[T11-05-08-12]

Function

Function

Def. A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

Def. A **function** is a set of ordered pairs such that each first element is paired with one and only one second element. That is, a relation F is a function if $(x_1, y_1) \in F$ and $(x_1, y_2) \in F$ implies $y_1 = y_2$.

Equivalently,

$$y_1 \neq y_2 \Longrightarrow x_1 \neq x_2$$

$$x_1 = x_2 \Longrightarrow y_1 = y_2$$

- **NB: The rule and the domain determine the function. $f(x) = x^2$, $\mathcal{D} = \{x : x \geq 0\}$ is not the same function as $g(x) = x^2$, $\mathcal{D} = \{x : x > 0\}$.**

- **Mention vertical line test.**

Domain and Range

Algebra

Let f and g be functions with domains A and B. Then,

$$(f + g)(x) = f(x) + g(x) \qquad \mathcal{D} = A \cap B$$

$$(f - g)(x) = f(x) - g(x) \qquad \mathcal{D} = A \cap B$$

$$(f\, g)(x) = f(x)\, g(x) \qquad \mathcal{D} = A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \mathcal{D} = \{A \cap B : g(x) \neq 0\}$$

Composition

Given two functions f and g, the **composite function** $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$.

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

In general, $f \circ g \neq g \circ f$.

[EX1] Suppose $f = x^2$ and $g = x - 3$. Then,

$$(f \circ g)(x) = f(g(x)) = f(x-3) = (x-3)^2$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3.$$

Note that $f \circ g \neq g \circ f$.

■ 1-1 correspondence

Def. A function *f* with domain A and range B is called a **one-to-one function** if no two elements of A are assigned the same element in B; that is,

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2$$

Equivalently,

$$x_1 \neq x_2 \Longrightarrow y_1 \neq y_2$$

$$y_1 = y_2 \Longrightarrow x_1 = x_2$$

■ Mention horizontal line test.

■ Inverse of a function

Def. Let *f* be a function $\{(x,\ y) : (x,\ y) \in f\}$ which is a one-to-one correspondence between its domain and its range. Then $\{(y,\ x) : (x,\ y) \in f^{-1}\}$ is also a function called **the inverse of *f***.

NB: the inverse of *f* is often written f^{-1}.

■ Identity element for functions

Let f^{-1} be the inverse of *f*. Then $f^{-1}(f(x)) = f(f^{-1}(x)) = x$.

NB: The inverse function f^{-1} of f undoes the work of f. If f takes $x_1 \in A$ to $y_1 \in B$, then f^{-1} takes $y_1 \in B$ to $x_1 \in A$. Result: $f^{-1}(f(x_1)) = x_1$. You are right back where you started. The identity element is the function $I(x) = x$.

NB: The domain of the inverse of f is the range of f. The range of a the inverse of f is the domain of f.