[10-12-01-S]

PART I: **Elementary Theory of Sets**

Chapter 1

Sets and Basic Operations on Sets

1.1 INTRODUCTION

The concept of a *set* appears in all branches of mathematics. This concept formalizes the idea of grouping objects together and viewing them as a single entity. This chapter introduces this notion of a set and its members. We also investigate three basic operations on sets, that is, the operations union, intersection, and complement.

Although logic is formally treated in Chapter 10, we indicate here the close relationship between set theory and logic by showing how Venn diagrams, pictures of sets, can be used to determine the validity of certain arguments. The relation between set theory and logic will be further explored when we discuss Boolean algebra in Chapter 11.

1.2 SETS AND ELEMENTS

A *set* may be viewed as any well-defined collection of objects; the objects are called the *elements* or *members* of the set.

Although we shall study sets as abstract entities, we now list ten examples of sets:

(1) The numbers 1, 3, 7, and 10.
(2) The solutions of the equation $x^2 - 3x - 2 = 0$.
(3) The vowels of the English alphabet: a, e, i, o, u.
(4) The people living on the earth.
(5) The students Tom, Dick, and Harry.
(6) The students absent from school.
(7) The countries England, France, and Denmark.
(8) The capital cities of Europe.
(9) The even integers: 2, 4, 6, ...
(10) The rivers in the United States.

Observe that the sets in the odd-numbered examples are *defined*, that is, specified or presented, by actually listing its members; and the sets in the even-numbered examples are defined by stating properties or rules which decide whether or not a particular object is a member of the set.

Notation

A set will usually be denoted by a capital letter, such as,

$$A, B, X, Y, \ldots,$$

whereas lower-case letters, $a, b, c, x, y, z, \ldots$ will usually be used to denote elements of sets.

There are essentially two ways to specify a particular set, as indicated above. One way, if possible, is to list its elements. For example,

$$A = \{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\}$$

means that A is the set whose elements are the letters a, e, i, o, u. Note that the elements are separated by commas and enclosed in braces { }. This is sometimes called the *tabular form* of a set.

The second way is to state those properties which characterize the elements in the set, that is, properties held by the members of the set but not by nonmembers. Consider, for example, the expression

$$B = \{x : x \text{ is an even integer}, x > 0\}$$

which reads:

"B is the set of x such that x is an even integer and $x > 0$"

It denotes the set B whose elements are the positive even integers. A letter, usually x, is used to denote a typical member of the set; the colon is read as "such that" and the comma as "and". This is sometimes called the *set-builder form* or *property method* of specifying a set.

Two sets A and B are *equal*, written $A = B$, if they both have the same elements, that is, if every element which belongs to A also belongs to B, and vice versa. The negation of $A = B$ is written $A \neq B$.

The statement "p is an element of A" or, equivalently, the statement "p belongs to A" is written

$$p \in A$$

We also write

$$a, b \in A$$

to state that both a and b belong to A. The statement that p is not an element of A, that is, the negation of $p \in A$, is written

$$p \notin A$$

Remark: It is common practice in mathematics to put a vertical line "|" or slanted line "/" through a symbol to indicate the opposite or negative meaning of the symbol.

EXAMPLE 1.1

(*a*) The set A above can also be written as

$$A = \{x : x \text{ is a letter in the English alphabet}, x \text{ is a vowel}\}$$

Observe that $b \notin A$, $e \in A$, and $p \notin$ A.

(*b*) We cannot list all the elements of the above set B, although we frequently specify the set by writing

$$B = \{2, 4, 6, \ldots\}$$

where we assume everyone knows what we mean. Observe that $8 \in B$, but $9 \notin B$.

(*c*) Let $E = \{x : x^2 - 3x + 2 = 0\}$. In other words, E consists of those numbers which are solutions of the equation $x^2 - 3x + 2 = 0$, sometimes called the *solution set* of the given equation. Since the solutions are 1 and 2, we could also write $E = \{1, 2\}$.

(*d*) Let $E = \{x : x^2 - 3x + 2 = 0\}$, $F = \{2, 1\}$, and $G = \{1, 2, 2, 1, 6/3\}$. Then $E = F = G$ since each consists precisely of the elements 1 and 2. Observe that a set does not depend on the way in which its elements are displayed. A set remains the same even if its elements are repeated or rearranged.

Some sets of numbers will occur very often in the text, and so we use special symbols for them. Unless otherwise specified, we will let:

$\mathbf{N}$ = the set of nonnegative integers: $0, 1, 2, \ldots$

$\mathbf{P}$ = the set of positive integers: $1, 2, 3, \ldots$

$\mathbf{Z}$ = the set of integers: $\ldots, -2, -1, 0, 1, 2, \ldots$

$\mathbf{Q}$ = the set of rational numbers

$\mathbf{R}$ = the set of real numbers

$\mathbf{C}$ = the set of complex numbers

Even if we can list the elements of a set, it may not be practical to do so. For example, we would not list the members of the set of people born in the world during the year 1976 although theoretically it is possible to compile such a list. That is, we describe a set by listing its elements only if the set contains a few elements; otherwise we describe a set by the property which characterizes its elements.

1.3 UNIVERSAL SET, EMPTY SET

All sets under investigation in any application of set theory are assumed to be contained in some large fixed set called the *universal set* or *universe*. For example, in plane geometry, the universal set consists of all the points in the plane, and in human population studies the universal set consists of all the people in the world. We will denote the universal set by

$$\mathbf{U}$$

unless otherwise specified.

Given a universal set **U** and a property P, there may be no element in **U** which has the property P. For example, the set

$$S = \{x : x \text{ is a positive integer, } x^2 = 3\}$$

has no elements since no positive integer has the required property. This set with no elements is called the *empty set* or *null set*, and is denoted by

$$\varnothing$$

(based on the Greek letter phi). There is only one empty set: If S and T are both empty, then $S = T$ since they have exactly the same elements, namely, none.

1.4 SUBSETS

Suppose every element in a set A is also an element of a set B; then A is called a *subset* of B. We also say that A is *contained* in B or B *contains* A. This relationship is written

$$A \subseteq B \quad \text{or} \quad B \supseteq A$$

If A is not a subset of B, that is, if at least one element of A does not belong to B, we write $A \not\subseteq B$ or $B \not\supseteq A$.

EXAMPLE 1.2

(*a*) Consider the sets

$$A = \{1, 3, 5, 8, 9\}, \qquad B = \{1, 2, 3, 5, 7\}, \qquad C = \{1, 5\}$$

Then $C \subseteq A$ and $C \subseteq B$ since 1 and 5, the elements of C, are also elements of A and B. But $B \not\subseteq A$ since some of its elements, e.g., 2 and 7, do not belong to A. Furthermore, since the elements in the sets A, B, C must also belong to the universal set **U**, it is clear that **U** must at least contain the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(*b*) Let **P**, **N**, **Z**, **Q**, **R** be defined as in Section 1.2. Then:

$$\mathbf{P} \subseteq \mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R}$$

(*c*) The set $E = \{2, 4, 6\}$ is a subset of the set $F = \{6, 2, 4\}$, since each number 2, 4, and 6 belonging to E also belongs to F. In fact, $E = F$. Similarly, it can be shown that every set is a subset of itself.

The following properties of sets should be noted:

(i) Every set A is a subset of the universal set $\mathbf{U}$ since, by definition, all the elements of A belong to $\mathbf{U}$. Also the empty set $\varnothing$ is a subset of A.

(ii) Every set A is a subset of itself since, trivially, the elements of A belong to A.

(iii) If every element of A belongs to a set B, and every element of B belongs to a set C, then clearly every element of A belongs to C. In other words, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(iv) If $A \subseteq B$ and $B \subseteq A$, then A and B have the same elements, i.e., $A = B$. Conversely, if $A = B$ then $A \subseteq B$ and $B \subseteq A$ since every set is a subset of itself.

We state these results formally.

Theorem 1.1: (i) For any set A, we have $\varnothing \subseteq A \subseteq \mathbf{U}$.
(ii) For any set A, we have $A \subseteq A$.
(iii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
(iv) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Proper Subset

If $A \subseteq B$, then it is still possible that $A = B$. When $A \subseteq B$ but $A \neq B$, we say that A is a *proper subset* of B. We will write $A \subset B$ when A is a proper subset of B. For example, suppose

$$A = \{1, 3\}, \qquad B = \{1, 2, 3\}, \qquad C = \{1, 3, 2\}$$

Then A and B are both subsets of C; but A is a proper subset of C, whereas B is not a proper subset of C.

Disjoint Sets

Two sets A and B are disjoint if they have no elements in common. For example, suppose

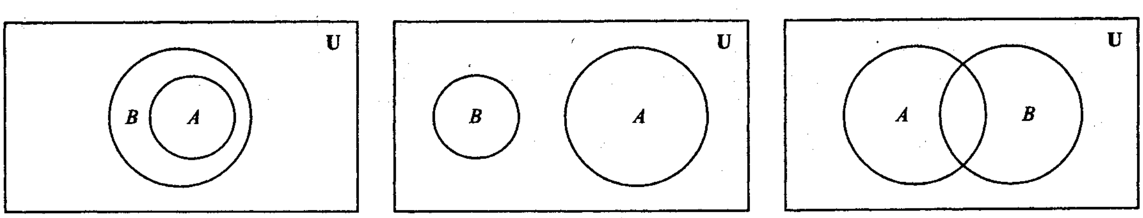
$$A = \{1, 2\}, \qquad B = \{2, 4, 6\}, \qquad C = \{4, 5, 6, 7\}$$

Note that A and B are not disjoint since they both contain the element 2. Similarly, B and C are not disjoint since they both contain the element 4, among others. On the other hand, A and C are disjoint since they have no element in common. We note that if two sets A and B are disjoint sets then neither is a subset of the other (unless one is the empty set).

1.5 VENN DIAGRAMS

A Venn diagram is a pictorial representation of sets where sets are represented by enclosed areas in the plane. The universal set $\mathbf{U}$ is represented by the points in a rectangle, and the other sets are represented by disks lying within the rectangle. If $A \subseteq B$, then the disk representing A will be entirely within the disk representing B, as in Fig. 1-1(*a*). If A and B are disjoint, i.e., have no elements in common, then the disk representing A will be separated from the disk representing B, as in Fig. 1-1(*b*).

On the other hand, if A and B are two arbitrary sets, it is possible that some elements are in A but not B, some elements are in B but not A, some are in both A and B, and some are in neither A nor B; hence, in general, we represent A and B as in Fig. 1-1(*c*).



(*a*) $A \subseteq B$ (*b*) A and B are disjoint (*c*)

Fig. 1-1

Problems

SETS AND SUBSETS

1.1. Which of these sets are equal: $\{r, t, s\}$, $\{s, t, r, s\}$, $\{t, s, t, r\}$, $\{s, r, s, t\}$?

1.2. List the elements of the following sets where $\mathbf{P} = \{1, 2, 3, \ldots\}$.

(*a*) $A = \{x : x \in \mathbf{P},\ 3 < x < 12\}$
(*b*) $B = \{x : x \in \mathbf{P},\ x \text{ is even},\ x < 15\}$
(*c*) $C = \{x : x \in \mathbf{P},\ 4 + x = 3\}$
(*d*) $D = \{x : x \in \mathbf{P},\ x \text{ is a multiple of } 5\}$.

1.3. Consider the following sets:

$$\varnothing, \quad A = \{1\}, \quad B = \{1, 3\}, \quad C = \{1, 5, 9\}, \quad D = \{1, 2, 3, 4, 5\},$$
$$E = \{1, 3, 5, 7, 9\}, \quad \mathbf{U} = \{1, 2, \ldots, 8, 9\}$$

Insert the correct symbol $\subseteq$ or $\not\subseteq$ between each pair of sets:

(*a*) $\varnothing, A$ (*c*) B, C (*e*) C, D (*g*) D, E
(*b*) A, B (*d*) B, E (*f*) C, E (*h*) $D, \mathbf{U}$

1.4. Show that $A = \{2, 3, 4, 5\}$ is not a subset of $B = \{x : x \in P,\ x \text{ is even}\}$.

1.5. Show that $A = \{2, 3, 4, 5\}$ is a proper subset of $C = \{1, 2, 3, \ldots, 8, 9\}$.

1.6. Determine whether or not each set is the null set:

(*a*) $X = \{x : x^2 = 9,\ 2x = 4\}$, (*b*) $Y = \{x : x \neq x\}$, (*c*) $Z = \{x : x + 8 = 8\}$.