Chapter 16

Complex numbers

Contents:

A COMPLEX NUMBERS AS 2-D VECTORS

Recall from **Chapter 8B** that a complex number can be written as $z = a + bi$ (Cartesian form) where $a = \text{Re}(z)$ and $b = \text{Im}(z)$ are both real numbers.

Hence, there exists a one-to-one relationship between any complex number $a+bi$ and any point (a, b) in the Cartesian Plane.

When we view points in the plane as complex numbers, we refer to the plane as the Argand diagram and the x-axis is called the "real axis" and y-axis is called the "imaginary axis".

Interestingly, all complex numbers with $b = 0$ (**real numbers**) lie on the **real** axis, and all complex numbers with $a = 0$ lie on the **imaginary** axis. The origin (0, 0) lies on both axes and it corresponds to $z = 0$, a real number.

Consequently all points on the **imaginary** axis (y-axis) except for the origin are known as **pure imaginary** complex numbers i.e., $z = bi$, $b \neq 0$.

Complex numbers that are neither **real** nor **pure imaginary** lie in one of the four quadrants (a, b both $\neq 0$).

Now recall from **Chapter 15** that any point P in the plane corresponds uniquely to a vector. The position vector of the point P is $\overrightarrow{\text{OP}}$. We also learned that vectors have magnitude and direction so we can in turn attribute magnitude and direction to complex numbers.

Also, note that the operations of (1) multiples of vectors and (2) addition and subtraction of vectors, give answers that correspond to the same answers for complex numbers.

Note: For those studying the option "sets, relations and groups" this means there is an **isomorphic relationship** between complex numbers and 2-D vectors under the **binary** operation of $+$.

So $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$ for vectors

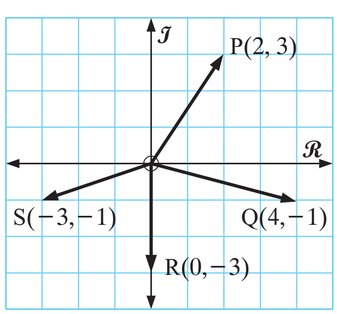
and $(a + bi) + (c + di) = (a + c) + (b + d)i$ for complex numbers

and $\begin{bmatrix} a+c \\ b+d \end{bmatrix} \equiv (a + c) + (b + d)i$

Note: $\equiv$ means "is equivalent to" or "corresponds to"

The **plane of complex numbers (complex plane**, or **Argand plane**), has a horizontal **real axis** and a vertical **imaginary axis**.

Notice that:



$\overrightarrow{\text{OP}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ represents $2 + 3i$

$\overrightarrow{\text{OQ}} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ represents $4 - i$

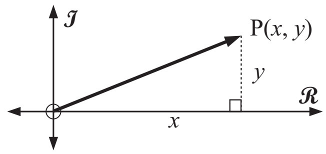
$\overrightarrow{\text{OR}} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$ represents $-3i$

$\overrightarrow{\text{OS}} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ represents $-3 - i$

$\mathcal{R}$ is the real axis, $\mathcal{I}$ is the imaginary axis.

Note: $a + bi$ is called the **Cartesian form** of a complex number as (a, b) is easily plotted on the **Cartesian plane**.

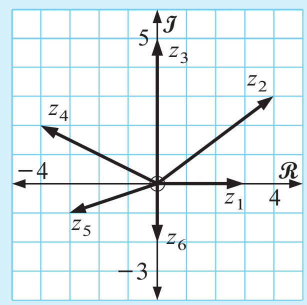
In general,



$\overrightarrow{OP} = \begin{bmatrix} x \\ y \end{bmatrix}$ represents $x + yi$

Example 1

Illustrate the position of $z_1 = 3$, $z_2 = 4 + 3i$, $z_3 = 5i$, $z_4 = -4 + 2i$, $z_5 = -3 - i$ and $z_6 = -2i$ in the complex plane.



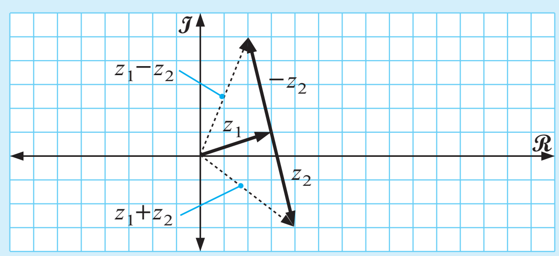
Example 2

If $z_1 = 3 + i$ and $z_2 = 1 - 4i$ find algebraically and vectorially:

a $z_1 + z_2$ **b** $z_1 - z_2$

a $z_1 + z_2$
$= 3 + i + 1 - 4i$
$= 4 - 3i$

b $z_1 - z_2$
$= 3 + i - (1 - 4i)$
$= 3 + i - 1 + 4i$
$= 2 + 5i$



Reminder:
Draw z_1 first and at its arrow end draw z_2.
$z_1 + z_2$ goes from the start of z_1 to the end of z_2.

EXERCISE 16A.1

1 On an Argand diagram, illustrate the complex numbers:

a $z_1 = 5$ **b** $z_2 = -1 + 2i$ **c** $z_3 = -6 - 2i$
d $z_4 = -6i$ **e** $z_5 = 2 - i$ **f** $z_6 = 4i$

2 If $z = 1 + 2i$ and $w = 3 - i$, find both algebraically and vectorially:

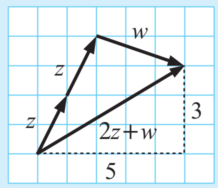
a $z + w$ **b** $z - w$ **c** $2z - w$ **d** $w - 3z$

Example 3

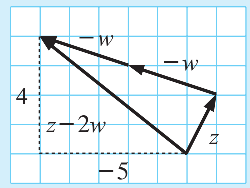
If $z = 1 + 2i$ and $w = 3 - i$, find both algebraically and vectorially:

a $2z + w$ **b** $z - 2w$

a $2z + w$
$= 2(1 + 2i) + 3 - i$
$= 2 + 4i + 3 - i$
$= 5 + 3i$



b $z - 2w$
$= 1 + 2i - 2(3 - i)$
$= 1 + 2i - 6 + 2i$
$= -5 + 4i$



3 If $z_1 = 4 - i$ and $z_2 = 2 + 3i$ find both algebraically and vectorially:

a $z_1 + 1$ **b** $z_1 + 2i$ **c** $z_2 + \frac{1}{2}z_1$ **d** $\dfrac{z_1 + 4}{2}$

4 If z is any complex number, explain with illustration how to find geometrically:

a $3z$ **b** $-2z$ **c** z^* **d** $3i - z$

e $2 - z$ **f** $z^* + i$ **g** $\dfrac{z+2}{3}$ **h** $\dfrac{z-4}{2}$

REPRESENTING CONJUGATES

If $z = x + iy$, then $z^* = x - iy$.

This means that if $\overrightarrow{OP} = [x, y]$ represents z then $\overrightarrow{OQ} = [x, -y]$ represents z^*.

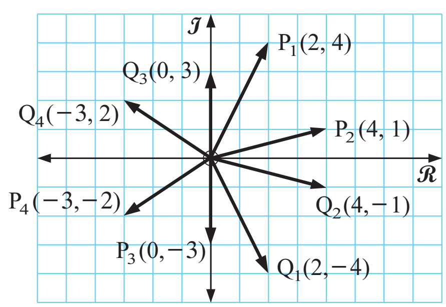
For example:

$\overrightarrow{OP_1}$ represents $2 + 4i$ and

$\overrightarrow{OQ_1}$ represents $2 - 4i$

$\overrightarrow{OP_3}$ represents $-3i$ and

$\overrightarrow{OQ_3}$ represents $3i$ etc.



If z is $\overrightarrow{OP}$, its conjugate z^* is $\overrightarrow{OQ}$ where $\overrightarrow{OQ}$ is a reflection of $\overrightarrow{OP}$ in the real axis.

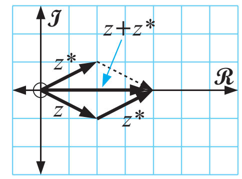
EXERCISE 16A.2

1 Show on an Argand diagram:

a $z = 3 + 2i$ and its conjugate $z^* = 3 - 2i$

b $z = -2 + 5i$ and its conjugate $z^* = -2 - 5i$

2 If $z = 2 - i$ we can add $z + z^*$ as shown in the diagram.
We notice that $z + z^*$ is 4 which is real.
Explain, by illustration, that $z + z^*$ is always real.



3 Explain, by illustration, that $z - z^*$ is always purely imaginary or zero. What distinguishes these two cases?

4 If z is real, what is z^*?

B MODULUS, ARGUMENT, POLAR FORM

MODULUS

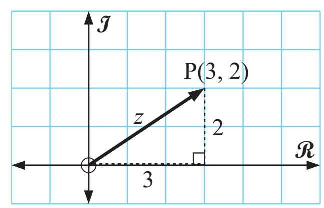
The modulus of the complex number z, written $|z|$, is the magnitude of the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ where $z = a + bi$. So:

The **modulus** of the complex number $z = a + bi$ is the real number $\sqrt{a^2 + b^2}$, and we write $|z| = \sqrt{a^2 + b^2}$ to represent the modulus of z.

Consider the complex number $z = 3 + 2i$.

The distance from P to O is its modulus, $|z|$.

So, $|z| = \sqrt{3^2 + 2^2}$ {Pythagoras}



Example 4

Find $|z|$ for z equal to:

a $3 + 2i$ **b** $3 - 2i$ **c** $-3 - 2i$

a $|z|$
$= \sqrt{3^2 + 2^2}$
$= \sqrt{13}$

b $|z|$
$= \sqrt{3^2 + (-2)^2}$
$= \sqrt{9 + 4}$
$= \sqrt{13}$

c $|z|$
$= \sqrt{(-3)^2 + (-2)^2}$
$= \sqrt{9 + 4}$
$= \sqrt{13}$

Note:
- $|z| = \sqrt{(\text{real part of } z)^2 + (\text{imaginary part of } z)^2}$, and is a positive real number.
- $|z|$ gives the distance of (a, b) from the origin if $z = a + bi$.

This is consistent with an earlier definition given for $|x|$ where $x \in \mathcal{R}$ as in **Chapter 8G**.

EXERCISE 16B.1

1 Find $|z|$ for z equal to:

a $3 - 4i$ **b** $5 + 12i$ **c** $-8 + 2i$ **d** $3i$ **e** -4

2 If $z = 2 + i$ and $w = -1 + 3i$ find:

a $\lvert z\rvert$	**b** $\lvert z^*\rvert$	**c** $\lvert z^*\rvert^2$	**d** zz^*
e $\lvert zw\rvert$	**f** $\lvert z\rvert\,\lvert w\rvert$	**g** $\left\lvert \dfrac{z}{w}\right\rvert$	**h** $\dfrac{\lvert z\rvert}{\lvert w\rvert}$
i $\lvert z^2\rvert$	**j** $\lvert z\rvert^2$	**k** $\lvert z^3\rvert$	**l** $\lvert z\rvert^3$

3 From **2**, suggest *five* possible rules for modulus.

4 If $z = a + bi$ is a complex number show that: **a** $|z^*| = |z|$ **b** $|z|^2 = zz^*$

Example 5

Prove that $|z_1 z_2| = |z_1| \times |z_2|$ for all complex numbers z_1 and z_2.

Let $z_1 = a + bi$ and $z_2 = c + di$ where a, b, c and d are real

$$\begin{aligned} \therefore\quad z_1 z_2 &= (a+bi)(c+di) \\ &= [ac - bd] + i[ad + bc] \end{aligned}$$

$$\begin{aligned} \text{Thus}\quad |z_1 z_2| &= \sqrt{(ac-bd)^2 + (ad+bc)^2} \\ &= \sqrt{a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2} \\ &= \sqrt{a^2(c^2+d^2) + b^2(c^2+d^2)} \\ &= \sqrt{(c^2+d^2)(a^2+b^2)} \\ &= \sqrt{a^2+b^2} \times \sqrt{c^2+d^2} \\ &= |z_1| \times |z_2| \end{aligned}$$

5 Use the result $|z_1 z_2| = |z_1|\,|z_2|$ to show that:

a $|z_1 z_2 z_3| = |z_1|\,|z_2|\,|z_3|$ and that $|z^3| = |z|^3$

b $|z_1 z_2 z_3 z_4| = |z_1|\,|z_2|\,|z_3|\,|z_4|$ and that $|z^4| = |z|^4$.

6 What is the generalisation of the results in **5**?

7 Simplify $\left|\dfrac{z}{w}\right| \times |w|$ using the result of **Example 5** and use it to show that

$\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ provided that $w \neq 0$.

8 Given $|z| = 3$, use the rules $|zw| = |z|\,|w|$ and $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ to find:

a $\lvert 2z\rvert$	**b** $\lvert -3z\rvert$	**c** $\lvert (1+2i)z\rvert$
d $\lvert iz\rvert$	**e** $\left\lvert \dfrac{1}{z}\right\rvert$	**f** $\left\lvert \dfrac{2i}{z^2}\right\rvert$

9 If $z = \cos\theta + i\sin\theta$, find $|z|$.

10 Use the result of **6** to find $|z^{20}|$ for $z = 1 - i\sqrt{3}$.

11 **a** If $w = \dfrac{z+1}{z-1}$ where $z = a + bi$, find w in the form $X + Yi$ when X and Y involve a and b.

b If $w = \dfrac{z+1}{z-1}$ and $|z| = 1$, find $\text{Re}(w)$.

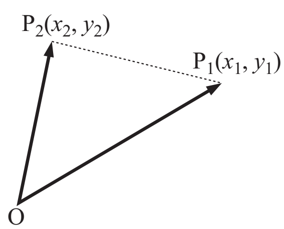
12 Use the Principle of mathematical induction to prove that:

a $|z_1 z_2 z_3 .... z_n| = |z_1|\,|z_2|\,|z_3| ..... |z_n|, \ n \in Z^+$ **b** $|z^n| = |z|^n, \ n \in Z^+$

SUMMARY OF MODULUS DISCOVERIES

- $|z^*| = |z|$ • $|z|^2 = zz^*$
- $|z_1 z_2| = |z_1|\,|z_2|$ and $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ provided $z_2 \neq 0$
- $|z_1 z_2 z_3 .... z_n| = |z_1|\,|z_2|\,|z_3| ..... |z_n|$ and $|z^n| = |z|^n$ for n a positive integer.

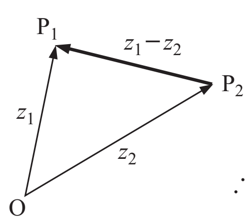
DISTANCE IN THE NUMBER PLANE



Suppose P_1 and P_2 are two points in the complex plane which correspond to z_1 represented by $\overrightarrow{OP_1}$ and z_2 represented by $\overrightarrow{OP_2}$.

Now $|z_1 - z_2| = |(x_1 + y_1 i) - (x_2 + y_2 i)|$
$= |(x_1 - x_2) + (y_1 - y_2)i|$
$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ which we recognise as the distance between P_1 and P_2.

Alternatively:



$\overrightarrow{P_2P_1} = \overrightarrow{P_2O} + \overrightarrow{OP_1}$
$= -z_2 + z_1$
$= z_1 - z_2$

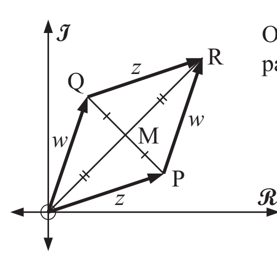
$\therefore \ |z_1 - z_2| = \left|\overrightarrow{P_2P_1}\right| =$ distance between P_1 and P_2.

Thus,

$|z_1 - z_2|$ is the distance between points P_1 and P_2, where $z_1 \equiv \overrightarrow{OP_1}$ and $z_2 \equiv \overrightarrow{OP_2}$.

Note: The point corresponding to $z_1 - z_2$ can be found by drawing a vector equal to $\overrightarrow{P_2P_1}$ emanating (starting) from the origin. Can you explain why?

CONNECTION TO COORDINATE GEOMETRY



OPRQ is a parallelogram

There is a clear connection between complex numbers, vector geometry and coordinate geometry.

Consider the following:

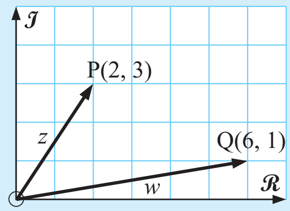
Notice that $\overrightarrow{OR} \equiv w + z$ and $\overrightarrow{OM} \equiv \dfrac{w+z}{2}$

as the diagonals of the parallelogram bisect each other.

$\overrightarrow{OR}$ and $\overrightarrow{PQ}$ give the diagonals of the parallelogram formed by w and z.

Example 6

P(2, 3) and Q(6, 1) are two points on the Cartesian plane. Use complex numbers to find **a** distance PQ **b** the midpoint of PQ.



a If $z = 2 + 3i$ and $w = 6 + i$

then $z - w = 2 + 3i - 6 - i$
$= -4 + 2i$

$\therefore\ |z - w| = \sqrt{(-4)^2 + 2^2} = \sqrt{20}$

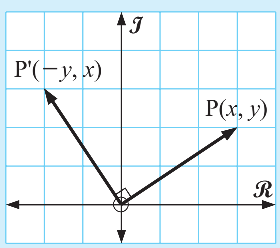
$\therefore$ PQ $= \sqrt{20}$ units

b $\dfrac{z + w}{2} = \dfrac{2 + 3i + 6 + i}{2} = 4 + 2i$

$\therefore$ the midpoint of PQ is (4, 2).

Example 7

What transformation moves z to iz?



If $z = x + iy$

then $iz = i(x + iy)$
$= xi + i^2y$
$= -y + xi$

we notice that $|z| = \sqrt{x^2 + y^2}$

and $|iz| = \sqrt{(-y)^2 + x^2}$
$= \sqrt{x^2 + y^2}$

So $\text{OP}' = \text{OP}$

$(x,\ y) \to (-y,\ x)$ under an anti-clockwise rotation of $\frac{\pi}{2}$ about O.

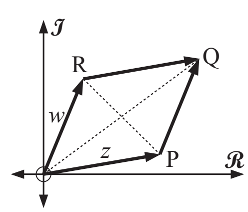
The transformation found in **Example 7** can be found more easily later. (See **Example 11**.)

EXERCISE 16B.2

1 Use complex numbers to find: **i** distance AB **ii** the midpoint of AB for

a A(3, 6) and B(−1, 2) **b** A(−4, 7) and B(1, −3)

2 OPQR is the parallelogram as shown. $\overrightarrow{\text{OP}}$ represents z and $\overrightarrow{\text{OR}}$ represents w where z and w are complex numbers.



a In terms of z and w, what are:

i $\overrightarrow{\text{OQ}}$ **ii** $\overrightarrow{\text{PR}}$?

b Explain from triangle OPQ, why $|z+w| \leqslant |z|+|w|$.
It is important to discuss when the equality case occurs.

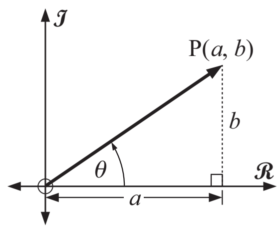
c Explain from triangle OPR, why $|z-w| \geqslant |w|-|z|$.
Once again discuss when the equality case occurs.

3 What transformation moves:

a z to z^* **b** z to $-z^*$ **c** z to $-z$ **d** z to $-iz$?

ARGUMENT

The direction of the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ can be described in a number of ways. One way is:



Suppose the complex number $z=a+bi$ is represented by vector $\overrightarrow{\text{OP}}$ as shown alongside.

$a+bi$ is the **Cartesian form** of z.

Suppose also that $\overrightarrow{\text{OP}}$ makes an angle of θ with the **positive real axis**.

The angle θ is called the **argument** of z, or simply $\arg z$.

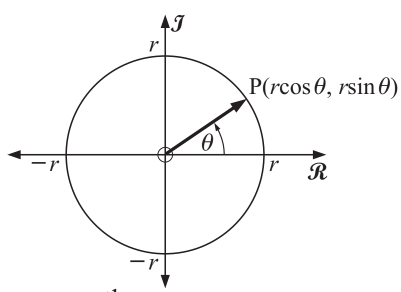
$\arg z=\theta$ has infinitely many possibilities i.e., $z \longmapsto \arg z=\theta$ is one-to-many and is not a function. Can you explain why?

To avoid the infinite number of possibilities for θ, we may choose to use $\theta \in]-\pi, \pi]$ which covers one full revolution and guarantees that $z \longmapsto \arg z=\theta$ is a function.

Note:
- Real numbers have argument of 0 *or* π.
- Pure imaginary numbers have argument of $\frac{\pi}{2}$ *or* $-\frac{\pi}{2}$.

POLAR FORM

Polar form is an alternative to Cartesian form, with many useful applications.



As P lies on a circle with centre O(0, 0) and radius r, its coordinates are $(r\cos\theta, r\sin\theta)$.

So, $z=r\cos\theta+ir\sin\theta$

i.e., $z=r(\cos\theta+i\sin\theta)$

But $r=|z|$ and if we represent $\cos\theta+i\sin\theta$ as $\text{cis}\,\theta$

then $z=|z|\,\text{cis}\,\theta$.

Consequently:

Any complex number z has **Cartesian form** $z=x+yi$ or **polar form** $z=|z|\,\text{cis}\,\theta$ where $|z|$ is the **modulus** of z, θ is the **argument** of z and $\text{cis}\,\theta=\cos\theta+i\sin\theta$.

As we develop this section we will observe that polar form is extremely powerful for dealing with multiplication and division of complex numbers as well as quickly finding powers and roots of numbers (see De Moivre's theorem). (Roots are really powers, why?)

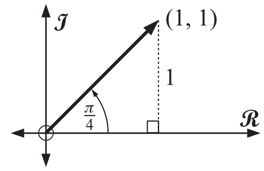
z can also be written as $z=re^{i\theta}$ (**Euler form**) where $r=|z|$ and $\theta=\arg z$.

This form is discovered in **Chapter 29**.

Consider the complex number $z = 1 + i$.

$|z| = \sqrt{2}$ and $\theta = \frac{\pi}{4}$ $\therefore$ $1 + i = \sqrt{2}\ \text{cis}\left(\frac{\pi}{4}\right)$

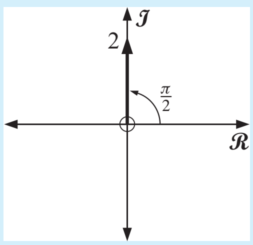
So, $\sqrt{2}\ \text{cis}\left(\frac{\pi}{4}\right)$ is the polar form of $1 + i$.



Example 8

Write in polar form: **a** $2i$ **b** -3 **c** $1 - i$

a

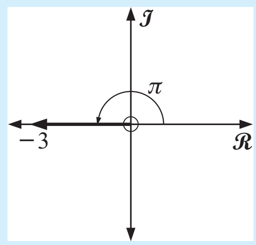


$|2i| = 2$

$\theta = \frac{\pi}{2}$

$\therefore$ $2i = 2\ \text{cis}\ \frac{\pi}{2}$

b

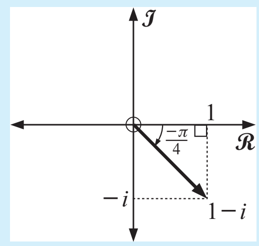


$|-3| = 3$

$\theta = \pi$

$\therefore$ $-3 = 3\ \text{cis}\ \pi$

c



$|1 - i| = \sqrt{1 + 1} = \sqrt{2}$

$\theta = -\frac{\pi}{4}$

$\therefore$ $1 - i = \sqrt{2}\ \text{cis}\left(-\frac{\pi}{4}\right)$

EXERCISE 16B.3

1 Find the modulus and argument of the following complex numbers and hence write them in polar form:

a 4 **b** $2i$ **c** -6 **d** $-3i$

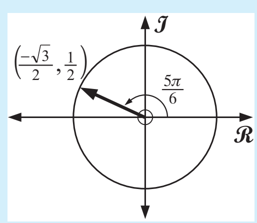
e $1 + i$ **f** $2 - 2i$ **g** $-\sqrt{3} + i$ **h** $2\sqrt{3} + 2i$

2 What complex number cannot be written in polar form? Why?

3 Convert $k + ki$ to polar form. [Careful! You must consider $k > 0$, $k = 0$, $k < 0$.]

Example 9

Convert $\sqrt{3}\ \text{cis}\left(\frac{5\pi}{6}\right)$ to Cartesian form.



$\sqrt{3}\ \text{cis}\left(\frac{5\pi}{6}\right)$

$= \sqrt{3}\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$

$= \sqrt{3}\left[-\frac{\sqrt{3}}{2} + i \times \frac{1}{2}\right]$

$= -\frac{3}{2} + \frac{\sqrt{3}}{2}i$

4 Convert to Cartesian form:

a $2\ \text{cis}\left(\frac{\pi}{2}\right)$ **b** $8\ \text{cis}\left(\frac{\pi}{4}\right)$ **c** $4\ \text{cis}\left(\frac{\pi}{6}\right)$

d $\sqrt{2}\ \text{cis}\left(-\frac{\pi}{4}\right)$ **e** $\sqrt{3}\ \text{cis}\left(\frac{2\pi}{3}\right)$ **f** $5\ \text{cis}\ \pi$

5 **a** Find the value of cis 0. **b** Find the modulus of $\text{cis}\ \theta$, i.e., $|\text{cis}\ \theta|$.

c Show that $\text{cis}\ \alpha\ \text{cis}\ \beta = \text{cis}\ (\alpha + \beta)$.

MULTIPLYING AND DIVIDING IN POLAR FORM

Cis θ has three useful properties. These are:

- $\text{cis }\theta \times \text{cis }\phi = \text{cis }(\theta+\phi)$
- $\dfrac{\text{cis }\theta}{\text{cis }\phi} = \text{cis }(\theta-\phi)$
- $\text{cis }(\theta + k2\pi) = \text{cis }\theta$ for all integers k.

The first two of these are similar to index laws: $a^\theta a^\phi = a^{\theta+\phi}$ and $\dfrac{a^\theta}{a^\phi} = a^{\theta-\phi}$.

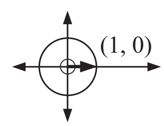
The three properties are easily proved:

Proof:

- $$\begin{aligned}\text{cis }\theta \times \text{cis }\phi &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\\ &= [\cos\theta\cos\phi - \sin\theta\sin\phi] + i[\sin\theta\cos\phi + \cos\theta\sin\phi]\\ &= \cos(\theta+\phi) + i\sin(\theta+\phi)\\ &= \text{cis }(\theta+\phi)\end{aligned}$$

- $$\begin{aligned}\frac{\text{cis }\theta}{\text{cis }\phi} &= \frac{\text{cis }\theta}{\text{cis }\phi} \times \frac{\text{cis }(-\phi)}{\text{cis }(-\phi)}\\ &= \frac{\text{cis }(\theta-\phi)}{\text{cis }0}\\ &= \text{cis }(\theta-\phi) \qquad \{\text{as } \text{cis }0 = 1\}\end{aligned}$$

- $$\begin{aligned}\text{cis }(\theta + k2\pi) &= \text{cis }\theta \times \text{cis }(k2\pi)\\ &= \text{cis }\theta \times 1\\ &= \text{cis }\theta\end{aligned}$$

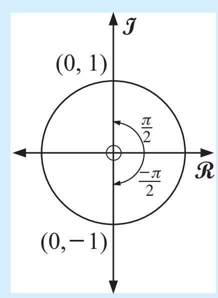


Example 10

Use the properties of cis to simplify: **a** $\text{cis }\left(\frac{\pi}{5}\right)\text{cis }\left(\frac{3\pi}{10}\right)$ **b** $\dfrac{\text{cis }\left(\frac{\pi}{5}\right)}{\text{cis }\left(\frac{7\pi}{10}\right)}$

a
$$\begin{aligned}&\text{cis }\left(\tfrac{\pi}{5}\right)\text{cis }\left(\tfrac{3\pi}{10}\right)\\ &= \text{cis }\left(\tfrac{\pi}{5} + \tfrac{3\pi}{10}\right)\\ &= \text{cis }\left(\tfrac{5\pi}{10}\right)\\ &= \text{cis }\tfrac{\pi}{2}\\ &= \cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}\\ &= 0 + i(1)\\ &= i\end{aligned}$$

b
$$\begin{aligned}&\frac{\text{cis }\left(\frac{\pi}{5}\right)}{\text{cis }\left(\frac{7\pi}{10}\right)}\\ &= \text{cis }\left(\tfrac{\pi}{5} - \tfrac{7\pi}{10}\right)\\ &= \text{cis }\left(-\tfrac{\pi}{2}\right)\\ &= \cos\left(-\tfrac{\pi}{2}\right) + i\sin\left(-\tfrac{\pi}{2}\right)\\ &= 0 + i(-1)\\ &= -i\end{aligned}$$



Example 11

What transformation moves z to iz?

(see **Example 7** earlier)

Let $z = r \text{ cis }\theta$, $i = 1 \text{ cis }\frac{\pi}{2}$.

$$\begin{aligned}\therefore\quad iz &= r \text{ cis }\theta \times \text{cis }\tfrac{\pi}{2}\\ &= r \text{ cis }\left(\theta + \tfrac{\pi}{2}\right)\end{aligned}$$

So, z has been rotated anti-clockwise by $\frac{\pi}{2}$ about O.

EXERCISE 16B.4

1 Use the properties of cis to simplify:

a $\text{cis}\,\theta\ \text{cis}\,2\theta$ **b** $\dfrac{\text{cis}\,3\theta}{\text{cis}\,\theta}$ **c** $[\text{cis}\,\theta]^3$

d $\text{cis}\left(\frac{\pi}{18}\right)\text{cis}\left(\frac{\pi}{9}\right)$ **e** $2\,\text{cis}\left(\frac{\pi}{12}\right)\text{cis}\left(\frac{\pi}{6}\right)$ **f** $2\,\text{cis}\left(\frac{2\pi}{5}\right)\times 4\,\text{cis}\left(\frac{8\pi}{5}\right)$

g $\dfrac{4\,\text{cis}\left(\frac{\pi}{12}\right)}{2\,\text{cis}\left(\frac{7\pi}{12}\right)}$ **h** $\dfrac{\sqrt{32}\,\text{cis}\left(\frac{\pi}{8}\right)}{\sqrt{2}\,\text{cis}\left(-\frac{7\pi}{8}\right)}$ **i** $\left[\sqrt{2}\,\text{cis}\left(\frac{\pi}{8}\right)\right]^4$

Example 12

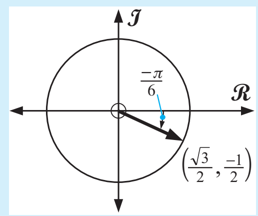
Simplify $\text{cis}\left(\frac{107\pi}{6}\right)$.

$\frac{107\pi}{6} = 17\frac{5}{6}\pi = 18\pi - \frac{\pi}{6}$

$\therefore\ \text{cis}\left(\frac{107\pi}{6}\right) = \text{cis}\left(18\pi - \frac{\pi}{6}\right)$

$= \text{cis}\left(-\frac{\pi}{6}\right)$ $\{\text{cis}\,(\theta + k2\pi) = \text{cis}\,\theta\}$

$= \frac{\sqrt{3}}{2} - \frac{1}{2}i$



2 Use the property $\text{cis}\,(\theta + k2\pi) = \text{cis}\,\theta$ to evaluate:

a $\text{cis}\,17\pi$ **b** $\text{cis}\,(-37\pi)$ **c** $\text{cis}\left(\frac{91\pi}{3}\right)$

3 If $z = 2\,\text{cis}\,\theta$:

a What is $|z|$ and $\arg z$?

b Write z^* in polar form.

c Write $-z$ in polar form. [**Note:** $-2\,\text{cis}\,\theta$ is not in polar form as the coefficient of $\text{cis}\,\theta$ must be positive, as it is a length.]

d Write $-z^*$ in polar form.

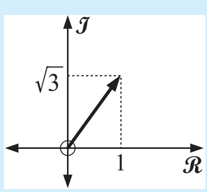
Example 13

a Write $z = 1 + \sqrt{3}i$ in polar form and then multiply it by $2\,\text{cis}\left(\frac{\pi}{6}\right)$.

b Illustrate what has happened on an Argand diagram.

c What transformations have taken place when multiplying by $2\,\text{cis}\left(\frac{\pi}{6}\right)$?

a



If $z = 1 + \sqrt{3}i$, then $|z| = \sqrt{1^2 + (\sqrt{3})^2}$
$= 2$

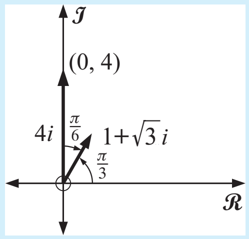
$\therefore\ z = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

i.e., $z = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

$z = 2\,\text{cis}\,\frac{\pi}{3}$

$$(1+\sqrt{3}i)\times 2\text{ cis }\tfrac{\pi}{6} = 2\text{ cis }\left(\tfrac{\pi}{3}\right)\times 2\text{ cis }\left(\tfrac{\pi}{6}\right)$$
$$= 4\text{ cis }\left(\tfrac{\pi}{3}+\tfrac{\pi}{6}\right)$$
$$= 4\text{ cis }\left(\tfrac{\pi}{2}\right)$$
$$= 4(0+1i)$$
$$= 4i$$

b



c When z was multiplied by $2\text{ cis }\left(\frac{\pi}{6}\right)$ its modulus (length) was doubled and it was rotated through $\frac{\pi}{6}$.

Note: Multiplying by r cis θ dilates the original complex number's vector representation by a factor of r and then rotates it through an angle of θ.

4 **a** Write i in polar form.
b $z = r$ cis θ is any complex number. Write iz in polar form.
c Explain why iz is the anti-clockwise rotation about 0 of z.
d What transformation maps z onto $-iz$? Give reasoning in polar form.

5 Write in polar form: **a** $\cos\theta - i\sin\theta$ **b** $\sin\theta - i\cos\theta$

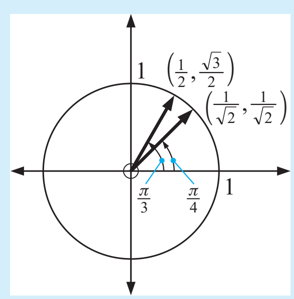
Use **a** above to complete this sentence:

If $z = r$ cis θ then $z^* = ...$ in polar form. Discuss.

Example 14

Use complex number methods to write $\cos\left(\frac{7\pi}{12}\right)$ and $\sin\left(\frac{7\pi}{12}\right)$ in simplest surd form.

$$\cos\left(\tfrac{7\pi}{12}\right)+i\sin\left(\tfrac{7\pi}{12}\right)$$
$$= \text{cis}\left(\tfrac{7\pi}{12}\right)$$
$$= \text{cis}\left(\tfrac{3\pi}{12}+\tfrac{4\pi}{12}\right)$$
$$= \text{cis }\tfrac{\pi}{4}\times\text{cis }\tfrac{\pi}{3} \qquad \{\text{cis }(\theta+\phi)=\text{cis }\theta\times\text{cis }\phi\}$$
$$= \left(\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right)\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)$$
$$= \left(\tfrac{1}{2\sqrt{2}}-\tfrac{\sqrt{3}}{2\sqrt{2}}\right)+i\left(\tfrac{\sqrt{3}}{2\sqrt{2}}+\tfrac{1}{2\sqrt{2}}\right)$$



Equating real parts: $\cos\frac{7\pi}{12} = \left(\frac{1-\sqrt{3}}{2\sqrt{2}}\right)\times\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$

Equating imaginary parts: $\sin\frac{7\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$

6 Use the method outlined above to find, in simplest surd form:

a $\cos\left(\frac{\pi}{12}\right)$ and $\sin\left(\frac{\pi}{12}\right)$ **b** $\cos\left(\frac{11\pi}{12}\right)$ and $\sin\left(\frac{11\pi}{12}\right)$.

PROPERTIES OF ARGUMENT

The basic properties of argument are:

- $\arg(zw) = \arg z + \arg w$
- $\arg\left(\dfrac{z}{w}\right) = \arg z - \arg w$
- $\arg(z^n) = n \arg z$

Notice that they are *identical to the laws of logarithms*, with arg replaced by log or ln.

Properties of modulus and argument can be proved jointly using polar form.

These properties lead to another form of expressing complex numbers, called Euler's form (see **Chapter 29**).

Example 15

Use polar form to establish that $|zw| = |z| \times |w|$ and $\arg(zw) = \arg z + \arg w$.

Let $z = |z| \text{ cis } \theta$ and $w = |w| \text{ cis } \phi$, say.

Now $zw = |z| \text{ cis } \theta \times |w| \text{ cis } \phi$

$= \underbrace{|z|\,|w|}_{\text{non-negative}} \text{ cis } (\theta + \phi)$ {property of cis}

$\therefore$ $|zw| = |z|\,|w|$ {the non-negative number multiplied by cis (.....)}

and $\arg(zw) = \theta + \phi = \arg z + \arg w$.

Example 16

If $z = \sqrt{2} \text{ cis } \theta$, find the modulus and argument of:

a $2z$ **b** iz **c** $(1-i)z$

a $2z = 2\sqrt{2} \text{ cis } \theta$ $\therefore$ $|2z| = 2\sqrt{2}$ and $\arg(2z) = \theta$

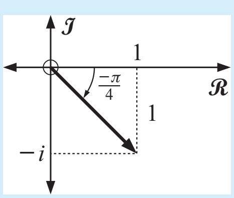
b $i = \text{cis } \frac{\pi}{2}$

$\therefore$ $iz = \text{cis } \frac{\pi}{2} \times \sqrt{2} \text{ cis } \theta$

$= \sqrt{2} \text{ cis } \left(\frac{\pi}{2} + \theta\right)$ So, $|iz| = \sqrt{2}$

and $\arg(iz) = \frac{\pi}{2} + \theta$

c



$1 - i = \sqrt{2} \text{ cis } \left(-\frac{\pi}{4}\right)$

$\therefore$ $(1-i)z = \sqrt{2} \text{ cis } \left(-\frac{\pi}{4}\right) \times \sqrt{2} \text{ cis } \theta$

$= 2 \text{ cis } \left(-\frac{\pi}{4} + \theta\right)$

$\therefore$ $|(1-i)z| = 2$ and $\arg((1-i)z) = \theta - \frac{\pi}{4}$

EXERCISE 16B.5

1 Use polar form to establish:

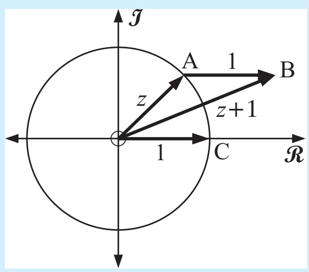
$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} \quad \text{and} \quad \arg\left(\frac{z}{w}\right) = \arg z - \arg w, \quad \text{provided} \quad w \neq 0.$$

2 Suppose $z = 3 \text{ cis } \theta$. Determine the modulus and argument of:

a $-z$ **b** z^* **c** iz **d** $(1+i)z$

Example 17

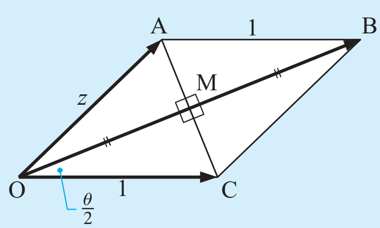
Suppose $z = \text{cis } \phi$ where ϕ is acute. Find the modulus and argument of $z + 1$.



$|z| = 1$

$\therefore$ z lies on the unit circle

$z + 1$ is $\overrightarrow{OB}$ {found vectorially}



OABC is a rhombus

$\therefore$ $\arg(z+1) = \frac{\theta}{2}$

{diagonals bisect the angles of the rhombus}

Also $\cos\left(\frac{\theta}{2}\right) = \frac{\text{OM}}{1}$

$\therefore$ $\text{OM} = \cos\left(\frac{\theta}{2}\right)$

$\therefore$ $\text{OB} = 2\cos\left(\frac{\theta}{2}\right)$

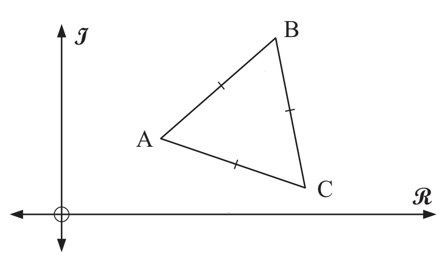
$\therefore$ $|z+1| = 2\cos\left(\frac{\theta}{2}\right)$

3 **a** If $z = \text{cis } \phi$ where ϕ is acute, determine the modulus and argument of $z - 1$.

b Using **a**, write $z - 1$ in polar form.

c Now write $(z-1)^*$ in polar form.

4 ABC is an equilateral triangle. Suppose z_1 represents $\overrightarrow{OA}$, z_2 represents $\overrightarrow{OB}$ and z_3 represents $\overrightarrow{OC}$.



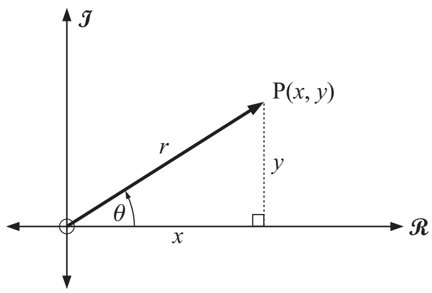
a Explain what vectors represent $z_2 - z_1$ and $z_3 - z_2$.

b Find $\left|\frac{z_2 - z_1}{z_3 - z_2}\right|$.

c Determine $\arg\left(\frac{z_2 - z_1}{z_3 - z_2}\right)$.

d Use **b** and **c** to find the value of $\left(\frac{z_2 - z_1}{z_3 - z_2}\right)^3$.

FURTHER CONVERSION BETWEEN CARTESIAN AND POLAR FORMS



If $\underbrace{z = x + iy}_{\text{Cartesian form}} = \underbrace{r \text{ cis } \theta}_{\text{Polar form}}$

$r = \sqrt{x^2 + y^2}$, $\tan\theta = \dfrac{y}{x}$,

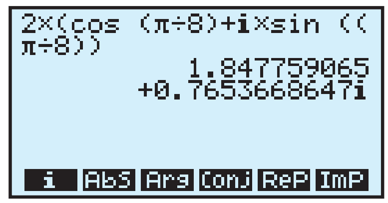
$\cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$, $\sin\theta = \dfrac{y}{\sqrt{x^2+y^2}}$

Polar to cartesian

$z = 2 \text{ cis } \left(\frac{\pi}{8}\right)$

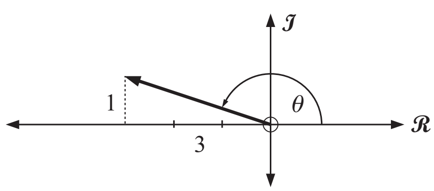
$= 2\left(\cos\frac{\pi}{8} + i \sin\frac{\pi}{8}\right)$

$\doteqdot 1.85 + 0.765i$



Cartesian to polar

$z = -3 + i$ has $r = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$ and $\cos\theta = \dfrac{-3}{\sqrt{10}}$, $\sin\theta = \dfrac{1}{\sqrt{10}}$



$\theta = \pi - \sin^{-1}\left(\frac{1}{\sqrt{10}}\right)$

$\left(or \quad \pi - \cos^{-1}\left(\frac{3}{\sqrt{10}}\right)\right)$

$\therefore \quad -3 + i \doteqdot \sqrt{10} \text{ cis } (2.82)$

Alternatively:

MATH CPX takes us to the complex number menu.

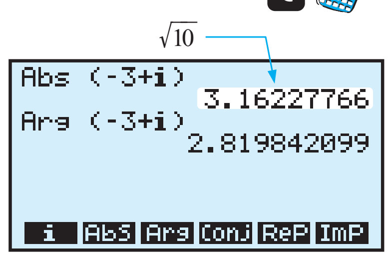
Pressing 5 brings up **abs(** for calculating the **modulus** (absolute value).

To find the modulus of $-3 + i$, press

(−) 3 + 2nd i) ENTER

MATH CPX then 4 brings up **angle(** for calculating the **argument**. To find the argument of $-3 + i$, press

(−) 3 + 2nd i) ENTER



EXERCISE 16B.6

1 Use your calculator to convert to Cartesian form:

a $\sqrt{3} \text{ cis } (2.5187)$ **b** $\sqrt{11} \text{ cis } \left(-\frac{3\pi}{8}\right)$ **c** $2.836\,49 \text{ cis } (-2.684\,32)$

2 Use your calculator to convert to polar form:

a $3-4i$ **b** $-5-12i$ **c** $-11.6814+13.2697i$

3 Add the following using $a+bi$ surd form and convert your answer to polar form:

a $3 \text{ cis}\left(\frac{\pi}{4}\right)+\text{cis}\left(\frac{-3\pi}{4}\right)$ **b** $2 \text{ cis}\left(\frac{2\pi}{3}\right)+5 \text{ cis}\left(\frac{-2\pi}{3}\right)$

4 Use the sum and product of roots to find the real quadratic equations with roots of:

a $2 \text{ cis}\left(\frac{2\pi}{3}\right),\ 2 \text{ cis}\left(\frac{4\pi}{3}\right)$ **b** $\sqrt{2} \text{ cis}\left(\frac{\pi}{4}\right),\ \sqrt{2} \text{ cis}\left(\frac{-\pi}{4}\right)$

Note: Answers, if approximate, should be given to 3 significant figures.

C De MOIVRE'S THEOREM

Squaring a complex number which is given in polar form gives us an indication of how we can find higher powers of that number.

We notice that if $z=|z| \text{ cis } \theta$

then
$$\begin{aligned} z^2 &= |z| \text{ cis } \theta \times |z| \text{ cis } \theta \\ &= |z|^2 \text{ cis } (\theta+\theta) \\ &= |z|^2 \text{ cis } 2\theta \end{aligned}$$
and
$$\begin{aligned} z^3 &= z^2 z \\ &= |z|^2 \text{ cis } 2\theta \times |z| \text{ cis } \theta \\ &= |z|^3 \text{ cis } (2\theta+\theta) \\ &= |z|^3 \text{ cis } 3\theta \end{aligned}$$

The generalisation of this process is: $(|z| \text{ cis } \theta)^n = |z|^n \text{ cis } n\theta$

Proof: (using mathematical induction)

P_n is: $(|z| \text{ cis } \theta)^n = |z|^n \text{ cis } n\theta$

(1) If $n=1$, then $(|z| \text{ cis } \theta)^1 = |z| \text{ cis } 1(\theta)$ $\therefore$ P_1 is true.

(2) If P_k is true, then $(|z| \text{ cis } \theta)^k = |z|^k \text{ cis } k\theta$ ...... $(*)$

$$\begin{aligned} \text{Thus } (|z| \text{ cis } \theta)^{k+1} &= (|z| \text{ cis } \theta)^k \times |z| \text{ cis } \theta && \{\text{index law}\} \\ &= |z|^k \text{ cis } k\theta \times |z| \text{ cis } \theta && \{\text{using } *\} \\ &= |z|^{k+1} \text{ cis } (k\theta+\theta) && \{\text{index law and cis property}\} \\ &= |z|^{k+1} \text{ cis } (k+1)\,\theta \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore$ P_n is true {Principle of mathematical induction}

We observe also that $\text{cis } (-n\theta) = \text{cis } (0-n\theta) = \dfrac{\text{cis } 0}{\text{cis } n\theta}$ $\left\{\text{as } \dfrac{\text{cis } \theta}{\text{cis } \phi} = \text{cis } (\theta-\phi)\right\}$

$$\begin{aligned} \therefore \text{ cis } (-n\theta) &= \frac{1}{\text{cis } n\theta} && \{\text{as cis } 0 = 1\} \\ &= \frac{1}{[\text{cis } \theta]^n} && \{\text{for } n \text{ a positive integer}\} \\ &= [\text{cis } \theta]^{-n} \end{aligned}$$

Also $[\text{cis}\,(\frac{\theta}{n})]^n = \text{cis}(n\,(\frac{\theta}{n})) = \text{cis}\,\theta$ and so $[\text{cis}\,\theta]^{\frac{1}{n}} = \text{cis}\,(\frac{\theta}{n})$

So DeMoivre's theorem seems to hold for any integer n and for $\frac{1}{n}$.

De MOIVRE'S THEOREM

$$(|z|\ \text{cis}\,\theta)^n = |z|^n\,\text{cis}\,n\theta \quad \text{for all rational } n.$$

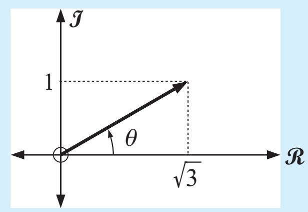
Example 18

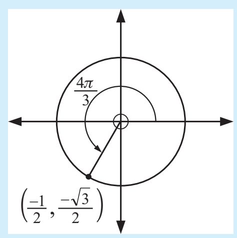
Find the exact value of $(\sqrt{3}+i)^8$ using De Moivre's theorem.
Check your answer by calculator.

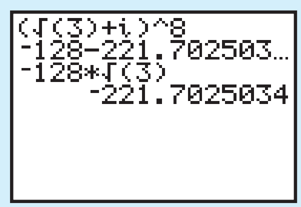
$\sqrt{3}+i$ has modulus $\sqrt{(\sqrt{3})^2+1^2} = \sqrt{4} = 2$

$$\therefore\quad \sqrt{3}+i = 2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$
$$= 2\ \text{cis}\ \frac{\pi}{6}$$

$$\therefore\quad (\sqrt{3}+i)^8 = \left(2\ \text{cis}\ \frac{\pi}{6}\right)^8$$
$$= 2^8\ \text{cis}\left(\frac{8\pi}{6}\right)$$
$$= 2^8\ \text{cis}\left(\frac{4\pi}{3}\right)$$
$$= 2^8\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$$
$$= -128-128\sqrt{3}i$$







EXERCISE 16C

1 Using De Moivre's theorem to find a simple answer for:

a $\left(\sqrt{2}\ \text{cis}\ \frac{\pi}{5}\right)^{10}$ **b** $\left(\text{cis}\ \frac{\pi}{12}\right)^{36}$ **c** $\left(\sqrt{2}\ \text{cis}\ \frac{\pi}{8}\right)^{12}$

d $\sqrt{5\ \text{cis}\ \frac{\pi}{7}}$ **e** $\sqrt[3]{8\ \text{cis}\ \frac{\pi}{2}}$ **f** $\left(8\ \text{cis}\ \frac{\pi}{5}\right)^{\frac{5}{3}}$

2 Use De Moivre's theorem to find the exact value of:

a $(1+i)^{15}$ **b** $(1-i\sqrt{3})^{11}$ **c** $(\sqrt{2}-i\sqrt{2})^{-19}$

d $(-1+i)^{-11}$ **e** $(\sqrt{3}-i)^{\frac{1}{2}}$ **f** $(2+2i\sqrt{3})^{-\frac{5}{2}}$

3 Use your calculator to check the answers to **2**.

4 **a** Recall that if $z = |z|\ \text{cis}\ \theta$ then $-\pi < \theta \leqslant \pi$.
Use De Moivre's theorem to find $\sqrt{z}$ in terms of $|z|$ and θ.

b What restrictions apply to $\theta = \arg(\sqrt{z})$?

c True or false? "$\sqrt{z}$ has non-negative real part."

5 Use De Moivre's theorem to explain why $|z^n| = |z|^n$ and $\arg(z^n) = n\arg z$.

6 Show that $\cos\theta - i\sin\theta = \text{cis}\,(-\theta)$. Hence, simplify $(\cos\theta - i\sin\theta)^{-3}$.

7 Write $z = 1+i$ in polar form and hence write z^n in polar form. Find all values of n for which: **a** z^n is real **b** z^n is purely imaginary.

8 If $|z| = 2$ and $\arg z = \theta$, determine the modulus and argument of:

a z^3 **b** iz^2 **c** $\dfrac{1}{z}$ **d** $-\dfrac{i}{z^2}$

9 If $z = \text{cis}\,\theta$, prove that $\dfrac{z^2-1}{z^2+1} = i\tan\theta$.

Example 19

By considering $\cos 2\theta + i\sin 2\theta$, deduce the double angle formulae: $\cos 2\theta$ and $\sin 2\theta$.

Now $\cos 2\theta + i\sin 2\theta$

$= \text{cis}\ 2\theta$

$= [\text{cis}\ \theta]^2$ {De Moivre's theorem}

$= [\cos\theta + i\sin\theta]^2$

$= [\cos^2\theta - \sin^2\theta] + i[2\sin\theta\cos\theta]$

Equating imaginary parts, $\sin 2\theta = 2\sin\theta\cos\theta$

Equating real parts, $\cos 2\theta = \cos^2\theta - \sin^2\theta$

10 By considering $\cos 3\theta + i\sin 3\theta$, deduce the formula:

a $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ **b** $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

11 **a** If $z = \text{cis}\ \theta$ prove that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$.

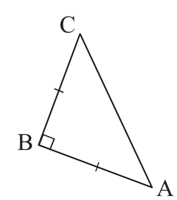
b Hence, explain why $z + \dfrac{1}{z} = 2\cos\theta$.

c Use the binomial theorem to expand $(z + \dfrac{1}{z})^3$, and simplify your result.

d By using **a**, **b** and **c** above show that $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$.

e Hence show the exact value of $\cos^3\frac{13\pi}{12}$ is $\frac{-5\sqrt{2}-3\sqrt{6}}{16}$ {**Note:** $\frac{13\pi}{12} = \frac{3\pi}{4} + \frac{\pi}{3}$}.

12 You are given that the points A, B, C in the Argand Diagram form an isosceles triangle with a right angle at B. Let the points A, B, and C be represented by the complex numbers z_1, z_2, and z_3 respectively.



a Show that $(z_1 - z_2)^2 = -(z_3 - z_2)^2$.

b If ABCD forms a square, what complex number represents the point D? Give your answer in terms of z_1, z_2 and z_3.

13 **a** Find a formula for

i $\cos 4\theta$ in terms of $\cos\theta$ **ii** $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$

b Show that if $z = \text{cis}\ \theta$ then $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$ and hence that $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$? (See question **11** above.)

D ROOTS OF COMPLEX NUMBERS

SOLVING $z^n = c$

We will examine solutions of equations of the form $z^n = c$ where n is a positive integer and c is either real or purely imaginary. However, the technique is satisfactory for all complex numbers c.

Definition: The nth roots of complex number c are the n solutions of $z^n = c$.

For example, the 4th roots of $2i$ are the four solutions of $z^4 = 2i$.

nth roots may be found by:
- factorisation
- using the 'nth root method' of complex numbers.

As factorisation can sometimes be very difficult or almost impossible, the 'nth root method' is most desirable in many cases.

Example 20

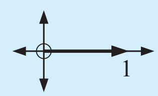
Find the four 4th roots of 1 by: **a** factorisation **b** the 'nth roots method'.

We need to find the 4 solutions of $z^4 = 1$.

a By factorisation

$$z^4 = 1$$
$$\therefore\ z^4 - 1 = 0$$
$$\therefore\ (z^2+1)(z^2-1) = 0$$
$$(z+i)(z-i)(z+1)(z-1) = 0$$
$$\therefore\ z = \pm i \text{ or } \pm 1$$

b (By the 'nth roots method')



$$z^4 = 1$$
$$\therefore\ z^4 = 1 \text{ cis } (0 + k2\pi) \quad \{\text{polar form}\}$$
$$\therefore\ z = [\text{cis } (k2\pi)]^{\frac{1}{4}}$$
$$\therefore\ z = \text{cis } \left(\frac{k2\pi}{4}\right) \quad \{\text{De Moivre}\}$$
$$\therefore\ z = \text{cis } \left(\frac{k\pi}{2}\right)$$
$$\therefore\ z = \text{cis } 0,\ \text{cis } \tfrac{\pi}{2},\ \text{cis } \pi,\ \text{cis } \tfrac{3\pi}{2}$$
$$\{\text{letting } k = 0, 1, 2, 3\}$$
$$\therefore\ z = 1, i, -1, -i$$

Note:
- The factorisation method is fine provided that the polynomial factorises easily, as in the example above.
- The substitution of $k = 0, 1, 2, 3$ to find the 4 roots could also be achieved by substituting any 4 consecutive integers for k. Why?

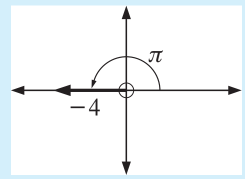
EXERCISE 16D

1. Find the three cube roots of 1 using: **a** factorisation **b** the 'nth roots method'.
2. Solve for z: **a** $z^3 = -8i$ **b** $z^3 = -27i$
3. Find the three cube roots of -1, and display them on an Argand diagram.
4. Solve for z: **a** $z^4 = 16$ **b** $z^4 = -16$
5. Find the four fourth roots of $-i$, and display them on an Argand diagram.

Example 21

Find the fourth roots of -4 in the form $a + bi$ and then factorise $z^4 + 4$ into linear factors. Hence, write $z^4 + 4$ as a product of real quadratic factors.

The fourth roots of -4 are solutions of



$$z^4 = -4$$

$$\therefore \quad z^4 = 4 \text{ cis } (\pi + k2\pi)$$

$$\therefore \quad z = [4 \text{ cis } (\pi + 2\pi)]^{\frac{1}{4}}$$

$$\therefore \quad z = 4^{\frac{1}{4}} \text{ cis } \left(\frac{\pi + k2\pi}{4}\right)$$

$$\therefore \quad z = 2^{\frac{1}{2}} \text{ cis } \tfrac{\pi}{4}, \quad 2^{\frac{1}{2}} \text{ cis } \tfrac{3\pi}{4}, \quad 2^{\frac{1}{2}} \text{ cis } \tfrac{5\pi}{4}, \quad 2^{\frac{1}{2}} \text{ cis } \tfrac{7\pi}{4} \quad \{\text{letting } k = 0, 1, 2, 3\}$$

$$\therefore \quad z = \sqrt{2}\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right), \sqrt{2}\left(-\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right), \sqrt{2}\left(-\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right), \sqrt{2}\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right)$$

$$\therefore \quad z = 1 + i, \quad -1 + i, \quad -1 - i, \quad 1 - i$$

Roots $1 \pm i$ have sum $= 2$ and product $= (1 + i)(1 - i) = 2$
and $\therefore$ come from the quadratic factor $z^2 - 2z + 2$.

Roots $-1 \pm i$ have sum $= -2$ and product $= (-1 + i)(-1 - i) = 2$
and $\therefore$ come from the quadratic factor $z^2 + 2z + 2$.

Thus $z^4 + 4 = (z^2 - 2z + 2)(z^2 + 2z + 2)$

6 Find the four solutions of $z^4 + 1 = 0$ giving each of them in the form $a + bi$ and display them on an Argand diagram. Hence, write $z^4 + 1$ as the product of two real quadratic factors.

Recall that a **real** polynomial of degree n has **exactly** n zeros that are real and/or occur in conjugate pairs (see **Chapter 8F**).

SUMMARY OF SOLUTIONS OF $z^n = c$ (nth roots of c)

- There are **exactly** n nth roots of c.
- If $c \in \mathcal{R}$, the complex roots must occur in conjugate pairs.
- If $c \notin \mathcal{R}$, the complex roots do not all occur in conjugate pairs.
- All the roots of z^n will have the same modulus which is $|c|^{\frac{1}{n}}$. Why?

  Thus on an Argand diagram, all the roots will be the same distance from the origin and hence lie on a circle radius $= |c|^{\frac{1}{n}}$.
- All the roots on the circle $r = |c|^{\frac{1}{n}}$ will be equally spaced around the circle.

  If you join all the points you will get a geometric shape that is a regular polygon.
  For example, $n = 3$ (equilateral Δ), $n = 4$ (square)
  $n = 5$ (regular pentagon) etc.
  $n = 6$ (regular hexagon)

E THE n^{th} ROOTS OF UNITY

The '**nth roots of unity**' are the solutions of $z^n = 1$.

Example 22

Find the three cube roots of unity and display them on an Argand diagram. If w is the root with smallest positive argument, show that the roots are 1, w and w^2 and that $1 + w + w^2 = 0$.

The cube roots of unity are the solutions of $z^3 = 1$.

But $1 = \text{cis } 0 = \text{cis}(0 + k2\pi)$

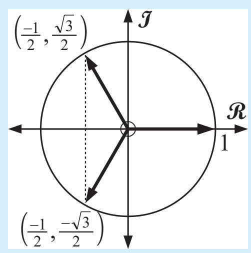
$\therefore \quad z^3 = \text{cis }(k2\pi)$

$\therefore \quad z = [\text{cis}(k2\pi)]^{\frac{1}{3}}$ {De Moivre's theorem}

i.e., $z = \text{cis}\left(\frac{k2\pi}{3}\right)$

$\therefore \quad z = \text{cis } 0, \text{ cis}\left(\frac{2\pi}{3}\right), \text{ cis}\left(\frac{4\pi}{3}\right)$ {letting $k = 0, 1, 2$}

$\therefore \quad z = 1, \quad -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad -\frac{1}{2} - \frac{\sqrt{3}}{2}i$



$w = \text{cis}\left(\frac{2\pi}{3}\right)$ and $w^2 = [\text{cis}\left(\frac{2\pi}{3}\right)]^2 = \text{cis}\left(\frac{4\pi}{3}\right)$

$\therefore$ the roots are 1, w and w^2 where $w = \text{cis}\left(\frac{2\pi}{3}\right)$

and $1 + w + w^2 = 1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 0$

EXERCISE 16E

1 In **Example 22** we showed that the cube roots of 1 are $1, w, w^2$ where $w = \text{cis}\left(\frac{2\pi}{3}\right)$.

a Use this fact to solve the following equations, giving your answers in terms of w:

i $(z+3)^3 = 1$ **ii** $(z-1)^3 = 8$ **iii** $(2z-1)^3 = -1$

2 Show by vector addition that $1 + w + w^2 = 0$ if 1, w and w^2 are the cube roots of unity.

3 In **Example 20** we showed that the four fourth roots of unity were $1, i, -1, -i$.

a Is it true that the four fourth roots of unity can be written in the form $1, w, w^2, w^3$ where $w = \text{cis } \frac{\pi}{2}$?

b Show that $1 + w + w^2 + w^3 = 0$.

c Show by vector addition that **b** is true.

4 **a** Find the 5 fifth roots of unity and display them on an Argand diagram.

b If w is the root with smallest positive argument show that the roots are $1, w, w^2, w^3$ and w^4.

c Simplify $(1+w+w^2+w^3+w^4)(1-w)$ and hence show that $1+w+w^2+w^3+w^4$ must be zero.

d Show by vector addition that $1 + w + w^2 + w^3 + w^4 = 0$.

5 If w is the nth root of unity with the smallest positive argument, i.e., $w = \text{cis}\left(\frac{2\pi}{n}\right)$, show that:

a the n roots of $z^n = 1$ are $1, w, w^2, w^3, ..., w^n$

b $1 + w + w^2 + w^3 + ... + w^{n-1} = 0$

Note: The roots of unity lie on the unit circle, equally spaced around the unit circle. Why?

REVIEW SET 16A

1 Find the real and imaginary parts of $(i - \sqrt{3})^5$.

2 If $z = x + yi$ and $P(x, y)$ moves in the complex plane, find the cartesian equation for: **a** $|z - i| = |z + 1 + i|$ **b** $z^* + iz = 0$

3 Find $|z|$ if z is a complex number and $|z + 16| = 4\,|z + 1|$.

4 Points A and B are the representations in the complex plane of the numbers $z = 2 - 2i$ and $w = -1 - \sqrt{3}i$ respectively.

a Given that the origin is O, find the angle AOB in radians, expressing your answer in terms of π.

b Calculate the argument of zw in radians, again expressing your answer in terms of π.

5 Write in polar form:

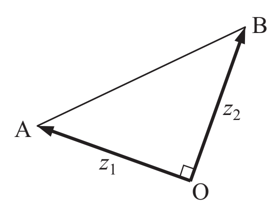
a $-5i$ **b** $2 - 2i\sqrt{3}$ **c** $k - ki$ where $k < 0$

6 Given that $z = (1 + bi)^2$, where b is real and positive, find the exact value of b if $\arg z = \frac{\pi}{3}$.

7 **a** Prove that $\text{cis}\,\theta \times \text{cis}\,\phi = \text{cis}\,(\theta + \phi)$.

b Write $(1 - i)z$ in polar form if $z = 2\sqrt{2}\,\text{cis}\,\alpha$, and hence find $\arg[(1 - i)z]$.

8 $z_1 \equiv \overrightarrow{OA}$ and $z_2 \equiv \overrightarrow{OB}$ represent two sides of a right angled isosceles triangle OAB.



a Determine the modulus and argument of $\dfrac{{z_1}^2}{{z_2}^2}$.

b Hence, deduce that ${z_1}^2 + {z_2}^2 = 0$.

REVIEW SET 16B

1 Let $z_1 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ and $z_2 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$.

Express $\left(\dfrac{z_1}{z_2}\right)^3$ in the form $z = a + bi$.

2 If $z = 4 + i$ and $w = 2 - 3i$, find:

a $2w^* - iz$ **b** $|w - z^*|$ **c** $|z^{10}|$ **d** $\arg(w - z)$

3 Find rationals a and b such that $\frac{2-3i}{2a+bi} = 3+2i$.

4 If $z = x + yi$ and $\text{P}(x, y)$ moves in the complex plane, find the Cartesian equation of the curve traced out by P if:

a $\arg(z - i) = \frac{\pi}{2}$ **b** $\left|\frac{z+2}{z-2}\right| = 2$

5 Write $2-2\sqrt{3}i$ in polar form and hence find all values of n for which $(2-2\sqrt{3}i)^n$ is real.

6 Determine the cube roots of -27.

7 If $z = 4 \text{ cis } \theta$, find the modulus and argument of:

a z^3 **b** $\frac{1}{z}$ **c** iz^*

8 If $z = \text{cis } \phi$, prove that:

a $|z| = 1$ **b** $z^* = \frac{1}{z}$ **c** $z^n - \frac{1}{z^n} = 2i \sin n\theta$

9 Prove the following:

a $\arg(z^n) = n \arg z$ for all complex numbers z and rational n.

b $\left(\frac{z}{w}\right)^* = \frac{z^*}{w^*}$ for all z and for all $w \neq 0$.

10 Find n given that each of the following can be written in the form $[\text{cis } \theta]^n$:

a $\cos 3\theta + i \sin 3\theta$ **b** $\frac{1}{\cos 2\theta + i \sin 2\theta}$ **c** $\cos\theta - i\sin\theta$

11 Determine the fifth roots of i.

12 If $z + \frac{1}{z}$ is real, prove that either $|z| = 1$ or z is real.

13 If w is the root of $z^5 = 1$ with smallest positive argument, find real quadratic equations with roots of:

a w and w^4 **b** $w + w^4$ and $w^2 + w^3$.

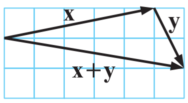
14 If $|z + w| = |z - w|$ prove that $\arg z$ and $\arg w$ differ by $\frac{\pi}{2}$.

REVIEW SET 16C

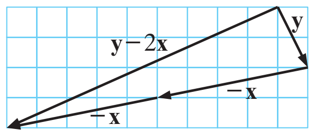
Click on the icon to obtain printable review sets and answers

REVIEW SET 16C

2 **a**



b



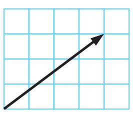
3 **a** $\overrightarrow{PQ}$ **b** $\overrightarrow{PR}$ **4** 4.845 km, 208^o **5** **a** $\overrightarrow{AC}$ **b** $\overrightarrow{AD}$

6 **a** $AB = \frac{1}{2}CD$, $AB \parallel CD$ **b** C is midpoint AB

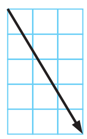
7 **a** $\mathbf{p}+\mathbf{r}=\mathbf{q}$ **b** $\mathbf{l}+\mathbf{m}=\mathbf{k}-\mathbf{j}+\mathbf{n}$

8 **a** $\mathbf{r}+\mathbf{q}$ **b** $-\mathbf{p}+\mathbf{r}+\mathbf{q}$ **c** $\mathbf{r}+\frac{1}{2}\mathbf{q}$ **d** $-\frac{1}{2}\mathbf{p}+\frac{1}{2}\mathbf{r}$

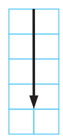
9 **a** $\begin{bmatrix}4\\3\end{bmatrix}$



b $\begin{bmatrix}3\\-5\end{bmatrix}$



c $\begin{bmatrix}0\\-4\end{bmatrix}$



10 **a** $\begin{bmatrix}-4\\-2\end{bmatrix}$ **b** $\begin{bmatrix}-1\\-13\end{bmatrix}$ **c** $\begin{bmatrix}-4\\8\end{bmatrix}$ **11** $\begin{bmatrix}1\\4\end{bmatrix}$

12 **a** $\sqrt{17}$ units **b** $\sqrt{13}$ units **c** $\sqrt{10}$ units **d** $\sqrt{109}$ units

13 **a** $\mathbf{p}+\mathbf{q}$ **b** $\frac{3}{2}\mathbf{p}+\frac{1}{2}\mathbf{q}$

14 **a** $\mathbf{x}=\begin{bmatrix}-1\\\frac{1}{3}\end{bmatrix}$ **b** $\mathbf{x}=\begin{bmatrix}1\\-10\end{bmatrix}$ **16** $r=4,\ s=7$

17 **a** $\mathbf{q}+\mathbf{r}$ **b** $\mathbf{r}+\mathbf{q}$, $DB = AC$, $DB \parallel AC$

REVIEW SET 15B

1 **a** $\overrightarrow{PQ}=\begin{bmatrix}-3\\12\\3\end{bmatrix}$ **b** $\sqrt{162}$ units **c** $\sqrt{61}$ units

2 **a** $\begin{bmatrix}3\\-3\\11\end{bmatrix}$ **b** $\begin{bmatrix}7\\-3\\-26\end{bmatrix}$ **c** $\sqrt{74}$ units **3** $\begin{bmatrix}8\\-8\\7\end{bmatrix}$

4 $m=5,\ n=-\frac{1}{2}$ **5** $2:3$ **6** $t=2\pm\sqrt{2}$ **7** 80.3^o

8 40.7^o **9** **a** $\begin{bmatrix}-6\\1\\3\end{bmatrix}$ **b** $\sqrt{46}$ units **c** $(-1,\ 3\frac{1}{2},\ \frac{1}{2})$

10 **a** -1 **b** $\begin{bmatrix}4\\-1\\7\end{bmatrix}$ **c** 60^o

11 $\measuredangle K \doteq 123.7^o$, $\measuredangle L \doteq 11.3^o$, $\measuredangle M \doteq 45.0^o$

12 63.95^o **13** $c=\frac{50}{3}$

14 **a** $\mathbf{a} \bullet \mathbf{b}$ is a scalar, so $\mathbf{a} \bullet \mathbf{b} \bullet \mathbf{c}$ is a scalar dotted with a vector, which is meaningless.
b $\mathbf{b} \times \mathbf{c}$ must be done first otherwise we have a scalar crossed with a vector which is meaningless.

15 **a** $k=\pm\frac{7}{\sqrt{33}}$ **b** $k=\pm\frac{1}{\sqrt{2}}$

REVIEW SET 15C

1 **a** -13 **b** -36 **3** $t=\frac{2}{3}$ or -3 **4** $k=6$

5 $k\begin{bmatrix}5\\4\end{bmatrix}$, $k\neq 0$ **6** $\measuredangle K = 64.44^o$, $\measuredangle L = 56.89^o$, $\measuredangle M = 58.67^o$

7 72.35^o or 107.65^o

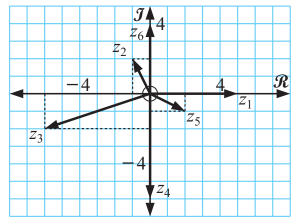
8 **a** **i** **(1)** $\mathbf{p}+\mathbf{q}$ **(2)** $\frac{1}{2}\mathbf{p}+\frac{1}{2}\mathbf{q}$
b **i** $\overrightarrow{AC}=-\mathbf{p}+\mathbf{r}$, $\overrightarrow{BC}=-\mathbf{q}+\mathbf{r}$

9 **a** $\begin{bmatrix}7\\-12\\-7\end{bmatrix}$ **b** $\begin{bmatrix}1\\-\frac{5}{3}\\-\frac{2}{3}\end{bmatrix}$ **c** $\begin{bmatrix}\frac{5}{14}\\-\frac{5}{7}\\-\frac{15}{14}\end{bmatrix}$

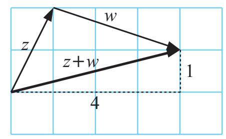
10 **a** ± 7 **b** $\frac{\sqrt{14}}{2}$ units^2 **c** $\frac{7}{6}$ units^3

EXERCISE 16A.1

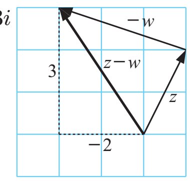
1



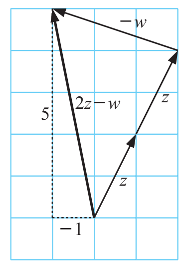
2 **a** $4+i$



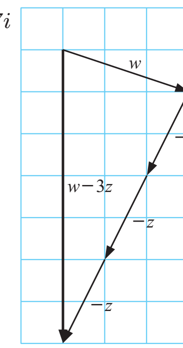
b $-2+3i$



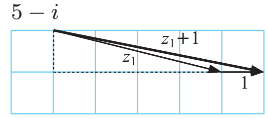
c $-1+5i$



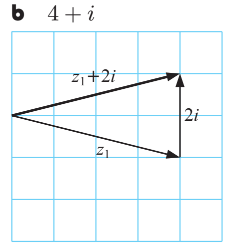
d $-7i$



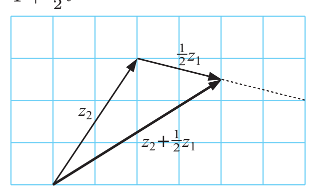
3 **a** $5-i$



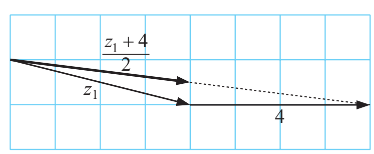
b $4+i$



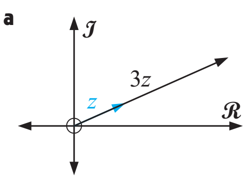
c $4+\frac{5}{2}i$



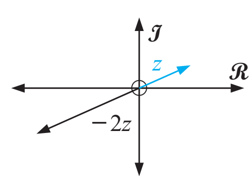
d $4-\frac{1}{2}i$



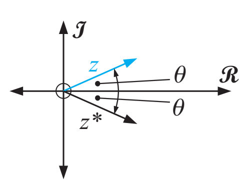
4 **a**



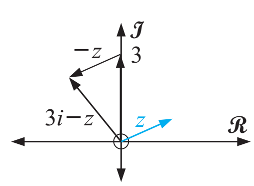
b



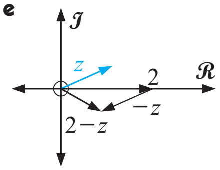
c



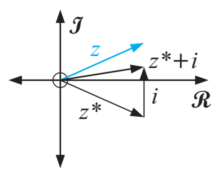
d



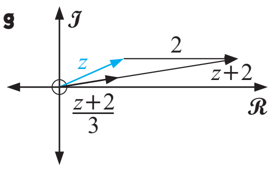
e



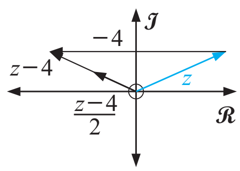
f



g

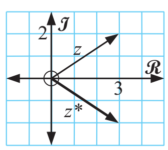


h

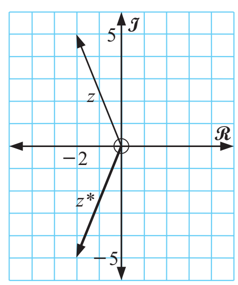


EXERCISE 16A.2

1 **a**



b



4 $z^* = z$

EXERCISE 16B.1

1 **a** 5 **b** 13 **c** $2\sqrt{17}$ **d** 3 **e** 4

2 **a** $\sqrt{5}$ **b** $\sqrt{5}$ **c** 5 **d** 5 **e** $5\sqrt{2}$ **f** $5\sqrt{2}$
g $\frac{1}{\sqrt{2}}$ **h** $\frac{1}{\sqrt{2}}$ **i** 5 **j** 5 **k** $5\sqrt{5}$ **l** $5\sqrt{5}$

3 • $|z^*| = |z|$ • $zz^* = |z|^2$ • $|zw| = |z|\,|w|$
• $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ • $|z^n| = |z|^n$

6 $|z_1z_2z_3....z_n| = |z_1|\,|z_2|\,.....\,|z_n|$ and that $|z^n| = |z|^n$

8 **a** 6 **b** 9 **c** $3\sqrt{5}$ **d** 3 **e** $\frac{1}{3}$ **f** $\frac{2}{9}$ **9** 1

10 $2^{20} = 1\,048\,576$

11 **a** $\left[\frac{a^2+b^2-1}{(a-1)^2+b^2}\right] + \left[\frac{-2b}{(a-1)^2+b^2}\right]i$ **b** 0

EXERCISE 16B.2

1 **a** **i** $4\sqrt{2}$ units **ii** $(1, 4)$ **b** **i** $5\sqrt{5}$ units **ii** $(-\frac{3}{2}, 2)$

2 **a** **i** $w + z$ **ii** $w - z$

3 **a** reflection in the $\mathcal{R}$-axis **b** reflection in the $\mathcal{J}$-axis
c anti-clockwise rotation of π about 0
d clockwise rotation of $\frac{\pi}{2}$ about 0

EXERCISE 16B.3

1 **a** 4 cis 0 **b** 2 cis $\frac{\pi}{2}$ **c** 6 cis π **d** 3 cis $\left(-\frac{\pi}{2}\right)$
e $\sqrt{2}$ cis $\frac{\pi}{4}$ **f** $2\sqrt{2}$ cis $\left(-\frac{\pi}{4}\right)$ **g** 2 cis $\left(\frac{5\pi}{6}\right)$ **h** 4 cis $\frac{\pi}{6}$

2 0 **3** $k\sqrt{2}$ cis $\frac{\pi}{4}$ if $k > 0$, $-k\sqrt{2}$ cis $\left(-\frac{3\pi}{4}\right)$ if $k < 0$,
not possible if $k = 0$

4 **a** $2i$ **b** $4\sqrt{2} + 4\sqrt{2}i$ **c** $2\sqrt{3} + 2i$ **d** $1 - i$
e $-\frac{\sqrt{3}}{2} + \frac{3}{2}i$ **f** -5 **5** **a** 1 **b** 1

EXERCISE 16B.4

1 **a** cis 3θ **b** cis 2θ **c** cis 3θ **d** $\frac{\sqrt{3}}{2} + \frac{1}{2}i$
e $\sqrt{2} + i\sqrt{2}$ **f** 8 **g** $-2i$ **h** -4 **i** $4i$

2 **a** -1 **b** -1 **c** $\frac{1}{2} + \frac{\sqrt{3}}{2}i$

3 **a** $|z| = 2$, $\arg(z) = \theta$ **b** 2 cis $(-\theta)$ **c** 2 cis $(\theta + \pi)$
d 2 cis $(\pi - \theta)$

4 **a** cis $\frac{\pi}{2}$ **b** r cis $\left(\theta + \frac{\pi}{2}\right)$ **d** clock. rotn. of $\frac{\pi}{2}$ about 0

5 **a** cis $(-\theta)$ **b** cis $\left(\theta - \frac{\pi}{2}\right)$ ..... then $z^* = r$ cis $(-\theta)$

6 **a** $\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{2}+\sqrt{6}}{4}$, $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$
b $\cos\left(\frac{11\pi}{12}\right) = \frac{-\sqrt{2}-\sqrt{6}}{4}$, $\sin\left(\frac{11\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$

EXERCISE 16B.5

2 **a** $|-z| = 3$, $\arg(-z) = \theta - \pi$ **b** $|z^*| = 3$, $\arg(z^*) = -\theta$
c $|iz| = 3$, $\arg(iz) = \theta + \frac{\pi}{2}$
d $|(1+i)z| = 3\sqrt{2}$, $\arg((1+i)z) = \theta + \frac{\pi}{4}$

3 **a** $|z - 1| = 2\sin\frac{\phi}{2}$, $\arg(z - 1) = \frac{\phi}{2} + \frac{\pi}{2}$
b $z - 1 = \left(2\sin(\frac{\phi}{2})\right)\text{cis}\left(\frac{\phi}{2} + \frac{\pi}{2}\right)$
c $(z-1)^* = \left(2\sin(\frac{\phi}{2})\right)\text{cis}\left(-\frac{\phi}{2} - \frac{\pi}{2}\right)$

4 **b** $\left|\frac{z_2-z_1}{z_3-z_2}\right| = 1$ **c** $\arg\left(\frac{z_2-z_1}{z_3-z_2}\right) = \frac{2\pi}{3}$ **d** 1

EXERCISE 16B.6

1 **a** $-1.41 + 1.01i$ **b** $1.27 - 3.06i$ **c** $-2.55 - 1.25i$

2 **a** 5 cis (-0.927) **b** 13 cis (-1.97) **c** 17.7 cis (2.29)

3 **a** 2 cis $\frac{\pi}{4}$ **b** $\sqrt{19}$ cis (-2.50)

4 **a** $a(x^2 + 2x + 4) = 0$, $a \neq 0$
b $a(x^2 - 2x + 2) = 0$, $a \neq 0$

EXERCISE 16C

1 **a** 32 **b** -1 **c** $-64i$ **d** $\sqrt{5}\,\text{cis}\left(\frac{\pi}{14}\right) \doteqdot (2.180 + 0.498i)$
e $\sqrt{3} + i$ **f** $16 + 16\sqrt{3}i$

2 **a** $128 - 128i$ **b** $1024 + 1024\sqrt{3}i$ **c** $\frac{1}{524\,288}\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$
d $\frac{1}{64}(1 - i)$ **e** $\sqrt{2}\cos\left(-\frac{\pi}{12}\right) + i\sqrt{2}\sin\left(-\frac{\pi}{12}\right)$
f $\frac{1}{64}(-\sqrt{3} - i)$

4 **a** $|z|^{\frac{1}{2}}$ cis $\frac{\theta}{2}$ **b** $-\frac{\pi}{2} < \theta \leqslant \frac{\pi}{2}$ **c** True **6** cis 3θ

7 $1 + i = \sqrt{2}$ cis $\left(\frac{\pi}{4}\right)$ $z^n = 2^{\frac{n}{2}}$ cis $\left(\frac{n\pi}{4}\right)$
a $n = 4k$, k any integer **b** $n = 2 + 4k$, $k \in Z$

8 **a** $\left|z^3\right| = 8$, $\arg(z^3) = 3\theta$
b $\left|iz^2\right| = 4$, $\arg(iz^2) = \frac{\pi}{2} + 2\theta$
c $\left|\frac{1}{z}\right| = \frac{1}{2}$ $\arg\left(\frac{1}{z}\right) = -\theta$
d $\left|\frac{-i}{z^2}\right| = \frac{1}{4}$ $\arg\left(\frac{-i}{z^2}\right) = -\frac{\pi}{2} - 2\theta$

11 **c** $\left(z + \frac{1}{z}\right)^3 = z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}$

12 **a** $\overrightarrow{AB} \equiv z_2 - z_1$, $\overrightarrow{BC} \equiv z_3 - z_2$ **Hint:** Notice that $\overrightarrow{BC}$ is a 90^o rotation of $\overrightarrow{BA}$ about B.
b $\overrightarrow{OD} \equiv z_1 + z_3 - z_2$

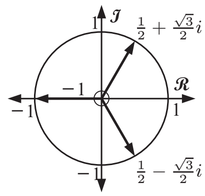
13 **a** **i** $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$
ii $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$
b **Hint:** When $n = 1$, $2i\sin\theta = z - \frac{1}{z}$.
Now cube both sides.

EXERCISE 16D

1 $1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ **2** **a** $\sqrt{3}-i,\ 2i,\ -\sqrt{3}-i$

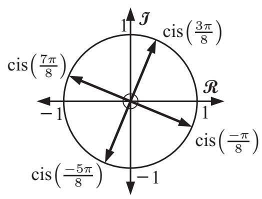
b $z = \frac{3\sqrt{3}}{2} - \frac{3}{2}i,\ 3i,\ -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$

3 $-1,\ \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

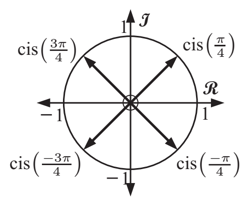


4 **a** $z = \pm 2,\ \pm 2i$ **b** $z = \sqrt{2} \pm i\sqrt{2},\ -\sqrt{2} \pm i\sqrt{2}$

5 $\text{cis}\left(\frac{3\pi}{8}\right),\ \text{cis}\left(\frac{7\pi}{8}\right),$
$\text{cis}\left(\frac{-\pi}{8}\right),\ \text{cis}\left(\frac{-5\pi}{8}\right)$



6 $z = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,$
$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,$
$-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,$ or
$-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$



$z^4 + 1 = (z^2 - \sqrt{2}z + 1)(z^2 + \sqrt{2}z + 1)$

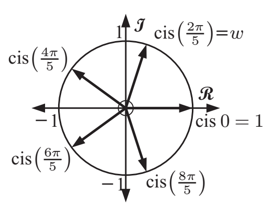
EXERCISE 16E

1 **a** **i** $z = w^n - 3$ $(n = 0, 1, 2)$ and $w = \text{cis}\,\frac{2\pi}{3}$

ii $z = 2w^n + 1$ $(n = 0, 1, 2)$ and $w = \text{cis}\,\frac{2\pi}{3}$

iii $z = \dfrac{1 - w^n}{2}$ $(n = 0, 1, 2)$ and $w = \text{cis}\,\frac{2\pi}{3}$

3 **a** Yes **4** **a** $z = \text{cis}\,0,$ $\text{cis}\left(\frac{2\pi}{5}\right),$ $\text{cis}\left(\frac{4\pi}{5}\right),$ $\text{cis}\left(\frac{6\pi}{5}\right),$ $\text{cis}\left(\frac{8\pi}{5}\right)$ **b**



c $1 - w^5$

5 **b** **Hint:** The LHS is a geometric series.

REVIEW SET 16A

1 Real part is $16\sqrt{3}$. Imaginary part is 16.

2 **a** $2x + 4y = -1$ **b** $y = x$ **3** $|z| = 4$

4 **a** $\frac{5\pi}{12}$ **b** $-\frac{11\pi}{12}$

5 **a** $5\ \text{cis}\left(-\frac{\pi}{2}\right)$ **b** $4\ \text{cis}\left(-\frac{\pi}{3}\right)$ **c** $-k\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$

6 $b = \frac{1}{\sqrt{3}}$

7 **b** $(1-i)z = 4\ \text{cis}\left(\alpha - \frac{\pi}{4}\right),$ $\arg((1-i)z) = \alpha - \frac{\pi}{4}$

8 **a** $\left|\frac{{z_1}^2}{{z_2}^2}\right| = 1,$ $\arg\left(\frac{{z_1}^2}{{z_2}^2}\right) = \pi$

REVIEW SET 16B

1 $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ **2** **a** $5 + 2i$ **b** $2\sqrt{2}$ **c** 17^5 **d** $\doteqdot -2.034$

3 $a = 0, b = -1$

4 **a** $x = 0,\ y > 1$ **b** $3x^2 + 3y^2 - 20x + 12 = 0$

5 $4\ \text{cis}\left(-\frac{\pi}{3}\right),\ n = 3k,\ k$ is an integer **6** $\frac{3}{2} \pm \frac{i3\sqrt{3}}{2},\ -3$

7 **a** $\left|z^3\right| = 64,\ \arg\left(z^3\right) = 3\theta$ **b** $\left|\frac{1}{z}\right| = \frac{1}{4},\ \arg\left(\frac{1}{z}\right) = -\theta$

c $|iz^*| = 4,\ \arg(iz^*) = \frac{\pi}{2} - \theta$

10 **a** $n = 3$ **b** $n = -2$ **c** $n = -1$

11 $\text{cis}\,\frac{\pi}{10},\ i,\ \text{cis}\,\frac{9\pi}{10},\ \text{cis}\,\frac{13\pi}{10},\ \text{cis}\,\frac{17\pi}{10}$

13 **a** $a(z^2 - 2\cos\left(\frac{2\pi}{5}\right)z + 1) = 0,\ a \neq 0$

b $a(z^2 + z - 1) = 0,\ a \neq 0$

EXERCISE 17A.1

1 **a** **i** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} + t\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ **ii** $x = 3 + t$, $y = -4 + 4t,\ t \in \mathcal{R}$

b **i** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + t\begin{bmatrix} -8 \\ 2 \end{bmatrix}$ **ii** $x = 5 - 8t$, $y = 2 + 2t,\ t \in \mathcal{R}$

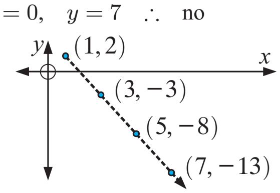
c **i** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ **ii** $x = -6 + 3t$, $y = 7t,\ t \in \mathcal{R}$

d **i** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 11 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ **ii** $x = -1 - 2t$, $y = 11 + t,\ t \in \mathcal{R}$

2 $x = -1 + 2\lambda,\ y = 4 - \lambda,\ \lambda \in \mathcal{R}$
Points are: $(-1, 4),\ (1, 3),\ (5, 1),\ (-3, 5),\ (-9, 8)$

3 **a** When $t = 1,\ x = 3,\ y = -2\ \therefore$ yes **b** $k = -5$
When $t = -2,\ x = 0,\ y = 7\ \therefore$ no

4 **a** $(1, 2)$ **b**



c $\sqrt{29}$ cm/s

EXERCISE 17A.2

1 **a** $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ $t \in \mathcal{R}$

b $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$ $t \in \mathcal{R}$

c $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$ $t \in \mathcal{R}$

2 **a** $x = 5 - t,\ y = 2 + 2t,\ z = -1 + 6t,\ t \in \mathcal{R}$
b $x = 2t,\ y = 2 - t,\ z = -1 + 3t,\ t \in \mathcal{R}$
c $x = 3,\ y = 2,\ z = -1 + t,\ t \in \mathcal{R}$

3 **a** $x = 1 - 2t,\ y = 2 + t,\ z = 1 + t,\ t \in \mathcal{R}$
b $x = 3t,\ y = 1,\ z = 3 - 4t,\ t \in \mathcal{R}$
c $x = 1,\ y = 2 - 3t,\ z = 5,\ t \in \mathcal{R}$
d $x = 5t,\ y = 1 - 2t,\ z = -1 + 4t,\ t \in \mathcal{R}$

4 **a** $(-\frac{1}{2}, \frac{9}{2}, 0)$ **b** $(0, 4, 1)$ **c** $(4, 0, 9)$

5 $(0, 7, 3)$ and $(\frac{20}{3}, -\frac{19}{3}, -\frac{11}{3})$

6 **a** $(1, 2, 3)$ **b** $(\frac{7}{3}, \frac{2}{3}, \frac{8}{3})$ **7** **a** $3\sqrt{3}$ units **b** $\sqrt{\frac{3}{2}}$ units

EXERCISE 17A.3

1 75.5^o **2** 75.7^o **3** $\begin{bmatrix} 5 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 4 \\ 10 \end{bmatrix} = 0\ \therefore$ perpendicular

4 28.6^o

EXERCISE 17B.1

1 **a** **i** $(-4, 3)$ **ii** $\begin{bmatrix} 12 \\ 5 \end{bmatrix}$ **iii** 13 m/s

b **i** $(0, -6)$ **ii** $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ **iii** 5 m/s