Theorem 1.8: Suppose A and B are finite sets. Then $A \cap B$ and $A \cup B$ are finite, and

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$$

That is, we find the number of elements in A or B (or both) by first adding $n(A)$ and $n(B)$ (inclusion) and then subtracting $n(A \cap B)$ (exclusion) since the elements in $A \cap B$ were counted twice.

We can apply this result to get a similar result for three sets.

Corollary 1.9: Suppose A, B, C are finite sets. Then $A \cup B \cup C$ is finite and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Mathematical induction (Section 1.11) may be used to further generalize this result to any finite number of finite sets.

EXAMPLE 1.6 Consider the following data among 110 students in a college dormitory:

30 students are on a list A (taking Accounting),
35 students are on a list B (taking Biology),
20 students are on both lists.

Find the number of students: (*a*) on list or B, (*b*) on exactly one of the two lists, (*c*) on neither list.

(*a*) We seek $n(A \cup B)$. By Theorem 1.8,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 30 + 35 - 20 = 45$$

In other words, we combine the two lists and then cross out the 20 student names which appear twice.

(*b*) List A contains 30 names and 20 of them are on list B; hence $30 - 20 = 10$ names are only on list A. That is,

$$n(A \backslash B) = n(A) - n(A \cup B) = 30 - 20 = 10$$

Similarly, list B contains 35 names and 20 of them are on list A; hence $35 - 20 = 15$ names are only on list B. That is,

$$n(B \backslash A) = n(B) - n(A \cup B) = 35 - 20 = 15$$

Thus there are $10 + 15 = 25$ students on exactly one of the two lists.

(*c*) The students on neither the A list nor the B list form the set $A^c \cap B^c$. By DeMorgan's law, $A^c \cap B^c = (A \cup B)^c$. Hence

$$n(A^c \cap B^c) = n((A \cup B)^c) = n(U) - n(A \cup B) = 110 - 45 = 65$$

EXAMPLE 1.7 Consider the following data for 120 mathematics students:

65 study French, 20 study French and German,
45 study German, 25 study French and Russian,
42 study Russian, 15 study German and Russian,
8 study all three languages

Let F, G, and R denote the sets of students studying French, German, and Russian, respectively.

(*a*) Find the number of students studying at least one of the three languages, i.e. find $n(F \cup G \cup R)$.

(*b*) Fill in the correct number of students in each of the eight regions of the Venn diagram of Fig. 1-5(*a*).

(*c*) Find the number k of students studying: (1) exactly one language, (2) exactly two languages.

(*a*) By Corollary 1.9,

$$\begin{aligned} n(F \cup G \cup R) &= n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) - n(F \cap G \cap R) \\ &= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100 \end{aligned}$$

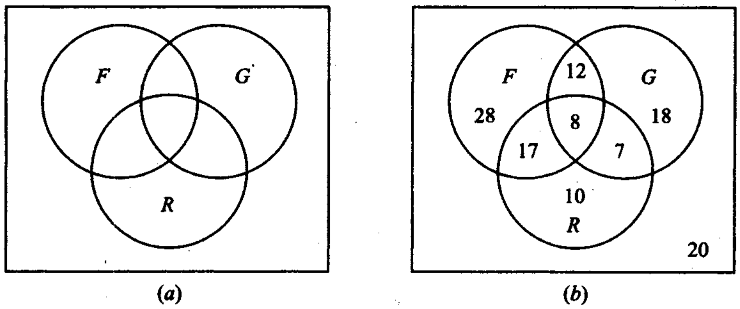

Fig. 1-5

(*b*) Using 8 study all three languages and 100 study at least one language, the remaining seven regions of the required Venn diagram Fig. 1-5(*b*) are obtained as follows:

$15 - 8 = 7$ study German and Russian but not French,
$25 - 8 = 17$ study French and Russian but not German,
$20 - 8 = 12$ study French and German but not Russian,
$42 - 17 - 8 - 7 = 10$ study only Russian,
$45 - 12 - 8 - 7 = 18$ study only German,
$65 - 12 - 8 - 17 = 28$ study only French,
$120 - 100 = 20$ do not study any of the languages.

(*c*) Use the Venn diagram of Fig. 1-5(*b*) to obtain:

(1) $k = 28 + 18 + 10 = 56$, (2) $k = 12 + 17 + 7 = 36$

1.9 CLASSES OF SETS, POWER SETS

Given a set S, we may wish to talk about some of its subsets. Thus we would be considering a "set of sets". Whenever such a situation arises, to avoid confusion, we will speak of a *class* of sets or a *collection* of sets. If we wish to consider some of the sets in a given class of sets, then we will use the term *subclass* or *subcollection*.

EXAMPLE 1.8 Suppose $S = \{1, 2, 3, 4\}$. Let $\mathscr{A}$ be the class of subsets of S which contain exactly three elements of S. Then

$$\mathscr{A} = [\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}]$$

The elements of $\mathscr{A}$ are the sets $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, and $\{2,3,4\}$.

Let $\mathscr{B}$ be the class of subsets of S which contain 2 and two other elements of S. Then

$$\mathscr{B} = [\{1,2,3\}, \{1,2,4\}, \{2,3,4\}]$$

The elements of $\mathscr{B}$ are $\{1,2,3\}$, $\{1,2,4\}$, and $\{2,3,4\}$. Thus $\mathscr{B}$ is a subclass of $\mathscr{A}$. (To avoid confusion, we will usually enclose the sets of a class in brackets instead of braces.)

Power Sets

For a given set S, we may speak about the class of all subsets of S. This class is called the *power set* of S, and it will be denoted by $\mathscr{P}(S)$. If S is finite, then so is $\mathscr{P}(S)$. In fact, the number of elements in $\mathscr{P}(S)$ is 2 raised to the power of $n(S)$; that is,

$$n(\mathscr{P}(S)) = 2^{n(S)}$$

(This is the reason $\mathscr{P}(S)$ is called the power set of S; it is also sometimes denoted by 2^S.)