a term $x^{n-r}y^r$ of the product arises from choosing the x in each of $n-r$ factors and the y in the other r factors. Since this can be done in $C(n, n-r) = C(n, r)$ ways, the simplified product will contain the term $C(n, r)x^{n-r}y^r$. We are able to conclude, independently of the derivation on page 452, that

$$(x+y)^n = \sum_{r=0}^{n} C(n, r)x^{n-r}y^r.$$

We now have two notations, $C(n, r)$ and $\binom{n}{r}$, for the same expression. In keeping with contemporary usage, we shall generally (though not invariably) prefer the latter.

The relationship between the binomial coefficient $\binom{n}{r}$ and the number of combinations $C(n, r)$ permits a proof of Pascal's Law that is rather more elegant than the one on page 451. In terms of combinations, rather than binomial coefficients, this law becomes the following.

▶ **Theorem.**

$$C(n, r-1) + C(n, r) = C(n+1, r).$$

Proof: Let E be a set containing $n+1$ elements. The number of r-element subsets of E is, by definition, $C(n+1, r)$. Now consider some fixed element of E; call it x. Each r-element subset of E either contains x or does not contain x. The number of those that do contain x is $C(n, r-1)$, because including x leaves n elements of E from which to choose the remaining $r-1$ elements of each subset. The number of those that do not contain x is $C(n, r)$, because excluding x leaves n elements of E from which to choose all r elements of each subset. Hence, by (*1*),

$$C(n, r-1) + C(n, r) = C(n+1, r).$$

Exercises [A]

1. Compute the number of 2-card subsets of a deck of 52 cards.
2. (a) A man has 12 friends whom he would like to invite to his home, but he has only room enough for 9 guests. In how many ways can he select the 9? (b) If he had only room enough for 3 guests, in how many ways could he select the 3?
3. Eighteen boys gather to play baseball. In how many ways can they be divided into 2 teams, 9 boys to a team?

4. Each of the 8 teams in a bowling league bowls against each of the others 4 times. How many matches are bowled?

5. (a) How many 3-element subsets can be chosen from $\{A, B, C, D, E, F\}$?
   (b) How many of these include A?
   (c) How many include A but not B?
   (d) How many include A and B?
   (e) How many exclude A and B?
   (f) How many include A or B?

6. A collector has ten antique coins. In how many ways can he select two or more to sell?

7. (a) How many committees of 6 people can be chosen from a group of 5 men and 6 women?
   (b) How many of these committees will consist of 3 men and 3 women?
   (c) Once a committee has been determined, how many ways are there of choosing a chairman and a secretary?

8. (a) How many lines are determined by n points, no three of them collinear?
   (b) How many diagonals has a polygon of n sides?

9. What is the maximum number of intersections of five lines?

10. (a) If no four of five given points are coplanar, how many lines do they determine?
    (b) How many planes do they determine?

11. If four lines are coplanar, no two of them parallel and no three concurrent, how many triangles do they form?

12. (a) If 16 points are marked on a circle, how many triangles have their vertices at these points?
    (b) If 16 points are marked on a square, 4 points on each side, how many triangles have their vertices at these points?

13. A "deck" of 12 cards is made up of the kings, queens, and jacks from an ordinary deck. (a) How many 3-card hands can be selected from this deck? (b) How many of these hands contain (i) 3 of a kind (3 cards of the same denomination)? (ii) 3 cards of the same suit? (iii) a sequence (J-Q-K)? (iv) a pair, but not 3 of a kind? (v) no pair?

14. A store stocks 4 kinds of candy bars. In how many ways can 3 candy bars be chosen?

15. How many rectangles are formed by the lines that bound the squares on an ordinary 64-square checkerboard?

16. In how many ways can 6 presents be distributed among 3 children so that each child gets 2 presents?

Exercises [B]

1. There are 8 seats in a row. In how many ways can 4 boys and 4 girls be seated so that each boy is next to at least one boy and each girl is next to at least one girl?

2. (a) Ten A's and x B's are to be placed in a row so that no two B's come together. What is the maximum value of x?

   (b) In how many ways can the A's and B's be so placed if $x = 4$?

3. How many different 5-card hands contain each of the following combinations?

   (a) A sequence, all in one suit. (The ace may be low, as in A-2-3-4-5, or high, as in 10-J-Q-K-A.)

   (b) Four of a kind (four cards of the same denomination).

   (c) Three of a kind, and a pair.

   (d) Five cards, all of one suit, but not in sequence.

   (e) A sequence, not all in one suit.

   (f) Three of a kind (three cards of one denomination, two cards of two other denominations).

   (g) Two pairs, but not three of a kind or four of a kind.

   (h) One pair, but nothing better.

4. How many bridge hands are distributed in a (a) 5-4-3-1 pattern? (b) 5-4-2-2 pattern?

5. Three dice can fall in $6^3 = 216$ ways. In how many of these ways do the dice show (a) three different faces? (b) two different faces? (c) all faces alike?

6. Use the following procedure to show that the expression (7) for the number of permutations of n objects, not all different, can be derived from (8):

   (a) In any arrangement of the n given objects, there are n places to be filled. In how many ways can the n_1 places to be filled by objects of the first kind be chosen?

(b) After all the objects of the first kind have been placed, how many places are left unfilled? In how many ways can n_2 of these places be chosen for the n_2 objects of the second kind?

(c) Continue the argument, finding the number of choices for the n_3 objects of the third kind, after the first two kinds have been placed, and so on to the kth kind.

(d) Now use the fundamental principle to find the total number of choices for all k kinds of objects. Express everything in factorial notation, and simplify.

7. A set consists of n A's and m B's. By considering in turn all r-element subsets that contain no A's, 1 A, 2 A's. · · ·, r A's, where $m \geq r$ and $n \geq r$, show that

$$\sum_{x=0}^{r} \binom{n}{x}\binom{m}{r-x} = \binom{m+n}{r}.$$

8. Use the result of Exercise 7 to show that $\sum_{x=0}^{n} \binom{n}{x}^2 = \binom{2n}{n}$.

9. Show that the coefficient of $a^x b^y c^z$ in the expansion of $(a + b + c)^n$ is $\dfrac{n!}{x!\,y!\,z!}$.

Pages 476–477

1. 23 3. 271 5. 271 7. (a) 16 (b) 12 9. 265 11. 8190 13. 59
15. 104

Pages 483–484

1. (a) A (b) B 3. $\emptyset$ 5. 7
7. (a) 1, 2, 3, 4, 5, 6 (b) 1, 4 (c) 1, 2, 3, 5, 6, 8 (d) 5, 8

Pages 488–489

1. (a) 120 (b) 2450 (c) 100 3. 1 5. $\frac{20!}{11!}$ 7. (a) 560 (b) 83160 9. 210
11. 5,652,770 13. 924 15. 2880 17. 56

Pages 489–490

1. (a) 21 (b) 21 (c) 21 3. (a) 120 (b) 120

Pages 494–496

1. 1326 3. 48620 5. (a) 20 (b) 10 (c) 6 (d) 4 (e) 4 (f) 16
7. (a) 462 (b) 200 (c) 30 9. 10 11. 4
13. (a) 220 (b) (i) 12 (ii) 4 (iii) 64 (iv) 144 (v) 64 15. 1296

Pages 496–497

1. 13,824 3. (a) 40 (b) 624 (c) 3744 (d) 5108 (e) 10,200 (f) 54,912
(g) 123,552 (h) 1,098,240 5. (a) 120 (b) 90 (c) 6

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1. 1120 3. 1536

Pages 501–502

1. (a) (b) (f) 3. (a) {(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)}
(b) (i) {(2, 1), (3, 1), (3, 2)} (ii) {(3, 1), (3, 2)} (iii) {(1, 2), (3, 2)}
(iv) {(1, 3), (3, 1)} (v) {(2, 1), (3, 1), (3, 2)} (vi) {(3, 1), (3, 2)}
(vii) {(1, 2), (1, 3), (3, 1), (3, 2)} (viii) $\emptyset$ (ix) {(3, 2)} (x) {(3, 2)}

Pages 508–509

9. $P(E) = \frac{1}{4}$, $P(\overline{E}) = \frac{3}{4}$ 13. (a) $\frac{1}{221}$ (b) $\frac{1}{17}$ (c) $\frac{1}{17}$ (d) $\frac{19}{34}$ (e) $\frac{33}{221}$
(f) $\frac{16}{17}$ (g) $\frac{29}{442}$ (h) $\frac{116}{221}$