SLANT ASYMPTOTES

Some curves have asymptotes that are oblique, that is, neither horizontal nor vertical. If

$$\lim_{x\to\infty} [f(x) - (mx + b)] = 0$$

then the line $y = mx + b$ is called a **slant asymptote** because the vertical distance between the curve $y = f(x)$ and the line $y = mx + b$ approaches 0. (A similar situation exists if we let $x \to -\infty$.) For rational functions, slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator. In such a case the equation of the slant asymptote can be found by long division as in the following example.

EXAMPLE 6 Sketch the graph of $f(x) = \dfrac{x^3}{x^2+1}$.

A. The domain is $\mathbb{R} = (-\infty, \infty)$.
B. The x- and y-intercepts are both 0.
C. Since $f(-x) = -f(x)$, f is odd and its graph is symmetric about the origin.
D. Since $x^2 + 1$ is never 0, there is no vertical asymptote. Since $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, there is no horizontal asymptote. But long division gives

$$f(x) = \frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$$

$$f(x) - x = -\frac{x}{x^2+1} = -\frac{\dfrac{1}{x}}{1 + \dfrac{1}{x^2}} \to 0 \quad \text{as} \quad x \to \pm\infty$$

So the line $y = x$ is a slant asymptote.

E. $$f'(x) = \frac{3x^2(x^2+1) - x^3 \cdot 2x}{(x^2+1)^2} = \frac{x^2(x^2+3)}{(x^2+1)^2}$$

Since $f'(x) > 0$ for all x (except 0), f is increasing on $(-\infty, \infty)$.

F. Although $f'(0) = 0$, f' does not change sign at 0, so there is no local maximum or minimum.

G. $$f''(x) = \frac{(4x^3 + 6x)(x^2+1)^2 - (x^4 + 3x^2) \cdot 2(x^2+1)2x}{(x^2+1)^4} = \frac{2x(3 - x^2)}{(x^2+1)^3}$$

Since $f''(x) = 0$ when $x = 0$ or $x = \pm\sqrt{3}$, we set up the following chart:

Interval	x	$3 - x^2$	$(x^2+1)^3$	$f''(x)$	f
$x < -\sqrt{3}$	$-$	$-$	$+$	$+$	CU on $(-\infty, -\sqrt{3})$
$-\sqrt{3} < x < 0$	$-$	$+$	$+$	$-$	CD on $(-\sqrt{3}, 0)$
$0 < x < \sqrt{3}$	$+$	$+$	$+$	$+$	CU on $(0, \sqrt{3})$
$x > \sqrt{3}$	$+$	$-$	$+$	$-$	CD on $(\sqrt{3}, \infty)$

The points of inflection are $(-\sqrt{3}, -3\sqrt{3}/4)$, $(0,0)$, and $(\sqrt{3}, 3\sqrt{3}/4)$.

H. The graph of f is sketched in Figure 15. ■

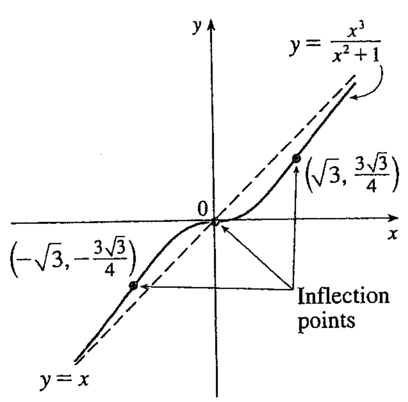


FIGURE 15

EXERCISES 4.5

1–44 ■ Discuss the curves under the headings A–H given in this section.

1. $y = 1 - 3x + 5x^2 - x^3$
2. $y = 2x^3 - 6x^2 - 18x + 7$
3. $y = x^4 - 6x^2$
4. $y = 4x^3 - x^4$
5. $y = \dfrac{x}{x - 1}$
6. $y = \dfrac{x}{(x - 1)^2}$
7. $y = \dfrac{1}{x^2 - 9}$
8. $y = \dfrac{x}{x^2 - 9}$
9. $y = \dfrac{1}{(x - 1)(x + 2)}$
10. $y = \dfrac{1}{x^2(x + 3)}$
11. $y = \dfrac{1 + x^2}{1 - x^2}$
12. $y = \dfrac{x^3 - 1}{x^3 + 1}$
13. $y = \dfrac{1}{x^3 - x}$
14. $y = \dfrac{1 - x^2}{x^3}$
15. $y = x\sqrt{x + 3}$
16. $y = \sqrt{x} - \sqrt{x - 1}$
17. $y = \sqrt{x^2 + 1} - x$
18. $y = \sqrt{\dfrac{x}{x - 5}}$
19. $y = \sqrt[4]{x^2 - 25}$
20. $y = x\sqrt{x^2 - 9}$
21. $y = \dfrac{\sqrt{1 - x^2}}{x}$
22. $y = \dfrac{x + 1}{\sqrt{x^2 + 1}}$
23. $y = x + 3x^{2/3}$
24. $y = x^{5/3} - 5x^{2/3}$
25. $y = x + \sqrt{|x|}$
26. $y = \sqrt[3]{(x^2 - 1)^2}$
27. $y = \cos x - \sin x$
28. $y = \sin x - \tan x$
29. $y = x \tan x, \quad -\pi/2 < x < \pi/2$
30. $y = 2x + \cot x, \quad 0 < x < \pi$
31. $y = \dfrac{x}{2} - \sin x, \quad 0 < x < 3\pi$
32. $y = 2 \sin x + \sin^2 x$
33. $y = 2 \cos x + \sin^2 x$
34. $y = \sin x - x$
35. $y = \sin 2x - 2 \sin x$
36. $y = \dfrac{\cos x}{2 + \sin x}$
37. $y = e^{-1/(x+1)}$
38. $y = xe^{x^2}$
39. $y = 1/(1 + e^{-x})$
40. $y = \ln(\cos x)$
41. $y = \ln(1 + x^2)$
42. $y = \ln(\tan^2 x)$
43. $y = \ln(x^2 - x)$
44. $y = x^{-\ln x}$

45–54 ■ Sketch the curve under the headings A–H using l'Hospital's Rule where appropriate.

45. $y = xe^{-x}$
46. $y = x^2 e^{-x}$
47. $y = x \ln x$
48. $y = (\ln x)/x$
49. $y = x^2 \ln x$
50. $y = x(\ln x)^2$
51. $y = xe^{-x^2}$
52. $y = e^x/x$
53. $y = xe^{1/x}$
54. $y = e^x - 3e^{-x} - 4x$

55–60 ■ Discuss each curve under the headings A–H. In D find an equation of the slant asymptote.

55. $y = \dfrac{x^3}{x^2 - 1}$
56. $y = x - \dfrac{1}{x}$
57. $xy = x^2 + 4$
58. $y = e^x - x$
59. $y = \dfrac{1}{x - 1} - x$
60. $y = \dfrac{x^2}{2x + 5}$

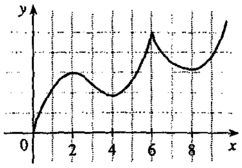
61. Show that the lines $y = (b/a)x$ and $y = -(b/a)x$ are slant asymptotes of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$.

62. Let $f(x) = (x^3 + 1)/x$. Show that
$$\lim_{x\to\pm\infty} [f(x) - x^2] = 0$$
This shows that the graph of f approaches the graph of $y = x^2$, and we say that the curve $y = f(x)$ is *asymptotic* to the parabola $y = x^2$. Use this fact to help sketch the graph of f.

63. Discuss the asymptotic behavior of $f(x) = (x^4 + 1)/x$ in the same manner as in Exercise 62. Then use your results to help sketch the graph of f.

64. Use the asymptotic behavior of $f(x) = \cos x + 1/x^2$ to sketch its graph without going through the curve-sketching procedure of this section.

33. Increasing on $[0,2]$, $[4,6]$, and $[8,\infty)$, decreasing on $[2,4]$ and $[6,8]$
(b) Local maxima at $x = 2, 6$, minimum at $x = 4$
(c) CU on $(3,6)$ and $(6,\infty)$, CD on $(0,3)$
(d) 3
(e) See graph at right.

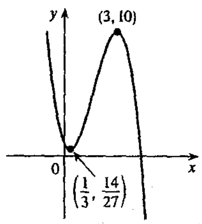


35. CU on $(0.1,\infty)$, CD on $(-\infty,0.1)$ **47.** IP, no extremum

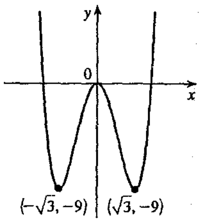
Exercises 4.5 ▪ page 287

Abbreviations: VA, vertical asymptote; HA, horizontal asymptote; IP, inflection point

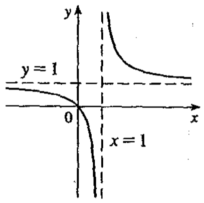
1. (a) $\mathbb{R}$ (b) y-intercept 1 (c) None
(d) None (e) Increasing on $[\frac{1}{3},3]$, decreasing on $(-\infty,\frac{1}{3}]$ and $[3,\infty)$
(f) Local minimum $f(\frac{1}{3}) = \frac{14}{27}$, local maximum $f(3) = 10$
(g) CD on $(\frac{5}{3},\infty)$, CU on $(-\infty,\frac{5}{3})$, IP $(\frac{5}{3},\frac{142}{27})$
(h) See graph at right.



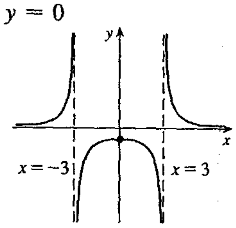
3. (a) $\mathbb{R}$ (b) y-intercept 0; x-intercepts 0, $\pm\sqrt{6}$
(c) About y-axis (d) None
(e) Increasing on $[-\sqrt{3},0]$ and $[\sqrt{3},\infty)$, decreasing on $(-\infty,-\sqrt{3}]$ and $[0,\sqrt{3}]$
(f) Local minima $f(\pm\sqrt{3}) = -9$, maximum $f(0) = 0$
(g) CU on $(-\infty,-1)$ and $(1,\infty)$, CD on $(-1,1)$, IP $(1,-5)$ and $(-1,-5)$
(h) See graph at right.



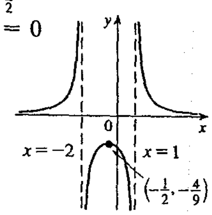
5. (a) $\{x \mid x \neq 1\}$
(b) x-intercept 0, y-intercept 0
(c) None (d) VA $x = 1$, HA $y = 1$
(e) Decreasing on $(-\infty,1)$ and $(1,\infty)$
(f) No maximum or minimum
(g) CD on $(-\infty,1)$, CU on $(1,\infty)$, no IP
(h) See graph at right.



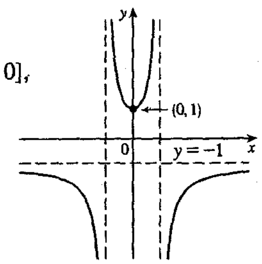
7. (a) $\{x \mid x \neq \pm 3\}$ (b) y-intercept $-\frac{1}{9}$
(c) About y-axis (d) VA $x = \pm 3$, HA $y = 0$
(e) Increasing on $(-\infty,-3)$ and $(-3,0]$, decreasing on $[0,3)$ and $(3,\infty)$
(f) Local maximum $f(0) = -\frac{1}{9}$
(g) CU on $(-\infty,-3)$ and $(3,\infty)$, CD on $(-3,3)$, no IP
(h) See graph at right.



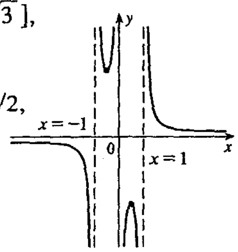
9. (a) $\{x \mid x \neq 1,-2\}$ (b) y-intercept $-\frac{1}{2}$
(c) None (d) VA $x = 1$, $x = -2$, HA $y = 0$
(e) Increasing on $(-\infty,-2)$ and $(-2,-\frac{1}{2}]$, decreasing on $[-\frac{1}{2},1)$ and $(1,\infty)$
(f) Local maximum $f(-\frac{1}{2}) = -\frac{4}{9}$
(g) CU on $(-\infty,-2)$ and $(1,\infty)$, CD on $(-2,1)$, no IP
(h) See graph at right.



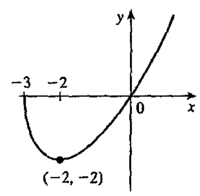
11. (a) $\{x \mid x \neq \pm 1\}$ (b) y-intercept 1
(c) About y-axis
(d) VA $x = \pm 1$, HA $y = -1$
(e) Decreasing on $(-\infty,-1)$ and $(-1,0]$, increasing on $[0,1)$ and $(1,\infty)$
(f) Local minimum $f(0) = 1$
(g) CD on $(-\infty,-1)$ and $(1,\infty)$, CU on $(-1,1)$, no IP
(h) See graph at right.



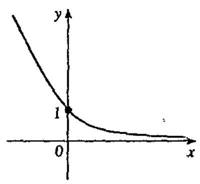
13. $\{x \mid x \neq 0,\pm 1\}$ (b) None (c) About $(0,0)$
(d) VA $x = -1$, $x = 0$, $x = 1$; HA $y = 0$
(e) Decreasing on $(-\infty,-1)$, $(-1,-1/\sqrt{3}]$, $[1/\sqrt{3},1)$, and $(1,\infty)$, increasing on $[-1/\sqrt{3},0)$ and $(0,1/\sqrt{3}]$
(f) Local minimum $f(-1/\sqrt{3}) = 3\sqrt{3}/2$, maximum $f(1/\sqrt{3}) = -3\sqrt{3}/2$
(g) CD on $(-\infty,-1)$, $(0,1)$, CU on $(-1,0)$, $(1,\infty)$
(h) See graph at right.



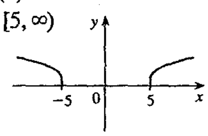
15. (a) $[-3,\infty)$
(b) x-intercepts 0, -3; y-intercept 0
(c) None (d) None
(e) Increasing on $[-2,\infty)$, decreasing on $[-3,-2]$
(f) Local minimum $f(-2) = -2$
(g) CU on $(-3,\infty)$
(h) See graph at right.



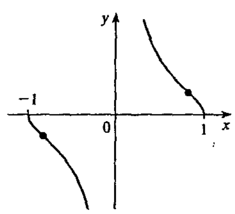
17. (a) $\mathbb{R}$ (b) y-intercept 1
(c) None (d) HA $y = 0$
(e) Decreasing on $(-\infty,\infty)$
(f) No local maximum or minimum
(g) CU on $(-\infty,\infty)$, no IP
(h) See graph at right.



19. (a) $\{x \mid |x| \geq 5\} = (-\infty,-5] \cup [5,\infty)$
(b) x-intercepts ± 5 (c) About y-axis (d) None
(e) Decreasing on $(-\infty,-5]$, increasing on $[5,\infty)$
(f) No local maximum or minimum
(g) CD on $(-\infty,-5)$ and $(5,\infty)$, no IP
(h) See graph at right.

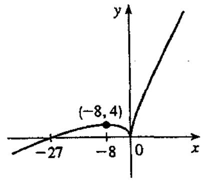


21. (a) $\{x \mid |x| \leq 1, x \neq 0\} = [-1,0) \cup (0,1]$
(b) x-intercepts ± 1 (c) About $(0,0)$
(d) VA $x = 0$
(e) Decreasing on $[-1,0)$ and $(0,1]$
(f) No local maximum or minimum
(g) CU on $(-1,-\sqrt{2/3})$ and $(0,\sqrt{2/3})$, CD on $(-\sqrt{2/3},0)$ and $(\sqrt{2/3},1)$, IP $(\pm\sqrt{2/3},\pm 1/\sqrt{2})$
(h) See graph at right.

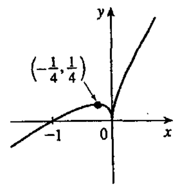


23. (a) $(-\infty,\infty)$ (b) x-intercepts 0, -27; y-intercept 0
(c) None (d) None

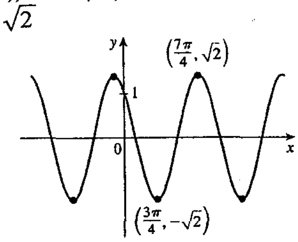
(e) Increasing on $(-\infty, -8]$ and $[0, \infty)$, decreasing on $[-8, 0]$
(f) Local minimum $f(0) = 0$, local maximum $f(-8) = 4$
(g) CD on $(-\infty, 0)$ and $(0, \infty)$
(h) See graph at right.



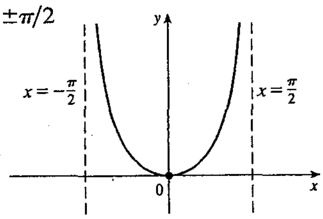
25. (a) $\mathbb{R}$ (b) x-intercepts $-1, 0$; y-intercept 0
(c) None (d) None
(e) Increasing on $(-\infty, -\frac{1}{4}]$ and $[0, \infty)$, decreasing on $[-\frac{1}{4}, 0]$
(f) Local maximum $f(-\frac{1}{4}) = \frac{1}{4}$, minimum $f(0) = 0$
(g) CD on $(-\infty, 0)$ and $(0, \infty)$, no IP
(h) See graph at right.



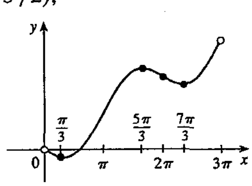
27. (a) $\mathbb{R}$
(b) y-intercept 1, x-intercepts $n\pi + (\pi/4)$ (n an integer)
(c) Period 2π (d) None
(e) Decreasing on $[2n\pi - (\pi/4), 2n\pi + (3\pi/4)]$, increasing on $[2n\pi + (3\pi/4), 2n\pi + (7\pi/4)]$
(f) Local minimum $f(2n\pi + (3\pi/4)) = -\sqrt{2}$, local maximum $f(2n\pi - (\pi/4)) = \sqrt{2}$
(g) CU on $(2n\pi + (\pi/4), 2n\pi + (5\pi/4))$, CD on $(2n\pi - (3\pi/4), 2n\pi + (\pi/4))$, IP $(n\pi + (\pi/4), 0)$
(h) See graph at right.



29. (a) $(-\pi/2, \pi/2)$ (b) x-intercept 0, y-intercept 0
(c) About y-axis (d) VA $x = \pm\pi/2$
(e) Decreasing on $(-\pi/2, 0]$, increasing on $[0, \pi/2)$
(f) Local minimum $f(0) = 0$
(g) CU on $(-\pi/2, \pi/2)$, no IP
(h) See graph at right.

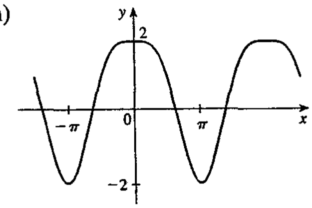


31. (a) $(0, 3\pi)$ (c) None (d) None
(e) Decreasing on $(0, \pi/3]$ and $[5\pi/3, 7\pi/3]$, increasing on $[\pi/3, 5\pi/3]$ and $[7\pi/3, 3\pi)$
(f) Local minima $f(\pi/3) = (\pi/6) - (\sqrt{3}/2)$, $f(7\pi/3) = (7\pi/6) - (\sqrt{3}/2)$, maximum $f(5\pi/3) = (5\pi/6) + (\sqrt{3}/2)$
(g) CU on $(0, \pi)$ and $(2\pi, 3\pi)$, CD on $(\pi, 2\pi)$, IP $(\pi, \pi/2)$, $(2\pi, \pi)$
(h) See graph at right.

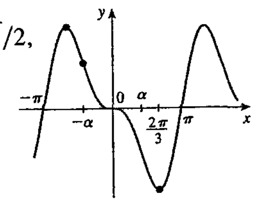


33. (a) $(-\infty, \infty)$ (b) y-intercept 2
(c) About y-axis, period 2π (d) None
(e) Increasing on $[(2n - 1)\pi, 2n\pi]$, decreasing on $[2n\pi, (2n + 1)\pi]$
(f) Maximum $f(2n\pi) = 2$, minimum $f((2n + 1)\pi) = -2$
(g) CD on $(2n\pi - (2\pi/3), (2n\pi + (2\pi/3))$, CU on remaining intervals, IP when $x = 2n\pi \pm (2\pi/3)$

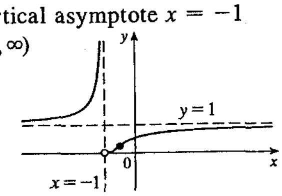
(h)



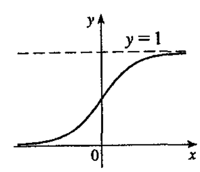
35. (a) $\mathbb{R}$ (b) y-intercept 0, x-intercept $n\pi$
(c) About $(0, 0)$, period 2π (d) None
(e) Increasing on $[-\pi, -2\pi/3]$ and $[2\pi/3, \pi]$, decreasing on $[-2\pi/3, 2\pi/3]$
(f) Local maximum $f(-2\pi/3) = 3\sqrt{3}/2$, local minimum $f(2\pi/3) = -3\sqrt{3}/2$
(g) CD on $(-\pi, -\alpha)$ and $(0, \alpha)$, where $\cos\alpha = \frac{1}{4}$, CU on $(-\alpha, 0)$ and (α, π), IP when $x = 0, \pm\alpha, \pi$
(h) See graph at right.



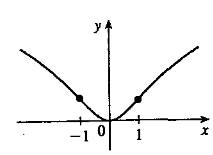
37. (a) $\{x \mid x \neq -1\}$ (b) y-intercept $1/e$ (c) None
(d) Horizontal asymptote $y = 1$, vertical asymptote $x = -1$
(e) Increasing on $(-\infty, -1)$ and $(-1, \infty)$
(f) No maximum or minimum
(g) CU on $(-\infty, -1)$ and $(-1, -\frac{1}{2})$, CD on $(-\frac{1}{2}, \infty)$, IP $(-\frac{1}{2}, 1/e^2)$
(h) See graph at right.



39. (a) $\mathbb{R}$ (b) y-intercept $\frac{1}{2}$ (c) None
(d) Horizontal asymptotes $y = 0$, $y = 1$
(e) Increasing on $\mathbb{R}$
(f) None
(g) CU on $(-\infty, 0)$, CD on $(0, \infty)$, IP $(0, \frac{1}{2})$
(h) See graph at right.



41. (a) $\mathbb{R}$ (b) x-intercept 0, y-intercept 0
(c) About y-axis (d) None
(e) Increasing on $[0, \infty)$, decreasing on $(-\infty, 0]$
(f) Local minimum $f(0) = 0$
(g) CD on $(-\infty, -1)$ and $(1, \infty)$, CU on $(-1, 1)$, IP $(\pm 1, \ln 2)$
(h) See graph at right.



43. (a) $(-\infty, 0) \cup (1, \infty)$
(b) x-intercepts $(1 \pm \sqrt{5})/2$
(c) None (d) VA $x = 0, x = 1$
(e) Decreasing on $(-\infty, 0)$, increasing on $(1, \infty)$
(f) None (g) CD on $(-\infty, 0)$ and $(1, \infty)$
(h) See graph at right.