrt{\dfrac{2 - \sqrt{3}}{2 + \sqrt{3}}}$ or $\sqrt{7 - 4\sqrt{3}}$
11. $\sqrt{\dfrac{2 + \sqrt{3}}{2 - \sqrt{3}}}$ or $\sqrt{7 + 4\sqrt{3}}$
13. $\dfrac{\sqrt{2 - \sqrt{2}}}{2}$
15. $\dfrac{\sqrt{2 - \sqrt{2}}}{2}$
17. $\dfrac{\sqrt{2 - \sqrt{3}}}{2}$
19. $\sqrt{\dfrac{2 - \sqrt{2}}{2 + \sqrt{2}}}$ or $\sqrt{3 - 2\sqrt{2}}$