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CALCULUS

EARLY TRANSCENDENTALS

Third Edition

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around the z-axis in the direction of a 90° counterclockwise rotation from the positive x-axis to the positive y-axis, then your thumb points in the positive direction of the z-axis.

The three coordinate axes determine the three **coordinate planes** illustrated in Figure 3(a). The xy-plane is the plane that contains the x- and y-axes; the yz-plane contains the y- and z-axes; the xz-plane contains the x- and z-axes. These three coordinate planes divide space into eight parts, called **octants.** The **first octant,** in the foreground, is determined by the positive axes.

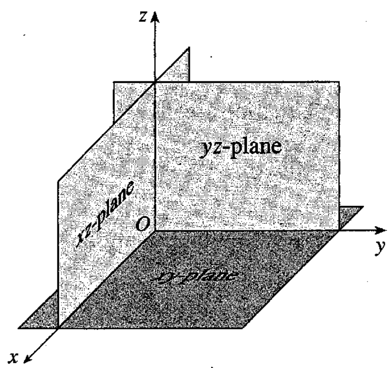
Because many people have some difficulty visualizing diagrams of three-dimensional figures, you may find it helpful to do the following [see Figure 3(b)]. Look at any bottom corner of a room and call the corner the origin. The wall on your left is in the xz-plane, the wall on your right is in the yz-plane, and the floor is in the xy-plane. The x-axis runs along the intersection of the floor and the left wall. The y-axis runs along the intersection of the floor and the right wall. The z-axis runs up from the floor toward the ceiling along the intersection of the two walls. You are situated in the first octant, and you can now imagine seven other rooms situated in the other seven octants (three on the same floor and four on the floor below), all connected by the common corner point O.

Now if P is any point in space, let a be the (directed) distance from the yz-plane to P, let b be the distance from the xz-plane to P, and let c be the distance from the xy-plane to P. We represent the point P by the ordered triple (a, b, c) of real numbers and we call a, b, and c the **coordinates** of P; a is the x-coordinate, b is the y-coordinate, and c is the z-coordinate. Thus, to locate the point (a, b, c) we can start at the origin O and move a units along the x-axis, then b units parallel to the y-axis, and then c units parallel to the z-axis as in Figure 4.

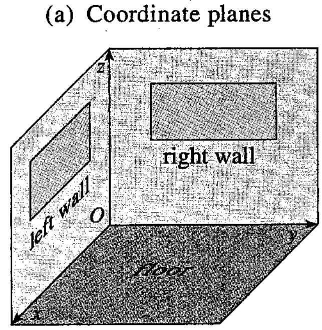
The point $P(a, b, c)$ determines a rectangular box as in Figure 5. If we drop a perpendicular from P to the xy-plane, we get a point Q with coordinates $(a, b, 0)$ called the **projection** of P on the xy-plane. Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of P on the yz-plane and xz-plane, respectively.

As numerical illustrations, the points $(-4, 3, -5)$ and $(3, -2, -6)$ are plotted in Figure 6.

The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ is the set of all ordered triples of real numbers and is denoted by $\mathbb{R}^3$. We have given a one-to-one correspondence between points P in space and ordered triples (a, b, c) in $\mathbb{R}^3$. It is called a



(a) Coordinate planes



(b)

FIGURE 3

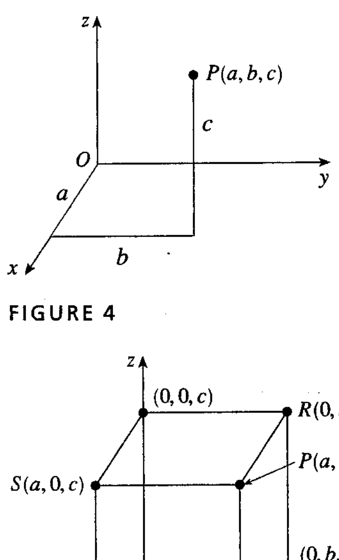


FIGURE 4

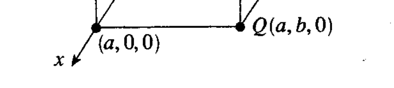


FIGURE 5

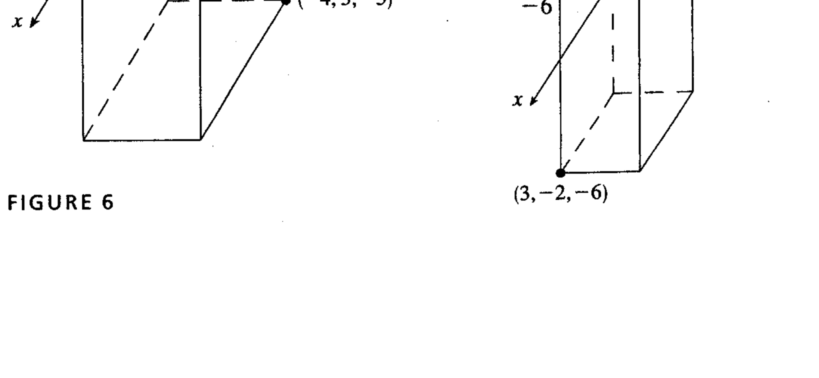


FIGURE 6

DISTANCE FORMULA IN THREE DIMENSIONS The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

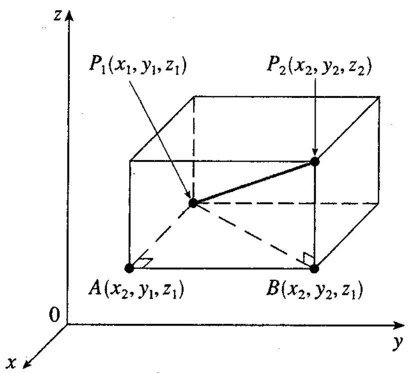


FIGURE 9

PROOF Construct a rectangular box as in Figure 9, where P_1 and P_2 are opposite vertices and the faces of the box are parallel to the coordinate planes. If $A(x_2, y_1, z_1)$ and $B(x_2, y_2, z_1)$ are the vertices of the box indicated in the figure, then

$$|P_1A| = |x_2 - x_1| \qquad |AB| = |y_2 - y_1| \qquad |BP_2| = |z_2 - z_1|$$

Because triangles P_1BP_2 and P_1AB are both right-angled, two applications of the Pythagorean Theorem give

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2 \quad \text{and} \quad |P_1B|^2 = |P_1A|^2 + |AB|^2$$

Combining these equations, we get

$$\begin{aligned} |P_1P_2|^2 &= |P_1A|^2 + |AB|^2 + |BP_2|^2 \\ &= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \end{aligned}$$

Therefore $\quad |P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ □

EXAMPLE 3 The distance from the point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$ is

$$\begin{aligned} |PQ| &= \sqrt{(1 - 2)^2 + (-3 + 1)^2 + (5 - 7)^2} \\ &= \sqrt{1 + 4 + 4} = 3 \end{aligned}$$ ■

EXAMPLE 4 Find an equation of a sphere with radius r and center $C(h, k, l)$.

SOLUTION By definition, a sphere is the set of all points $P(x, y, z)$ whose distance from C is r (see Figure 10). Thus P is on the sphere if and only if $|PC| = r$. Squaring both sides, we have $|PC|^2 = r^2$ or

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$ ■

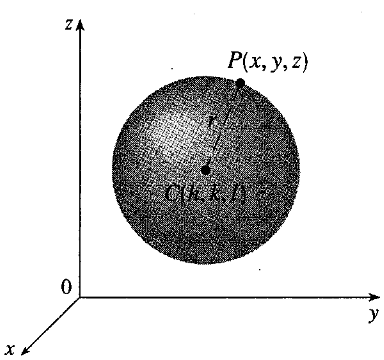


FIGURE 10

This result is worth remembering:

EQUATION OF A SPHERE An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin O, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

EXAMPLE 5 Show that $x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$ is the equation of a sphere, and find its center and radius.

SOLUTION We can rewrite the given equation in the form of an equation of a sphere if we complete squares:

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1) = -6 + 4 + 9 + 1$$

$$(x + 2)^2 + (y - 3)^2 + (z + 1)^2 = 8$$

Comparing this equation with the standard form, we see that it is the equation of a sphere with center $(-2, 3, -1)$ and radius $\sqrt{8} = 2\sqrt{2}$. ■

EXERCISES 11.1

1–4 ■ Draw a rectangular box that has P and Q as opposite vertices and has its faces parallel to the coordinate planes. Then find (a) the coordinates of the other six vertices of the box and (b) the length of the diagonal of the box.

1. $P(0, 0, 0)$, $Q(2, 3, 5)$
2. $P(0, 0, 0)$, $Q(-4, -1, 2)$
3. $P(1, 1, 2)$, $Q(3, 4, 5)$
4. $P(4, 3, 0)$, $Q(1, 6, -4)$

5–8 ■ Find the lengths of the sides of the triangle ABC and determine whether the triangle is isosceles, a right triangle, both, or neither.

5. $A(2, 1, 0)$, $B(3, 3, 4)$, $C(5, 4, 3)$
6. $A(5, 5, 1)$, $B(3, 3, 2)$, $C(1, 4, 4)$
7. $A(-2, 6, 1)$, $B(5, 4, -3)$, $C(2, -6, 4)$
8. $A(3, -4, 1)$, $B(5, -3, 0)$, $C(6, -7, 4)$

9–10 ■ Determine whether the given points are collinear.

9. $P(1, 2, 3)$, $Q(0, 3, 7)$, $R(3, 5, 11)$
10. $K(0, 3, -4)$, $L(1, 2, -2)$, $M(3, 0, 1)$

11–14 ■ Find the equation of the sphere with center C and radius r.

11. $C(0, 1, -1)$, $r = 4$
12. $C(-1, 2, 4)$, $r = \frac{1}{2}$
13. $C(-6, -1, 2)$, $r = 2\sqrt{3}$
14. $C(1, 2, -3)$, $r = 7$

15–20 ■ Show that the given equation represents a sphere, and find its center and radius.

15. $x^2 + y^2 + z^2 + 2x + 8y - 4z = 28$
16. $x^2 + y^2 + z^2 = 6x + 4y + 10z$
17. $x^2 + y^2 + z^2 + x - 2y + 6z - 2 = 0$
18. $2x^2 + 2y^2 + 2z^2 + 4y - 2z = 1$
19. $x^2 + y^2 + z^2 = x$
20. $x^2 + y^2 + z^2 + ax + by + cz + d = 0$, where $a^2 + b^2 + c^2 > 4d$

21. Prove that the midpoint of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$
22. Find an equation of a sphere if one of its diameters has endpoints $(2, 1, 4)$ and $(4, 3, 10)$.
23. Find the lengths of the medians of the triangle with vertices $A(1, 2, 3)$, $B(-2, 0, 5)$, and $C(4, 1, 5)$.
24. Find an equation of the sphere that has center $(1, 2, 3)$ and passes through the point $(-1, 1, -2)$.
25. Find equations of the spheres with center $(2, -3, 6)$ that touch (a) the xy-plane, (b) the yz-plane, (c) the xz-plane.
26. Consider the points P such that the distance from P to $A(-1, 5, 3)$ is twice the distance from P to $B(6, 2, -2)$. Show that the set of all such points is a sphere, and find its center and radius.
27. Find an equation of the set of all points equidistant from the points $A(-1, 5, 3)$ and $B(6, 2, -2)$. Describe the set.

28–43 ■ Describe in words the region of $\mathbb{R}^3$ represented by the equation or inequality.

28. $x = 0$
29. $x = 9$
30. $z = -8$
31. $y > 2$
32. $z \leqslant 0$
33. $z = x$
34. $y = z$
35. $x^2 + y^2 = 1$
36. $y^2 + z^2 \leqslant 4$
37. $x^2 + y^2 + z^2 > 1$
38. $1 \leqslant x^2 + y^2 + z^2 \leqslant 25$
39. $x^2 + y^2 + z^2 - 2z < 3$
40. $xy = 0$
41. $xy = 1$
42. $xyz = 0$
43. $|z| \leqslant 2$

44–47 ■ Write inequalities to describe the given region.

44. The solid rectangular box in the first octant bounded by the planes $x = 1$, $y = 2$, and $z = 3$

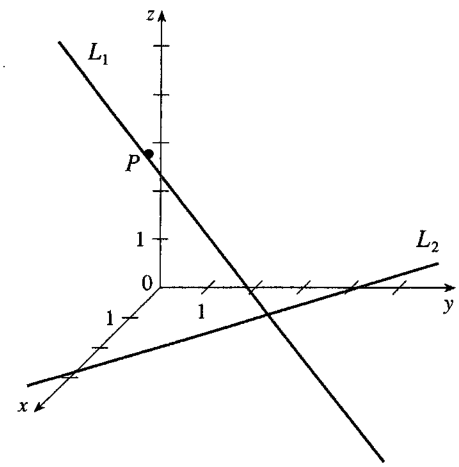
45. The half-space consisting of all points to the left of the xz-plane

46. The solid upper hemisphere of the sphere of radius 2 centered at the origin

47. The region consisting of all points between (but not on) the spheres of radius r and R centered at the origin, where $r < R$

48. Use a computer with three-dimensional graphing software to graph the sphere with center the origin and radius 1. (You may need to graph the upper and lower hemispheres separately.)

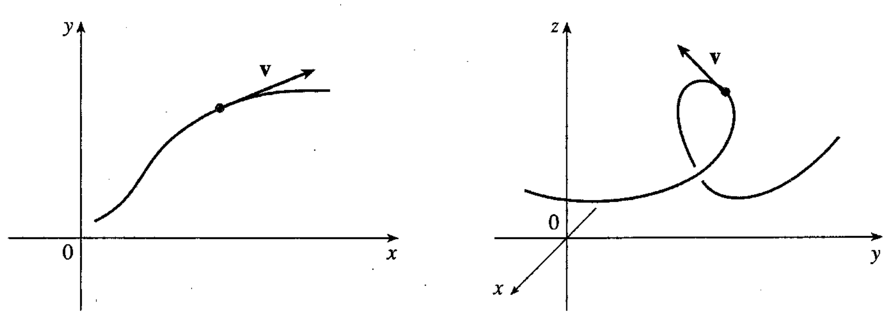
49. The figure shows a line L_1 in space and a second line L_2, which is the projection of L_1 on the xy-plane. (In other words, the points on L_2 are directly beneath, or above, the points on L_1.)
   (a) Find the coordinates of the point P.
   (b) Locate on the diagram the points A, B, and C, where the line L_1 intersects the xy-plane, the yz-plane, and the xz-plane, respectively.



50. Find the volume of the intersection of the spheres $x^2 + y^2 + z^2 + 4x - 2y + 4z + 5 = 0$ and $x^2 + y^2 + z^2 = 4$.

11.2 VECTORS

The term **vector** is used by scientists to indicate a quantity (such as velocity or force) that has both magnitude and direction. A vector is often represented by an arrow or a directed line segment. The length of the arrow represents the magnitude of the vector and the arrow points in the direction of the vector. For instance, Figure 1(a) shows a particle moving along a path in the plane and its velocity vector **v** at a specific location of the particle. Here the length of the arrow represents the speed of the particle and it points in the direction that the particle is moving.



(a) Two dimensions

(b) Three dimensions

FIGURE 1 The velocity vector of a particle

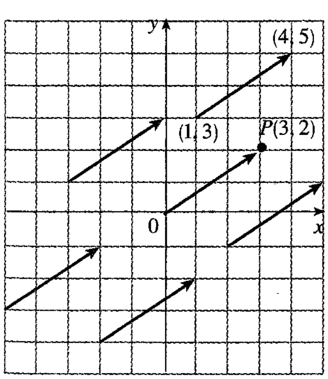
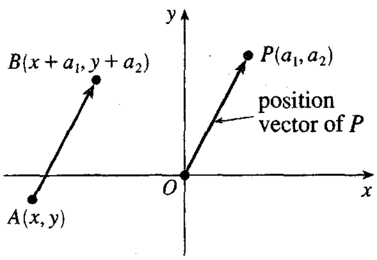


FIGURE 2
Representations of the vector $\mathbf{v} = \langle 3, 2 \rangle$

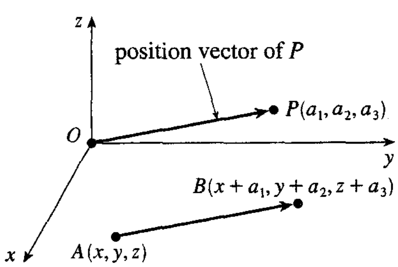
Figure 1(b) shows the path of a particle moving in space. Here the velocity vector **v** is a three-dimensional vector. (This application of vectors will be studied in detail in Section 11.9.)

Notice that all of the arrows in Figure 2 are equivalent in the sense that they have the same length and point in the same direction even though they are in different locations. All of the directed line segments have the property that the terminal point is reached from the initial point by a displacement of three units to the right and two upward. We regard each of the directed line segments as equivalent representations of a single entity called a **vector.** In other words, we can regard a vector $\mathbf{v}$ as a set of equivalent directed line segments. These line segments are characterized by the numbers 3 and 2, and we symbolize this situation by writing $\mathbf{v} = \langle 3, 2\rangle$. Thus a two-dimensional vector can be thought of as an ordered pair of real numbers. We use the notation $\langle a, b\rangle$ for the ordered pair that refers to a vector so as not to confuse it with the ordered pair (a, b) that refers to a point in the plane. A vector can be indicated by printing a letter in boldface ($\mathbf{v}$) or by putting an arrow above the letter ($\vec{v}$).

> **(1) DEFINITION** A **two-dimensional vector** is an ordered pair $\mathbf{a} = \langle a_1, a_2\rangle$ of real numbers. A **three-dimensional vector** is an ordered triple $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ of real numbers. The numbers a_1, a_2, and a_3 are called the **components** of $\mathbf{a}$.



(a) Representations of $\mathbf{a} = \langle a_1, a_2\rangle$



(b) Representations of $\mathbf{a} = \langle a_1, a_2, a_3\rangle$

FIGURE 3

A **representation** of the vector $\mathbf{a} = \langle a_1, a_2\rangle$ is a directed line segment $\overrightarrow{AB}$ from any point $A(x, y)$ to the point $B(x + a_1, y + a_2)$. A particular representation of $\mathbf{a}$ is the directed line segment $\overrightarrow{OP}$ from the origin to the point $P(a_1, a_2)$, and $\langle a_1, a_2\rangle$ is called the **position vector** of the point $P(a_1, a_2)$. Likewise, in three dimensions, the vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ is the position vector of the point $P(a_1, a_2, a_3)$ (see Figure 3).

Observe that if $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ is a vector that has the representation $\overrightarrow{AB}$, where the initial point is $A(x_1, y_1, z_1)$ and the terminal point is $B(x_2, y_2, z_2)$, then we must have $x_1 + a_1 = x_2$, $y_1 + a_2 = y_2$, and $z_1 + a_3 = z_2$ and so $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, and $a_3 = z_2 - z_1$. Thus we have the following:

> **(2)** Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{AB}$ is
>
> $$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$$

EXAMPLE 1 Find the vector represented by the directed line segment with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.

SOLUTION By (2), the vector corresponding to $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle -2 - 2, 1 - (-3), 1 - 4\rangle = \langle -4, 4, -3\rangle$$ ■

The **magnitude** or **length** of the vector $\mathbf{v}$ is the length of any of its representations and is denoted by the symbol $|\mathbf{v}|$ or $\|\mathbf{v}\|$. By using the distance formula to compute the length of a segment OP, we obtain the following:

> **(3)** The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2\rangle$ is
>
> $$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$
>
> The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ is
>
> $$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

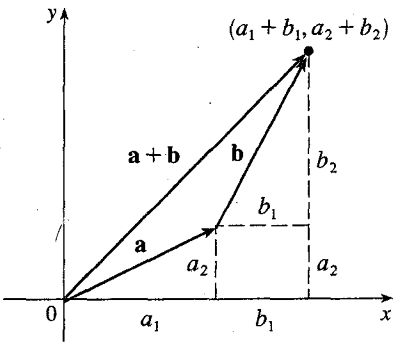


FIGURE 4
Triangle Law

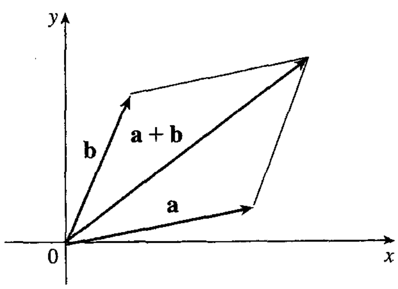


FIGURE 5
Parallelogram Law

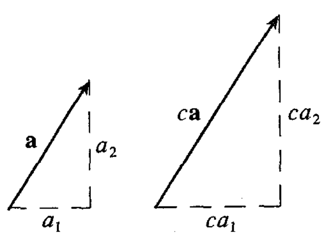


FIGURE 6

The only vector with length 0 is the **zero vector** $\mathbf{0} = \langle 0, 0\rangle$ (or $\mathbf{0} = \langle 0, 0, 0\rangle$). This vector is also the only vector with no specific direction.

According to the following definition, we add vectors by adding the corresponding components of two vectors.

(4) VECTOR ADDITION If $\mathbf{a} = \langle a_1, a_2\rangle$ and $\mathbf{b} = \langle b_1, b_2\rangle$, then the vector $\mathbf{a} + \mathbf{b}$ is defined by

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2\rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3\rangle + \langle b_1, b_2, b_3\rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$$

Definition 4 is illustrated geometrically in Figure 4 for the two-dimensional case. You can see why the definition of vector addition is sometimes called the **Triangle Law.** Alternatively, another interpretation of vector addition is shown in Figure 5 and is called the **Parallelogram Law.**

It is possible to multiply a vector by a real number c. (In this context we call the real number c a **scalar** to distinguish it from a vector.) For instance, we want $2\mathbf{a}$ to be the same vector as $\mathbf{a} + \mathbf{a}$, so

$$2\langle a_1, a_2\rangle = \langle a_1, a_2\rangle + \langle a_1, a_2\rangle = \langle 2a_1, 2a_2\rangle$$

In general, we multiply a vector by a scalar by multiplying each component by that scalar.

(5) MULTIPLICATION OF A VECTOR BY A SCALAR If c is a scalar and $\mathbf{a} = \langle a_1, a_2\rangle$, then the vector $c\mathbf{a}$ is defined by

$$c\mathbf{a} = \langle ca_1, ca_2\rangle$$

Similarly, for three-dimensional vectors,

$$c\langle a_1, a_2, a_3\rangle = \langle ca_1, ca_2, ca_3\rangle$$

Definition 5 is illustrated by Figure 6.

Let us see how the scalar multiple $c\mathbf{a}$ compares with the original vector $\mathbf{a}$. If $\mathbf{a} = \langle a_1, a_2\rangle$, then

$$|c\mathbf{a}| = \sqrt{(ca_1)^2 + (ca_2)^2} = \sqrt{c^2(a_1^2 + a_2^2)}$$
$$= \sqrt{c^2}\sqrt{a_1^2 + a_2^2} = |c||\mathbf{a}|$$

so the length of $c\mathbf{a}$ is $|c|$ times the length of $\mathbf{a}$.

If $a_1 \neq 0$, we can talk about the slope of $\mathbf{a}$ as being a_2/a_1. But, if $c \neq 0$, then the slope of $c\mathbf{a}$ is $ca_2/ca_1 = a_2/a_1$, the same as the slope of $\mathbf{a}$. If $c > 0$, then a_1 and ca_1 have the same sign. Also, a_2 and ca_2 have the same sign. This means that $\mathbf{a}$ and $c\mathbf{a}$ have the same direction. On the other hand, if $c < 0$, then a_1 and ca_1 have opposite signs, as do a_2 and ca_2, so $\mathbf{a}$ and $c\mathbf{a}$ have opposite directions. In particular, the vector $-\mathbf{a} = (-1)\mathbf{a}$

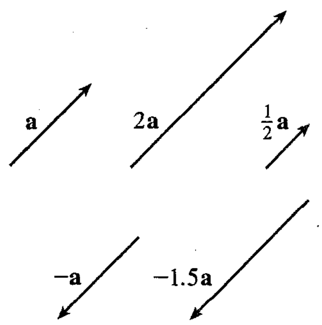


FIGURE 7
Scalar multiples of **a**

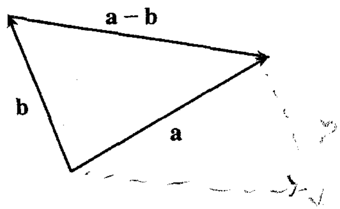


FIGURE 8

has the same length as **a** but points in the opposite direction. Illustrations of representations are shown in Figure 7.

Although we have been considering two-dimensional vectors, it is also true for three-dimensional vectors that $c\mathbf{a}$ is a vector that is $|c|$ times as long as **a** and has the same direction as **a** if $c > 0$ and the opposite direction if $c < 0$. Two vectors **a** and **b** are called **parallel** if $\mathbf{b} = c\mathbf{a}$ for some scalar c.

By the **difference** $\mathbf{a} - \mathbf{b}$ of two vectors, we mean

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

so if $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

Since $(\mathbf{a} - \mathbf{b}) + \mathbf{b} = \mathbf{a}$, the vector $\mathbf{a} - \mathbf{b}$, when added to **b**, gives **a**. This is illustrated in Figure 8 by means of the Triangle Law.

EXAMPLE 2 If $\mathbf{a} = \langle 4, 0, 3 \rangle$ and $\mathbf{b} = \langle -2, 1, 5 \rangle$, find $|\mathbf{a}|$ and the vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $3\mathbf{b}$, and $2\mathbf{a} + 5\mathbf{b}$.

SOLUTION

$$|\mathbf{a}| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5$$

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= \langle 4, 0, 3 \rangle + \langle -2, 1, 5 \rangle \\ &= \langle 4 - 2, 0 + 1, 3 + 5 \rangle = \langle 2, 1, 8 \rangle\end{aligned}$$

$$\begin{aligned}\mathbf{a} - \mathbf{b} &= \langle 4, 0, 3 \rangle - \langle -2, 1, 5 \rangle \\ &= \langle 4 - (-2), 0 - 1, 3 - 5 \rangle = \langle 6, -1, -2 \rangle\end{aligned}$$

$$3\mathbf{b} = 3\langle -2, 1, 5 \rangle = \langle 3(-2), 3(1), 3(5) \rangle = \langle -6, 3, 15 \rangle$$

$$\begin{aligned}2\mathbf{a} + 5\mathbf{b} &= 2\langle 4, 0, 3 \rangle + 5\langle -2, 1, 5 \rangle \\ &= \langle 8, 0, 6 \rangle + \langle -10, 5, 25 \rangle = \langle -2, 5, 31 \rangle\end{aligned}$$

■

We denote by V_2 the set of all two-dimensional vectors and by V_3 the set of all three-dimensional vectors. More generally, we will later need to consider the set V_n of all n-dimensional vectors. An n-dimensional vector is an ordered n-tuple:

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$$

where $a_1, a_2, \ldots, a_n$ are real numbers that are called the components of **a**. Addition and scalar multiplication are defined in terms of components just as for the cases $n = 2$ and $n = 3$.

(6) PROPERTIES OF VECTORS If **a**, **b**, and **c** are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	**2.** $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$	**4.** $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$	**6.** $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$	**8.** $1\mathbf{a} = \mathbf{a}$

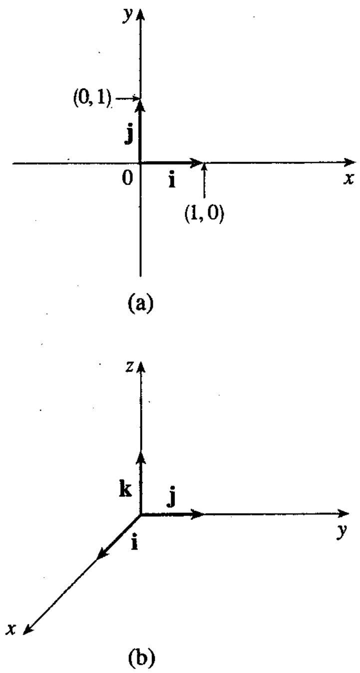


FIGURE 9
Standard basis vectors in V_2 and V_3

The eight properties of vectors in Theorem 6 can be readily verified using Definitions 4 and 5. For instance, here is the verification of Property 1 for the case $n = 2$:

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle \\ &= \langle b_1 + a_1, b_2 + a_2 \rangle = \langle b_1, b_2 \rangle + \langle a_1, a_2 \rangle \\ &= \mathbf{b} + \mathbf{a}\end{aligned}$$

The remaining proofs are left as exercises.

Three vectors in V_3 play a special role. Let

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

Then **i**, **j**, and **k** are vectors that have length 1 and point in the directions of the positive x-, y-, and z-axes. Similarly, in two dimensions we define $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ (see Figure 9).

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\begin{aligned}\mathbf{a} &= \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1\langle 1, 0, 0 \rangle + a_2\langle 0, 1, 0 \rangle + a_3\langle 0, 0, 1 \rangle\end{aligned}$$

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \tag{7}$$

Thus any vector in V_3 can be expressed in terms of the **standard basis vectors i, j,** and **k**. For instance,

$$\langle 1, -2, 6 \rangle = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

Similarly, in two dimensions, we can write

$$\mathbf{a} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j} \tag{8}$$

See Figure 10 for the geometric interpretation of Equations 8 and 7 and compare with Figure 9.

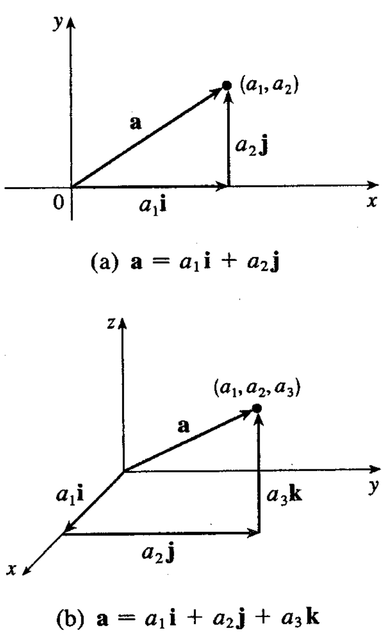


FIGURE 10

EXAMPLE 3 If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 7\mathbf{k}$, express the vector $2\mathbf{a} + 3\mathbf{b}$ in terms of **i**, **j**, and **k**.

SOLUTION Using Properties 1, 2, 5, 6, and 7 of Theorem 6, we have

$$\begin{aligned}2\mathbf{a} + 3\mathbf{b} &= 2(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + 3(4\mathbf{i} + 7\mathbf{k}) \\ &= 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k} + 12\mathbf{i} + 21\mathbf{k} = 14\mathbf{i} + 4\mathbf{j} + 15\mathbf{k}\end{aligned}$$ ■

A **unit vector** is a vector whose length is 1. For instance, **i**, **j**, and **k** are all unit vectors. In general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as **a** is

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|} \tag{9}$$

In order to verify this, we let $c = 1/|\mathbf{a}|$. Then $\mathbf{u} = c\mathbf{a}$ and c is a positive scalar, so **u** has the same direction as **a**. Also

$$|\mathbf{u}| = |c\mathbf{a}| = |c||\mathbf{a}| = \frac{1}{|\mathbf{a}|}|\mathbf{a}| = 1$$

EXAMPLE 4 Find the unit vector in the direction of the vector $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

SOLUTION The given vector has length

$$|2\mathbf{i} - \mathbf{j} - 2\mathbf{k}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

so, by Equation 9, the unit vector with the same direction is

$$\tfrac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}$$ ■

We conclude this section by considering one of the many applications of vectors in physics and engineering. A force is represented by a vector because it has both a magnitude (measured in pounds or newtons) and a direction. If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

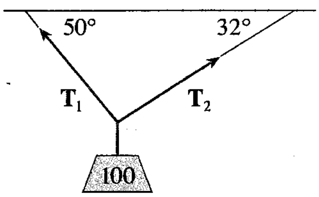


FIGURE 11

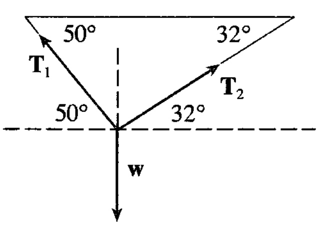


FIGURE 12

EXAMPLE 5 A 100-lb weight hangs from two wires as shown in Figure 11. Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires and their magnitudes.

SOLUTION We first express $\mathbf{T}_1$ and $\mathbf{T}_2$ in terms of their horizontal and vertical components. From Figure 12 we see that

$$\mathbf{T}_1 = -|\mathbf{T}_1|\cos 50°\,\mathbf{i} + |\mathbf{T}_1|\sin 50°\,\mathbf{j} \tag{10}$$

$$\mathbf{T}_2 = |\mathbf{T}_2|\cos 32°\,\mathbf{i} + |\mathbf{T}_2|\sin 32°\,\mathbf{j} \tag{11}$$

The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight $\mathbf{w}$ and so we must have

$$\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} = 100\,\mathbf{j}$$

Thus

$$(-|\mathbf{T}_1|\cos 50° + |\mathbf{T}_2|\cos 32°)\,\mathbf{i} + (|\mathbf{T}_1|\sin 50° + |\mathbf{T}_2|\sin 32°)\,\mathbf{j} = 100\,\mathbf{j}$$

Equating components, we get

$$-|\mathbf{T}_1|\cos 50° + |\mathbf{T}_2|\cos 32° = 0$$

$$|\mathbf{T}_1|\sin 50° + |\mathbf{T}_2|\sin 32° = 100$$

Solving the first of these equations for $|\mathbf{T}_2|$ and substituting into the second, we get

$$|\mathbf{T}_1|\sin 50° + \frac{|\mathbf{T}_1|\cos 50°}{\cos 32°}\sin 32° = 100$$

So the magnitudes of the tensions are

$$|\mathbf{T}_1| = \frac{100}{\sin 50° + \tan 32°\cos 50°} \approx 85.64 \text{ lb}$$

and

$$|\mathbf{T}_2| = \frac{|\mathbf{T}_1|\cos 50°}{\cos 32°} \approx 64.91 \text{ lb}$$

Substituting these values in (10) and (11), we obtain the tension vectors

$$\mathbf{T}_1 \approx -55.05\,\mathbf{i} + 65.60\,\mathbf{j} \qquad \mathbf{T}_2 \approx 55.05\,\mathbf{i} + 34.40\,\mathbf{j}$$ ■

EXERCISES 11.2

1–6 ■ Find a vector **a** with representation given by the directed line segment $\overrightarrow{AB}$. Draw $\overrightarrow{AB}$ and the equivalent representation starting at the origin.

1. $A(1,3)$, $B(4,4)$ **2.** $A(-3,4)$, $B(-1,0)$

3. $A(3,-1)$, $B(3,-3)$ **4.** $A(4,-1)$, $B(1,2)$

5. $A(0,3,1)$, $B(2,3,-1)$ **6.** $A(1,-2,0)$, $B(1,-2,3)$

7–10 ■ Find the sum of the given vectors and illustrate geometrically.

7. $\langle 2,3\rangle$, $\langle 3,-4\rangle$ **8.** $\langle -1,2\rangle$, $\langle 5,3\rangle$

9. $\langle 1,0,1\rangle$, $\langle 0,0,1\rangle$ **10.** $\langle 0,3,2\rangle$, $\langle 1,0,-3\rangle$

11–18 ■ Find $|\mathbf{a}|$, $\mathbf{a}+\mathbf{b}$, $\mathbf{a}-\mathbf{b}$, $2\mathbf{a}$, and $3\mathbf{a}+4\mathbf{b}$.

11. $\mathbf{a} = \langle 5,-12\rangle$, $\mathbf{b} = \langle -2,8\rangle$

12. $\mathbf{a} = \langle -1,2\rangle$, $\mathbf{b} = \langle 4,3\rangle$

13. $\mathbf{a} = \langle 2,-3,6\rangle$, $\mathbf{b} = \langle 1,1,4\rangle$

14. $\mathbf{a} = \langle 3,2,-1\rangle$, $\mathbf{b} = \langle 0,6,7\rangle$

15. $\mathbf{a} = \mathbf{i} - \mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{j}$

16. $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j}$

17. $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

18. $\mathbf{a} = 6\mathbf{i} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$

19–24 ■ Find a unit vector that has the same direction as the given vector.

19. $\langle 1,2\rangle$ **20.** $\langle 3,-5\rangle$

21. $\langle -2,4,3\rangle$ **22.** $\langle 1,-4,8\rangle$

23. $\mathbf{i} + \mathbf{j}$ **24.** $2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$

25–26 ■ Express **i** and **j** in terms of **a** and **b**.

25. $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = \mathbf{i} - \mathbf{j}$

26. $\mathbf{a} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = 3\mathbf{i} + \mathbf{j}$

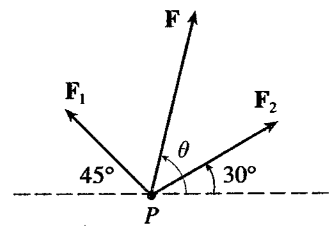
27. If A, B, and C are the vertices of a triangle, find $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$.

28. Let C be the point on the line segment AB that is twice as far from B as it is from A. If $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, and $\mathbf{c} = \overrightarrow{OC}$, show that $\mathbf{c} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$.

29. (a) Draw the vectors $\mathbf{a} = \langle 3,2\rangle$, $\mathbf{b} = \langle 2,-1\rangle$, and $\mathbf{c} = \langle 7,1\rangle$.
(b) Show, by means of a sketch, that there are scalars s and t such that $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$.
(c) Use the sketch to estimate the values of s and t.
(d) Find the exact values of s and t.

30. Suppose that **a** and **b** are nonzero vectors that are not parallel and **c** is any vector in the plane determined by **a** and **b**. Give a geometric argument to show that **c** can be written as $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ for suitable scalars s and t. Then give an argument using components.

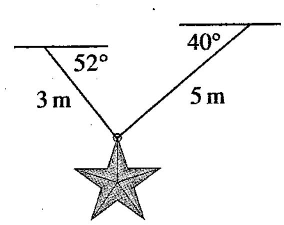
31. Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ with magnitudes 10 lb and 12 lb act on an object at a point P as shown in the figure. Find the resultant force **F** acting at P as well as its magnitude and its direction. (Indicate the direction by finding the angle θ shown in the figure.)



32. Velocities have both direction and magnitude and thus are vectors. The magnitude of a velocity vector is called *speed.* Suppose that a wind is blowing from the direction N45°W at a speed of 50 km/h. (This means that the direction from which the wind blows is 45° west of the northerly direction.) A pilot is steering a plane in the direction N60°E at an airspeed (speed in still air) of 250 km/h. The *true course,* or *track,* of the plane is the direction of the resultant of the velocity vectors of the plane and the wind. The *ground speed* of the plane is the magnitude of the resultant. Find the true course and the ground speed of the plane.

33. A woman walks due west on the deck of a ship at 3 mi/h. The ship is moving north at a speed of 22 mi/h. Find the speed and direction of the woman relative to the surface of the water.

34. Ropes 3 m and 5 m in length are fastened to a holiday decoration that is suspended over a town square. The decoration has a mass of 5 kg. The ropes, fastened at different heights, make angles of 52° and 40° with the horizontal. Find the tension in each wire and the magnitude of each tension.



35. If $\mathbf{r} = \langle x,y,z\rangle$ and $\mathbf{r}_0 = \langle x_0,y_0,z_0\rangle$, describe the set of all points (x,y,z) such that $|\mathbf{r} - \mathbf{r}_0| = 1$.

36. If $\mathbf{r} = \langle x,y\rangle$, $\mathbf{r}_1 = \langle x_1,y_1\rangle$, and $\mathbf{r}_2 = \langle x_2,y_2\rangle$, describe the set of all points (x,y) such that $|\mathbf{r} - \mathbf{r}_1| + |\mathbf{r} - \mathbf{r}_2| = k$, where $k > |\mathbf{r}_1 - \mathbf{r}_2|$.

37. Prove Property 2 of Theorem 6 for the case $n = 2$.

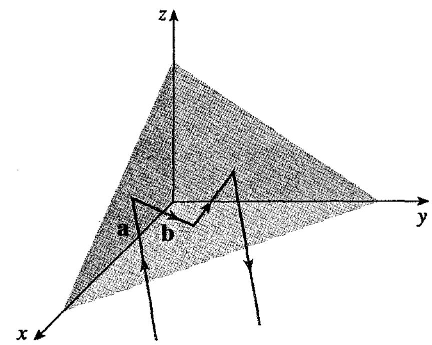
38. Prove Property 5 of Theorem 6 for the case $n = 3$.

39. Prove Property 6 of Theorem 6 for the case $n = 3$.

40. Use vectors to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

41. A quadrilateral has one pair of opposite sides parallel and of equal length. Use vectors to prove that the other pair of opposite sides is parallel and of equal length.

42. Suppose the three coordinate planes are all mirrored and a light ray given by the vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ first strikes the xz-plane, as shown in the figure. Use the fact that the angle of incidence equals the angle of reflection to show that the direction of the reflected ray is given by $\mathbf{b} = \langle a_1, -a_2, a_3 \rangle$. Deduce that, after being reflected by all three mutually perpendicular mirrors, the resulting ray is parallel to the initial ray. (American space scientists used this principle, together with laser beams and an array of corner mirrors on the moon, to calculate very precisely the distance from the earth to the moon.)



11.3 THE DOT PRODUCT

So far we have added two vectors and multiplied a vector by a scalar. The question arises: Is it possible to multiply two vectors so that their product is a useful quantity? One such product is the dot product, whose definition follows. Another is the cross product, which is discussed in the next section.

(1) DEFINITION If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of **a** and **b** is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Thus, to find the dot product of **a** and **b** we multiply corresponding components and add. The result is not a vector. It is a real number, that is, a scalar. For this reason, the dot product is sometimes called the **scalar product** (or **inner product**). Although Definition 1 is given for three-dimensional vectors, the dot product of two-dimensional vectors is defined in a similar fashion:

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

EXAMPLE 1

$$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1) = 2$$

$$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\tfrac{1}{2} \rangle = (-1)(6) + 7(2) + 4(-\tfrac{1}{2}) = 6$$

$$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = 1(0) + 2(2) + (-3)(-1) = 7$$

■

The dot product obeys many of the laws that hold for ordinary products of real numbers. These are stated in the following theorem.

(2) PROPERTIES OF THE DOT PRODUCT If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

These properties are easily proved using Definition 1. For instance, here are the proofs of Properties 1 and 3:

1. $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2$

3. $$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle \\ &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\ &= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \\ &= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\ &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \end{aligned}$$

The proofs of the remaining properties are left as exercises.

The dot product $\mathbf{a} \cdot \mathbf{b}$ can be given a geometric interpretation in terms of the **angle θ between a and b,** which is defined to be the angle between the representations of $\mathbf{a}$ and $\mathbf{b}$ that start at the origin, where $0 \leqslant \theta \leqslant \pi$. In other words, θ is the angle between the line segments $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in Figure 1. Note that if $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors, then $\theta = 0$ or π.

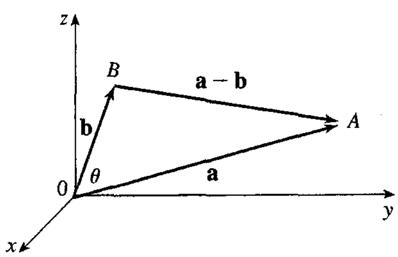


FIGURE 1

The formula in the following theorem is used by physicists as the *definition* of the dot product.

(3) THEOREM If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

PROOF If we apply the Law of Cosines to triangle OAB in Figure 1, we get

$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta \tag{4}$$

(Observe that the Law of Cosines still applies in the limiting cases when $\theta = 0$ or π, or $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.) But $|OA| = |\mathbf{a}|$, $|OB| = |\mathbf{b}|$, and $|AB| = |\mathbf{a} - \mathbf{b}|$, so Equation 4 becomes

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta \tag{5}$$

Using Properties 1, 2, and 3 of the dot product, we can rewrite the left side of this equation as follows:

$$\begin{aligned} |\mathbf{a} - \mathbf{b}|^2 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &= |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 \end{aligned}$$

Therefore, Equation 5 gives

$$|\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

Thus
$$-2\mathbf{a} \cdot \mathbf{b} = -2|\mathbf{a}||\mathbf{b}|\cos\theta$$

or
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

□

(6) COROLLARY If θ is the angle between the nonzero vectors **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

EXAMPLE 2 Find the angle between the vectors $\mathbf{a} = \langle 2, 2, -1\rangle$ and $\mathbf{b} = \langle 5, -3, 2\rangle$.

SOLUTION Since

$$|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3 \qquad \text{and} \qquad |\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

and since $\mathbf{a}\cdot\mathbf{b} = 2(5) + 2(-3) + (-1)(2) = 2$

we have, from Corollary 6,

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{2}{3\sqrt{38}}$$

So the angle between **a** and **b** is

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46 \quad (\text{or } 84°)$$

■

Two nonzero vectors **a** and **b** are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \pi/2$. Then Theorem 3 gives

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\pi/2) = 0$$

and conversely if $\mathbf{a}\cdot\mathbf{b} = 0$, then $\cos\theta = 0$, so $\theta = \pi/2$. The zero vector **0** is considered to be perpendicular to all vectors. Therefore

(7) **a** and **b** are orthogonal if and only if $\mathbf{a}\cdot\mathbf{b} = 0$.

EXAMPLE 3 Show that $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

SOLUTION Since

$$(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})\cdot(5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 2(5) + 2(-4) + (-1)(2) = 0$$

these vectors are perpendicular by (7). ■

DIRECTION ANGLES AND DIRECTION COSINES

The **direction angles** of a nonzero vector **a** are the angles α, β, and γ in the interval $[0, \pi]$ that **a** makes with the positive x-, y-, and z-axes (see Figure 2).

The cosines of these direction angles, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, are called the **direction cosines** of the vector **a**. Using Corollary 6 with **b** replaced by **i**, we obtain

(8)
$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}$$

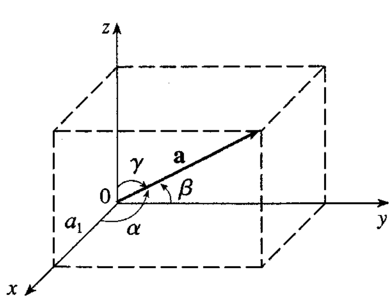


FIGURE 2

(This can also be seen directly from Figure 2.) Similarly, we also have

(9)
$$\cos\beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos\gamma = \frac{a_3}{|\mathbf{a}|}$$

By squaring the expressions in Equations 8 and 9 and adding, we see that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \qquad (10)$$

We can also use Equations 8 and 9 to write

$$\begin{aligned}\mathbf{a} = \langle a_1, a_2, a_3\rangle &= \langle |\mathbf{a}|\cos\alpha, |\mathbf{a}|\cos\beta, |\mathbf{a}|\cos\gamma\rangle \\ &= |\mathbf{a}|\langle\cos\alpha, \cos\beta, \cos\gamma\rangle\end{aligned}$$

Therefore

$$\frac{1}{|\mathbf{a}|}\mathbf{a} = \langle\cos\alpha, \cos\beta, \cos\gamma\rangle \qquad (11)$$

which says that the direction cosines of **a** are the components of the unit vector in the direction of **a**.

EXAMPLE 4 Find the direction angles of the vector $\mathbf{a} = \langle 1, 2, 3\rangle$.

SOLUTION Since $|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, Equations 8 and 9 give

$$\cos\alpha = \frac{1}{\sqrt{14}} \qquad \cos\beta = \frac{2}{\sqrt{14}} \qquad \cos\gamma = \frac{3}{\sqrt{14}}$$

and so

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) \approx 74° \qquad \beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 58° \qquad \gamma = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 37°$$

■

PROJECTIONS

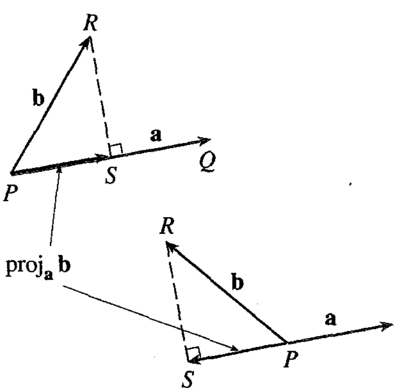


FIGURE 3
Vector projections

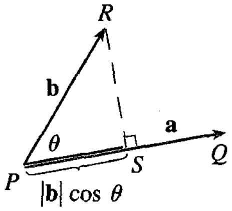


FIGURE 4
Scalar projection

Figure 3 shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors **a** and **b** with the same initial point P. If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the **vector projection** of **b** onto **a** and is denoted by $\text{proj}_\mathbf{a}\mathbf{b}$. The **scalar projection** of **b** onto **a** (also called the **component of b along a**) is defined to be the number $|\mathbf{b}|\cos\theta$, where θ is the angle between **a** and **b** (see Figure 4). This is denoted by $\text{comp}_\mathbf{a}\mathbf{b}$. Observe that it is negative if $\pi/2 < \theta \leq \pi$. The equation

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{a}|(|\mathbf{b}|\cos\theta)$$

shows that the dot product of **a** and **b** can be interpreted as the length of **a** times the scalar projection of **b** onto **a**. Since

$$|\mathbf{b}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|}\cdot\mathbf{b}$$

the component of **b** along **a** can be computed by taking the dot product of **b** with the unit vector in the direction of **a**. To summarize:

Scalar projection of **b** onto **a**:	$\text{comp}_\mathbf{a}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$
Vector projection of **b** onto **a**:	$\text{proj}_\mathbf{a}\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$

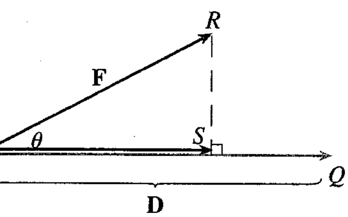
EXAMPLE 5 Find the scalar projection and vector projection of $\mathbf{b} = \langle 1, 1, 2\rangle$ onto $\mathbf{a} = \langle -2, 3, 1\rangle$.

SOLUTION Since $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$, the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

The vector projection is this scalar projection times the unit vector in the direction of $\mathbf{a}$:

$$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \frac{3}{\sqrt{14}}\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14}\mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right\rangle$$ ■



GURE 5

One use of projections occurs in physics in calculating work. In Section 6.4 we defined the work done by a constant force F in moving an object through a distance d as $W = Fd$, but this applies only when the force is directed along the line of motion of the object. Suppose, however, that the constant force is a vector $\mathbf{F} = \overrightarrow{PR}$ pointing in some other direction as in Figure 5. If the force moves the object from P to Q, then the **displacement vector** is $\mathbf{D} = \overrightarrow{PQ}$. The work done by this force is defined to be the product of the component of the force along $\mathbf{D}$ and the distance moved:

$$W = (|\mathbf{F}|\cos\theta)\,|\mathbf{D}|$$

But then, from Theorem 3, we have

$$W = |\mathbf{F}||\mathbf{D}|\cos\theta = \mathbf{F}\cdot\mathbf{D} \tag{12}$$

Thus the work done by a constant force $\mathbf{F}$ is the dot product $\mathbf{F}\cdot\mathbf{D}$, where $\mathbf{D}$ is the displacement vector.

EXAMPLE 6 A force is given by a vector $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$. Find the work done.

SOLUTION The displacement vector is $\mathbf{D} = \overrightarrow{PQ} = \langle 2, 5, 2\rangle$, so by Equation 12, the work done is

$$W = \mathbf{F}\cdot\mathbf{D} = \langle 3, 4, 5\rangle\cdot\langle 2, 5, 2\rangle$$
$$= 6 + 20 + 10 = 36$$

If the unit of length is meters and the magnitude of the force is measured in newtons, then the work done is 36 joules. ■

ERCISES 11.3

3 ■ Find $\mathbf{a}\cdot\mathbf{b}$.

$\mathbf{a} = \langle 2, 5\rangle$, $\mathbf{b} = \langle -3, 1\rangle$

$\mathbf{a} = \langle -2, -8\rangle$, $\mathbf{b} = \langle 6, -4\rangle$

$\mathbf{a} = \langle 4, 7, -1\rangle$, $\mathbf{b} = \langle -2, 1, 4\rangle$

$\mathbf{a} = \langle -1, -2, -3\rangle$, $\mathbf{b} = \langle 2, 8, -6\rangle$

5. $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$

6. $\mathbf{a} = \mathbf{i} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j}$

7. $|\mathbf{a}| = 2$, $|\mathbf{b}| = 3$, the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\pi/3$

8. $|\mathbf{a}| = 6$, $|\mathbf{b}| = \frac{1}{3}$, the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\pi/4$

9. If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, show that $\mathbf{a}\cdot\mathbf{i} = a_1$, $\mathbf{a}\cdot\mathbf{j} = a_2$, and $\mathbf{a}\cdot\mathbf{k} = a_3$.

10. (a) Show that $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$.
(b) Show that $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$.

11–16 ■ Find the angle between the vectors. (First find an exact expression and then approximate to the nearest degree.)

11. $\mathbf{a} = \langle 1, 2, 2 \rangle$, $\mathbf{b} = \langle 3, 4, 0 \rangle$
12. $\mathbf{a} = \langle 6, 0, 2 \rangle$, $\mathbf{b} = \langle 5, 3, -2 \rangle$
13. $\mathbf{a} = \langle 1, 2 \rangle$, $\mathbf{b} = \langle 12, -5 \rangle$
14. $\mathbf{a} = \langle 3, 1 \rangle$, $\mathbf{b} = \langle 2, 4 \rangle$
15. $\mathbf{a} = 6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
16. $\mathbf{a} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{j} - 3\mathbf{k}$

17–18 ■ Find, correct to the nearest degree, the three angles of the triangle with the given vertices.

17. $A(1, 2, 3)$, $B(6, 1, 5)$, $C(-1, -2, 0)$
18. $P(0, -1, 6)$, $Q(2, 1, -3)$, $R(5, 4, 2)$

19–24 ■ Determine whether the given vectors are orthogonal, parallel, or neither.

19. $\mathbf{a} = \langle 2, -4 \rangle$, $\mathbf{b} = \langle -1, 2 \rangle$
20. $\mathbf{a} = \langle 2, -4 \rangle$, $\mathbf{b} = \langle 4, 2 \rangle$
21. $\mathbf{a} = \langle 2, 8, -3 \rangle$, $\mathbf{b} = \langle -1, 2, 5 \rangle$
22. $\mathbf{a} = \langle -1, 5, 2 \rangle$, $\mathbf{b} = \langle 4, 2, -3 \rangle$
23. $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$
24. $\mathbf{a} = 2\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = -3\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$

25–28 ■ Find the values of x such that the given vectors are orthogonal.

25. $x\mathbf{i} - 2\mathbf{j}$, $x\mathbf{i} + 8\mathbf{j}$
26. $x\mathbf{i} + 2x\mathbf{j}$, $x\mathbf{i} - 2\mathbf{j}$
27. $\langle x, 1, 2 \rangle$, $\langle 3, 4, x \rangle$
28. $\langle x, x, -1 \rangle$, $\langle 1, x, 6 \rangle$

29. Find a unit vector that is orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$.

30. For what values of c is the angle between the vectors $\langle 1, 2, 1 \rangle$ and $\langle 1, 0, c \rangle$ equal to $60°$?

31–35 ■ Find the direction cosines and direction angles of the vector. (Give the direction angles correct to the nearest degree.)

31. $\langle 1, 2, 2 \rangle$
32. $\langle -4, -1, 2 \rangle$
33. $-8\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$
34. $3\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$
35. $\langle 2, 1.2, 0.8 \rangle$

36. If a vector has direction angles $\alpha = \pi/4$ and $\beta = \pi/3$, find the third direction angle γ.

37–42 ■ Find the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$.

37. $\mathbf{a} = \langle 2, 3 \rangle$, $\mathbf{b} = \langle 4, 1 \rangle$
38. $\mathbf{a} = \langle 3, -1 \rangle$, $\mathbf{b} = \langle 2, 3 \rangle$
39. $\mathbf{a} = \langle 4, 2, 0 \rangle$, $\mathbf{b} = \langle 1, 1, 1 \rangle$
40. $\mathbf{a} = \langle -1, -2, 2 \rangle$, $\mathbf{b} = \langle 3, 3, 4 \rangle$
41. $\mathbf{a} = \mathbf{i} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j}$
42. $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$

43. Show that the vector

$$\text{orth}_{\mathbf{a}}\mathbf{b} = \mathbf{b} - \text{proj}_{\mathbf{a}}\mathbf{b}$$

is orthogonal to $\mathbf{a}$. (It is called an **orthogonal projection** of $\mathbf{b}$.)

44. For the vectors in Exercise 38, find $\text{orth}_{\mathbf{a}}\mathbf{b}$ and illustrate by drawing the vectors $\mathbf{a}$, $\mathbf{b}$, $\text{proj}_{\mathbf{a}}\mathbf{b}$, and $\text{orth}_{\mathbf{a}}\mathbf{b}$.

45. If $\mathbf{a} = \langle 3, 0, -1 \rangle$, find a vector $\mathbf{b}$ such that $\text{comp}_{\mathbf{a}}\mathbf{b} = 2$.

46. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors.
(a) Under what circumstances is $\text{comp}_{\mathbf{a}}\mathbf{b} = \text{comp}_{\mathbf{b}}\mathbf{a}$?
(b) Under what circumstances is $\text{proj}_{\mathbf{a}}\mathbf{b} = \text{proj}_{\mathbf{b}}\mathbf{a}$?

47. A constant force with vector representation $\mathbf{F} = 10\mathbf{i} + 18\mathbf{j} - 6\mathbf{k}$ moves an object along a straight line from the point $(2, 3, 0)$ to the point $(4, 9, 15)$. Find the work done if the distance is measured in meters and the magnitude of the force is measured in newtons.

48. Find the work done by a force of 20 lb acting in the direction N50°W in moving an object 4 ft due west.

49. A woman exerts a horizontal force of 25 lb on a crate as she pushes it up a ramp that is 10 ft long and inclined at an angle of 20° above the horizontal. Find the work done on the box.

50. A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 50 N. The handle of the wagon is at an angle of 30° above the horizontal. How much work is done?

51. Which of the following expressions have no meaning?
(a) $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ (b) $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$
(c) $|\mathbf{a}|(\mathbf{b} \cdot \mathbf{c})$ (d) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$
(e) $\mathbf{a} \cdot \mathbf{b} + \mathbf{c}$ (f) $|\mathbf{a}| \cdot (\mathbf{b} + \mathbf{c})$

52. Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

53. Use a scalar projection to show that the distance from a point $P_1(x_1, y_1)$ to the line with equation $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

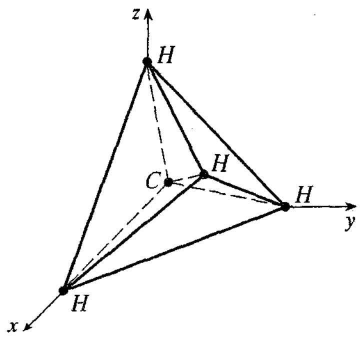
Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

54. If $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, show that the vector equation $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ represents a sphere, and find its center and radius.

55. Find the angle between a diagonal of a cube and one of its edges.

56. Find the angle between a diagonal of a cube and a diagonal of one of its faces.

57. A molecule of methane, CH_4, is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The *bond angle* is the angle formed by the H—C—H combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is about 109.5°. [*Hint:* Take the vertices of the tetrahedron to be the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$ as shown in the figure. Then the centroid is $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$.]



58. If $\mathbf{c} = |\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$, where **a**, **b**, and **c** are all nonzero vectors, show that **c** bisects the angle between **a** and **b**.

59. Prove Properties 2 and 5 of the dot product (Theorem 2).

60. Prove Property 4 of the dot product (Theorem 2).

61. Use Theorem 3 to prove the Cauchy-Schwarz Inequality:

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$$

62. The Triangle Inequality for vectors is

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

(a) Give a geometric interpretation of the Triangle Inequality.
(b) Use the Cauchy-Schwarz Inequality from Exercise 61 to prove the Triangle Inequality. [*Hint:* Use the fact that $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$ and use Property 3 of the dot product.]

63. The Parallelogram Law states that

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$$

(a) Give a geometric interpretation of the Parallelogram Law.
(b) Prove the Parallelogram Law. (See the hint in Exercise 62.)

11.4 THE CROSS PRODUCT

The **cross product a × b** of two vectors **a** and **b**, unlike the dot product, is a vector. For this reason it is also called the **vector product.** Note that **a × b** is defined only when **a** and **b** are *three-dimensional* vectors.

(1) DEFINITION If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of **a** and **b** is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$$

This may seem like a strange way of defining a product. The reason for the particular form of Definition 1 is that the cross product defined in this way has many useful properties, as we will soon see. In particular, we will show that the vector **a × b** is perpendicular to both **a** and **b**.

In order to make Definition 1 easier to remember, we use the notation of determinants. A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For example,

$$\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} = 2(4) - 1(-6) = 14$$

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

(2) $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Observe that each term on the right side of Equation 2 involves a number a_i in the first row of the determinant, and a_i is multiplied by the second-order determinant obtained from the left side by deleting the row and column in which a_i appears. Notice also the minus sign in the second term. For example,

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$
$$= 1(0 - 4) - 2(6 + 5) + (-1)(12 - 0) = -38$$

If we now rewrite Definition 1 using second-order determinants and the standard basis vectors **i**, **j**, and **k**, we see that the cross product of $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is

(3) $$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

In view of the similarity between Equations 2 and 3, we often write

(4) $$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Although the first row of the symbolic determinant in Equation 4 consists of vectors, if we expand it as if it were an ordinary determinant using the rule in Equation 2, we obtain Equation 3. The symbolic formula in Equation 4 is probably the easiest way of remembering and computing cross products.

EXAMPLE 1 If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$
$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}\mathbf{k}$$
$$= (-15 - 28)\mathbf{i} - (-5 - 8)\mathbf{j} + (7 - 6)\mathbf{k} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$ ∎

EXAMPLE 2 Show that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector **a** in V_3.

SOLUTION If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then

$$\mathbf{a} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$
$$= (a_2a_3 - a_3a_2)\mathbf{i} - (a_1a_3 - a_3a_1)\mathbf{j} + (a_1a_2 - a_2a_1)\mathbf{k}$$
$$= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$ ∎

One of the most important properties of the cross product is given by the following theorem.

(5) THEOREM The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

PROOF In order to show that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$, we compute their dot product as follows:

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\ &= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1) \\ &= a_1a_2b_3 - a_1b_2a_3 - a_1a_2b_3 + b_1a_2a_3 + a_1b_2a_3 - b_1a_2a_3 \\ &= 0\end{aligned}$$

A similar computation shows that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$. Therefore, $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. □

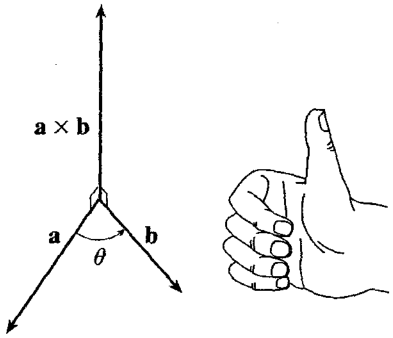


FIGURE 1

If $\mathbf{a}$ and $\mathbf{b}$ are represented by directed line segments with the same initial point (as in Figure 1), then Theorem 5 says that the cross product $\mathbf{a} \times \mathbf{b}$ points in a direction perpendicular to the plane through $\mathbf{a}$ and $\mathbf{b}$. It turns out that the direction of $\mathbf{a} \times \mathbf{b}$ is given by the *right-hand rule:* If the fingers of your right hand curl in the direction of a rotation (through an angle less than 180°) from $\mathbf{a}$ to $\mathbf{b}$, then your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

Now that we know the direction of the vector $\mathbf{a} \times \mathbf{b}$, the remaining thing we need to complete its geometric description is its length $|\mathbf{a} \times \mathbf{b}|$. This is given by the following theorem.

(6) THEOREM If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

PROOF From the definitions of the cross product and length of a vector, we have

$$\begin{aligned}|\mathbf{a} \times \mathbf{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\ &= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 \\ &\qquad + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\ &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta \qquad \text{(by Theorem 11.3.3)} \\ &= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2\theta) \\ &= |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta\end{aligned}$$

Taking square roots and observing that $\sqrt{\sin^2\theta} = \sin\theta$ because $\sin\theta \geqslant 0$ when $0 \leqslant \theta \leqslant \pi$, we have

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

□

Geometric characterization of $\mathbf{a} \times \mathbf{b}$

Since a vector is completely determined by its magnitude and direction, we can now say that $\mathbf{a} \times \mathbf{b}$ is the vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, whose orientation is determined by the right-hand rule, and whose length is $|\mathbf{a}||\mathbf{b}|\sin\theta$. In fact, that is exactly how physicists *define* $\mathbf{a} \times \mathbf{b}$.

(7) COROLLARY Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

PROOF Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\theta = 0$ or π. In either case $\sin\theta = 0$. □

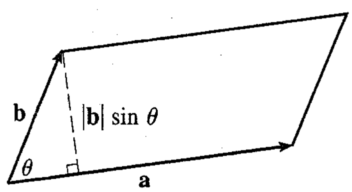


FIGURE 2

The geometric interpretation of Theorem 6 can be seen by looking at Figure 2. If $\mathbf{a}$ and $\mathbf{b}$ are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\mathbf{a}|$, altitude $|\mathbf{b}|\sin\theta$, and area

$$A = |\mathbf{a}|(|\mathbf{b}|\sin\theta) = |\mathbf{a} \times \mathbf{b}|$$

Thus **the length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.**

EXAMPLE 3 Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

SOLUTION By (11.2.2) the vectors that correspond to the directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are $\mathbf{a} = \langle -3, 1, -7\rangle$ and $\mathbf{b} = \langle 0, -5, -5\rangle$. We compute the cross product of these vectors:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= (-5 - 35)\,\mathbf{i} - (15 - 0)\,\mathbf{j} + (15 - 0)\,\mathbf{k} = -40\,\mathbf{i} - 15\,\mathbf{j} + 15\,\mathbf{k}$$

The area A of triangle PQR is half the area of the parallelogram with adjacent sides PQ and PR. Thus

$$A = \tfrac{1}{2}|\mathbf{a} \times \mathbf{b}| = \tfrac{1}{2}\sqrt{(-40)^2 + (-15)^2 + 15^2} = \frac{5\sqrt{82}}{2}$$ ■

If we apply Theorems 5 and 6 to the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ using $\theta = \pi/2$, we obtain

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

Observe that

$$\mathbf{i} \times \mathbf{j} \neq \mathbf{j} \times \mathbf{i}$$

Thus the cross product is not commutative. Also

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

whereas

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$$

So the associative law for multiplication does not usually hold; that is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

However, some of the usual laws of algebra do hold for cross products. The following theorem summarizes the properties of vector products.

(8) THEOREM If **a**, **b**, and **c** are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

These properties can be proved by writing the vectors in terms of their components and using the definition of a cross product. We give the proof of Property 5 and leave the remaining proofs as exercises.

PROOF OF PROPERTY 5 If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, then

$$\begin{aligned} (9) \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\ &= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 \\ &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \end{aligned}$$ □

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ that occurs in Property 5 is called the **scalar triple product** of the vectors **a**, **b**, and **c**. Notice from Equation 9 that we can write the scalar triple product as a determinant:

$$(10) \qquad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

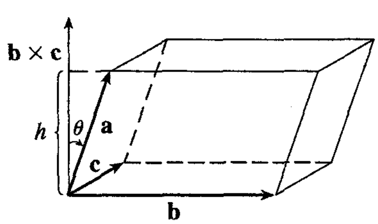


FIGURE 3

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors **a**, **b**, and **c** (Figure 3). The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$. If θ is the angle between **a** and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$. (We must use $|\cos\theta|$ instead of $\cos\theta$ in case $\theta > \pi/2$.) Thus the volume of the parallelepiped is

$$V = Ah = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Thus we have proved the following:

(11) The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

EXAMPLE 4 Use the scalar triple product to show that the vectors $\mathbf{a} = \langle 1, 4, -7 \rangle$, $\mathbf{b} = \langle 2, -1, 4 \rangle$, and $\mathbf{c} = \langle 0, -9, 18 \rangle$ are coplanar; that is, they lie in the same plane.

SOLUTION We use Equation 10 to compute their scalar triple product:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7\begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

$$= 1(18) - 4(36) - 7(-18) = 0$$

Therefore, by (11) the volume of the parallelepiped determined by **a**, **b**, and **c** is 0. This means that **a**, **b**, and **c** are coplanar. ■

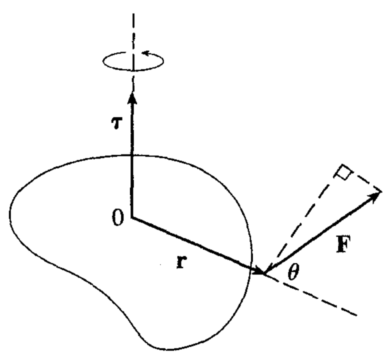


FIGURE 4

The idea of a cross product occurs often in physics. In particular, we consider a force **F** acting on a rigid body at a point given by a position vector **r** (see Figure 4). The **torque** $\boldsymbol{\tau}$ (relative to the origin) is defined to be the cross product of the position and force vectors

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

and measures the tendency of the body to rotate about the origin. The direction of the torque vector indicates the axis of rotation. According to Theorem 6, the magnitude of the torque vector is

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}|\sin\theta$$

where θ is the angle between the position and force vectors. Observe that the only component of **F** that can cause a rotation is the one perpendicular to **r**, that is, $|\mathbf{F}|\sin\theta$. The magnitude of the torque is equal to the area of the parallelogram determined by **r** and **F**.

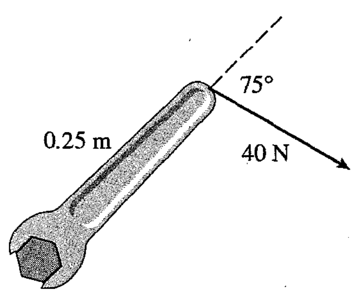


FIGURE 5

EXAMPLE 5 A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown in Figure 5. Find the magnitude of the torque about the center of the bolt.

SOLUTION The magnitude of the torque vector is

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}|\sin 75° = (0.25)(40)\sin 75°$$

$$= 10\sin 75° \approx 9.66 \text{ N-m} = 9.66 \text{ J}$$

If the bolt is right-threaded, then the torque vector itself is

$$\boldsymbol{\tau} = |\boldsymbol{\tau}|\mathbf{n} \approx 9.66\mathbf{n}$$

where **n** is a unit vector directed down into the page. ■

EXERCISES 11.4

1–7 ■ Find the cross product $\mathbf{a} \times \mathbf{b}$.

1. $\mathbf{a} = \langle 1, 0, 1 \rangle$, $\mathbf{b} = \langle 0, 1, 0 \rangle$
2. $\mathbf{a} = \langle 2, 4, 0 \rangle$, $\mathbf{b} = \langle -3, 1, 6 \rangle$
3. $\mathbf{a} = \langle -2, 3, 4 \rangle$, $\mathbf{b} = \langle 3, 0, 1 \rangle$

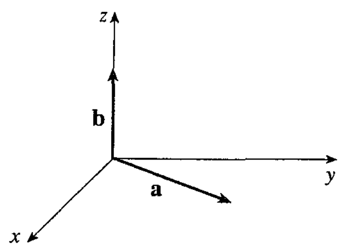
4. $\mathbf{a} = \langle 1, 2, -3 \rangle, \quad \mathbf{b} = \langle 5, -1, -2 \rangle$

5. $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

6. $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} - \mathbf{j} + 7\mathbf{k}$

7. $\mathbf{a} = 2\mathbf{i} - \mathbf{k}, \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j}$

8. The figure shows a vector $\mathbf{a}$ in the xy-plane and a vector $\mathbf{b}$ in the direction of $\mathbf{k}$. Their lengths are $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 2$.
(a) Find $|\mathbf{a} \times \mathbf{b}|$.
(b) Use the right-hand rule to decide whether the components of $\mathbf{a} \times \mathbf{b}$ are positive, negative, or 0.



9. If $\mathbf{a} = \langle 0, 1, 2 \rangle$ and $\mathbf{b} = \langle 3, 1, 0 \rangle$, find $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$.

10. If $\mathbf{a} = \langle -4, 0, 3 \rangle$, $\mathbf{b} = \langle 2, -1, 0 \rangle$, and $\mathbf{c} = \langle 0, 2, 5 \rangle$, show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

11. Find two unit vectors orthogonal to both $\langle 1, -1, 1 \rangle$ and $\langle 0, 4, 4 \rangle$.

12. Find two unit vectors orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$.

13. Show that $\mathbf{0} \times \mathbf{a} = \mathbf{0} = \mathbf{a} \times \mathbf{0}$ for any vector $\mathbf{a}$ in V_3.

14. Show that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$ for all vectors $\mathbf{a}$ and $\mathbf{b}$ in V_3.

15. Prove Property 1 of Theorem 8.

16. Prove Property 2 of Theorem 8.

17. Prove Property 3 of Theorem 8.

18. Prove Property 4 of Theorem 8.

19. Find the area of the parallelogram with vertices $A(0, 1)$, $B(3, 0)$, $C(5, -2)$, and $D(2, -1)$.

20. Find the area of the parallelogram with vertices $P(0, 0, 0)$, $Q(5, 0, 0)$, $R(2, 6, 6)$, and $S(7, 6, 6)$.

21–24 ■ (a) Find a vector orthogonal to the plane through the points P, Q, and R, and (b) find the area of triangle PQR.

21. $P(1, 0, 0), \quad Q(0, 2, 0), \quad R(0, 0, 3)$

22. $P(1, 0, -1), \quad Q(2, 4, 5), \quad R(3, 1, 7)$

23. $P(0, 0, 0), \quad Q(1, -1, 1), \quad R(4, 3, 7)$

24. $P(-4, -4, -4), \quad Q(0, 5, -1), \quad R(3, 1, 2)$

25–26 ■ Find the volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

25. $\mathbf{a} = \langle 1, 0, 6 \rangle, \quad \mathbf{b} = \langle 2, 3, -8 \rangle, \quad \mathbf{c} = \langle 8, -5, 6 \rangle$

26. $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}, \quad \mathbf{b} = \mathbf{i} - \mathbf{j}, \quad \mathbf{c} = 2\mathbf{i} + 3\mathbf{k}$

27–28 ■ Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS.

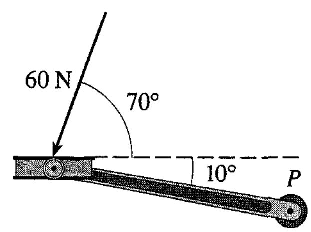
27. $P(1, 1, 1), \quad Q(2, 0, 3), \quad R(4, 1, 7), \quad S(3, -1, -2)$

28. $P(0, 1, 2), \quad Q(2, 4, 5), \quad R(-1, 0, 1), \quad S(6, -1, 4)$

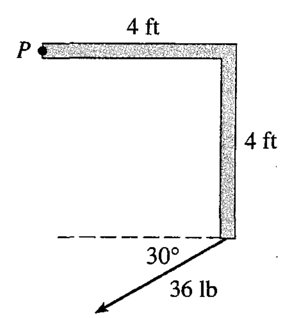
29. Use the scalar triple product to verify that the vectors $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j}$, and $\mathbf{c} = 7\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ are coplanar.

30. Use the scalar triple product to verify that the points $P(1, 0, 1)$, $Q(2, 4, 6)$, $R(3, -1, 2)$, and $S(6, 2, 8)$ are coplanar.

31. A bicycle pedal is pushed by a foot with a 60-N force as shown. The shaft of the pedal is 18 cm long. Find the magnitude of the torque about P.



32. Find the magnitude of the torque about P if a 36-lb force is applied as shown.



33. (a) Let P be a point not on the line L that passes through the points Q and R. Show that the distance d from the point P to the line L is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}$$

where $\mathbf{a} = \overrightarrow{QR}$ and $\mathbf{b} = \overrightarrow{QP}$.

(b) Use the formula in part (a) to find the distance from the point $P(1, 1, 1)$ to the line through $Q(0, 6, 8)$ and $R(-1, 4, 7)$.

34. (a) Let P be a point not on the plane that passes through the points Q, R, and S. Show that the distance d from P to the plane is

$$d = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|}$$

where $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{QS}$, and $\mathbf{c} = \overrightarrow{QP}$.

(b) Use the formula in part (a) to find the distance from the point $P(2, 1, 4)$ to the plane through the points $Q(1, 0, 0)$, $R(0, 2, 0)$, and $S(0, 0, 3)$.

35. Prove that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$.

36. The product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is called the **vector triple product** of **a**, **b**, and **c**. Prove the following formula for the vector triple product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

37. Prove that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$

38. Prove that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$$

39. Suppose that $\mathbf{a} \neq \mathbf{0}$.

(a) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

(b) If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

(c) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

40. If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are noncoplanar vectors, let

$$\mathbf{k}_1 = \frac{\mathbf{v}_2 \times \mathbf{v}_3}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \qquad \mathbf{k}_2 = \frac{\mathbf{v}_3 \times \mathbf{v}_1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

$$\mathbf{k}_3 = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

(These vectors occur in the study of crystallography. Vectors of the form $n_1\mathbf{v}_1 + n_2\mathbf{v}_2 + n_3\mathbf{v}_3$, where each n_i is an integer, form a *lattice* for a crystal. Vectors written similarly in terms of $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ form the *reciprocal lattice.*)

(a) Show that $\mathbf{k}_i$ is perpendicular to $\mathbf{v}_j$ if $i \neq j$.

(b) Show that $\mathbf{k}_i \cdot \mathbf{v}_i = 1$ for $i = 1, 2, 3$.

(c) Show that $\mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3) = \dfrac{1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$.

11.5 EQUATIONS OF LINES AND PLANES

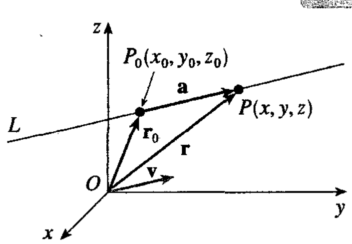


FIGURE 1

A line in the xy-plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given. The equation of the line can then be written using the point-slope form.

Likewise, a line L in three-dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and the direction of L. In three dimensions the direction of a line is conveniently described by a vector, so we let $\mathbf{v}$ be a vector parallel to L. Let $P(x, y, z)$ be an arbitrary point on L and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P (that is, they have representations $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$). If $\mathbf{a}$ is the vector with representation $\overrightarrow{P_0P}$, as in Figure 1, then the Triangle Law for vector addition gives $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$. But, since $\mathbf{a}$ and $\mathbf{v}$ are parallel vectors, there is a scalar t such that $\mathbf{a} = t\mathbf{v}$. Thus

(1) $$\boxed{\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}}$$

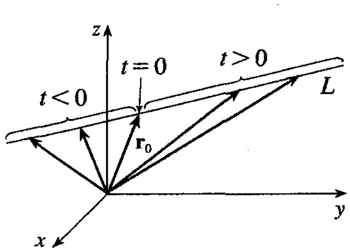


FIGURE 2

which is a **vector equation** of L. Each value of the **parameter** t gives the position vector $\mathbf{r}$ of a point on L. As Figure 2 indicates, positive values of t correspond to points on L that lie on one side of P_0, whereas negative values of t correspond to points that lie on the other side of P_0.

If the vector $\mathbf{v}$ that gives the direction of L is written in component form as $\mathbf{v} = \langle a, b, c \rangle$, then we have $t\mathbf{v} = \langle ta, tb, tc \rangle$. We can also write $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, so the vector equation (1) becomes

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Two vectors are equal if and only if corresponding components are equal. Therefore, we have the three scalar equations:

(2) $$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

where $t \in \mathbb{R}$. These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$. Each value of the parameter t gives a point (x, y, z) on L.

EXAMPLE 1

(a) Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.
(b) Find two other points on the line.

Figure 3 shows the line L in Example 1 and its relation to the given point and to the vector that gives its direction.

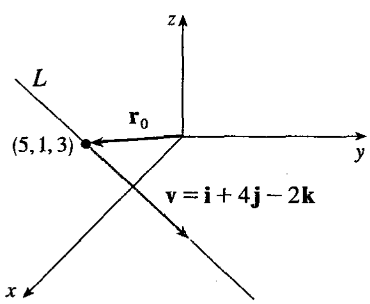


FIGURE 3

SOLUTION

(a) Here $\mathbf{r}_0 = \langle 5, 1, 3\rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, so the vector equation (1) becomes

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

or

$$\mathbf{r} = (5 + t)\mathbf{i} + (1 + 4t)\mathbf{j} + (3 - 2t)\mathbf{k}$$

Parametric equations are

$$x = 5 + t \qquad y = 1 + 4t \qquad z = 3 - 2t$$

(b) Choosing the parameter value $t = 1$ gives $x = 6$, $y = 5$, and $z = 1$, so $(6, 5, 1)$ is a point on the line. Similarly, $t = -1$ gives the point $(4, -3, 5)$. ■

The vector equation and parametric equations of a line are not unique. If we change the point or the parameter or choose a different parallel vector, then the equations change. For instance, if, instead of $(5, 1, 3)$, we choose the point $(6, 5, 1)$ in Example 1, then the parametric equations of the line become

$$x = 6 + t \qquad y = 5 + 4t \qquad z = 1 - 2t$$

Or, if we stay with the point $(5, 1, 3)$ but choose the parallel vector $2\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$, we arrive at the equations

$$x = 5 + 2t \qquad y = 1 + 8t \qquad z = 3 - 4t$$

In general, if a vector $\mathbf{v} = \langle a, b, c\rangle$ is used to describe the direction of a line L, then the numbers a, b, and c are called **direction numbers** of L. Since any vector parallel to $\mathbf{v}$ could also be used, we see that any three numbers proportional to a, b, and c could also be used as a set of direction numbers for L.

Another way of describing a line L is to eliminate the parameter t from Equations 2. If none of a, b, or c is 0, we can solve each of these equations for t, equate the results, and obtain

(3) $$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These equations are called **symmetric equations** of L. Notice that the numbers a, b, and c that appear in the denominators of Equations 3 are direction numbers of L, that is, components of a vector parallel to L. If one of a, b, or c is 0, we can still eliminate t. For instance, if $a = 0$, we could write the equations of L as

$$x = x_0 \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This means that L lies in the vertical plane $x = x_0$.

EXAMPLE 2

(a) Find parametric equations and symmetric equations of the line that passes through the points $A(2, 4, -3)$ and $B(3, -1, 1)$.

(b) At what point does this line intersect the xy-plane?

SOLUTION

(a) We are not explicitly given a vector parallel to the line, but observe that the vector $\mathbf{v}$ with representation $\overrightarrow{AB}$ is parallel to the line and

$$\mathbf{v} = \langle 3 - 2, -1 - 4, 1 - (-3) \rangle = \langle 1, -5, 4 \rangle$$

Thus direction numbers are $a = 1$, $b = -5$, and $c = 4$. Taking the point $(2, 4, -3)$ as P_0, we see that parametric equations (2) are

$$x = 2 + t \qquad y = 4 - 5t \qquad z = -3 + 4t$$

Figure 4 shows the line L in Example 2 and the point P where it intersects the xy-plane.

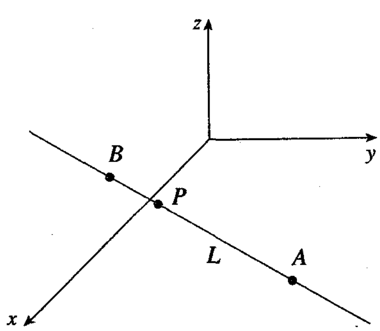


FIGURE 4

and symmetric equations (3) are

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{z + 3}{4}$$

(b) The line intersects the xy-plane when $z = 0$, so we put $z = 0$ in the symmetric equations and obtain

$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{3}{4}$$

This gives $x = \frac{11}{4}$ and $y = \frac{1}{4}$, so the line intersects the xy-plane at the point $\left(\frac{11}{4}, \frac{1}{4}, 0\right)$. ■

In general, the procedure of Example 2 shows that direction numbers of the line L through the points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ are $x_1 - x_0$, $y_1 - y_0$, and $z_1 - z_0$ and so symmetric equations of L are

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

The lines L_1 and L_2 in Example 3, shown in Figure 5, are skew lines.

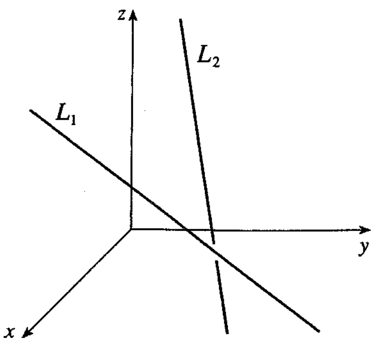


FIGURE 5

EXAMPLE 3 Show that the lines L_1 and L_2 with parametric equations

$$x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$

$$x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

are **skew lines;** that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).

SOLUTION The lines are not parallel because the corresponding vectors $\langle 1, 3, -1 \rangle$ and $\langle 2, 1, 4 \rangle$ are not parallel. (Their components are not proportional.) If L_1 and L_2 had a point of intersection, there would be values of t and s such that

$$1 + t = 2s$$

$$-2 + 3t = 3 + s$$

$$4 - t = -3 + 4s$$

But if we solve the first two equations, we get $t = \frac{11}{5}$ and $s = \frac{8}{5}$, and these values do not satisfy the third equation. Therefore, there are no values of t and s that satisfy the three equations. Thus L_1 and L_2 do not intersect. Hence L_1 and L_2 are skew lines. ■

PLANES

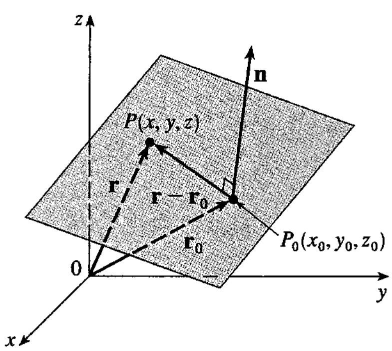


FIGURE 6

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n}$ that is orthogonal to the plane. This orthogonal vector $\mathbf{n}$ is called a **normal vector.** Let $P(x, y, z)$ be an arbitrary point in the plane and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P. Then the vector $\mathbf{r} - \mathbf{r}_0$ is represented by $\overrightarrow{P_0P}$ (see Figure 6). The normal vector $\mathbf{n}$ is orthogonal to every vector in the given plane. In particular, $\mathbf{n}$ is orthogonal to $\mathbf{r} - \mathbf{r}_0$ and so we have

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \tag{4}$$

which can be rewritten as

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0 \tag{5}$$

Either Equation 4 or Equation 5 is called a **vector equation of the plane.**

To obtain a scalar equation for the plane, we write $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$. Then the vector equation (4) becomes

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{6}$$

Equation 6 is the **scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$.**

EXAMPLE 4 Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$. Find the intercepts and sketch the plane.

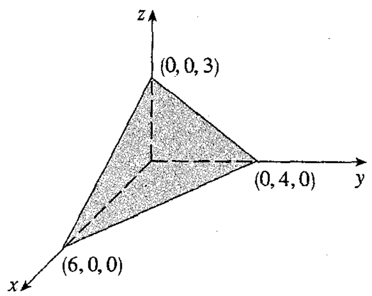


FIGURE 7

SOLUTION Putting $a = 2$, $b = 3$, $c = 4$, $x_0 = 2$, $y_0 = 4$, and $z_0 = -1$ in Equation 6, we see that an equation of the plane is

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0$$

or

$$2x + 3y + 4z = 12$$

To find the x-intercept we set $y = z = 0$ in this equation and obtain $x = 6$. Similarly, the y-intercept is 4 and the z-intercept is 3. This enables us to sketch the portion of the plane that lies in the first octant (see Figure 7). ■

By collecting terms in Equation 6 as we did in Example 4, we can rewrite the equation of a plane as

(7)
$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$. Equation 7 is called a **linear equation** in x, y, and z. Conversely, it can be shown that if a, b, and c are not all 0, then the linear equation (7) represents a plane with normal vector $\langle a, b, c\rangle$. (See Exercise 71.)

Figure 8 shows the portion of the plane in Example 5 that is enclosed by triangle PQR.

EXAMPLE 5 Find an equation of the plane that passes through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$.

SOLUTION The vectors $\mathbf{a}$ and $\mathbf{b}$ corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\mathbf{a} = \langle 2, -4, 4\rangle \qquad \mathbf{b} = \langle 4, -1, -2\rangle$$

Since both $\mathbf{a}$ and $\mathbf{b}$ lie in the plane, their cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane and can be taken as the normal vector. Thus

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\mathbf{i} + 20\mathbf{j} + 14\mathbf{k}$$

and an equation of the plane is

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$$

or

$$6x + 10y + 7z = 50$$ ■

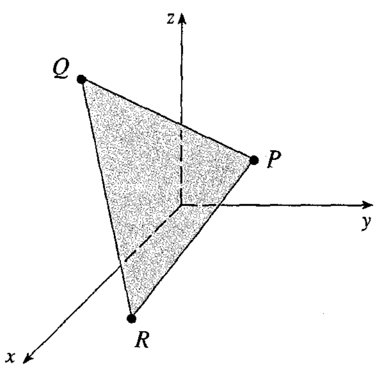


FIGURE 8

EXAMPLE 6 Find the point at which the line with parametric equations $x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

SOLUTION We substitute the expressions for x, y, and z from the parametric equations into the equation of the plane:

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18$$

This simplifies to $-10t = 20$, so $t = -2$. Therefore, the point of intersection occurs when the parameter value is $t = -2$. Then $x = 2 + 3(-2) = -4$, $y = -4(-2) = 8$, $z = 5 - 2 = 3$, and so the point of intersection is $(-4, 8, 3)$. ■

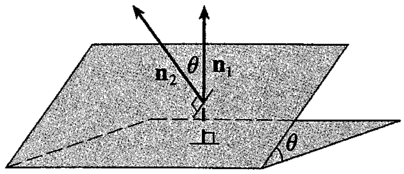


FIGURE 9

Two planes are **parallel** if their normal vectors are parallel. For instance, the planes $x + 2y - 3z = 4$ and $2x + 4y - 6z = 3$ are parallel because their normal vectors are $\mathbf{n}_1 = \langle 1, 2, -3 \rangle$ and $\mathbf{n}_2 = \langle 2, 4, -6 \rangle$ and $\mathbf{n}_2 = 2\mathbf{n}_1$. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors (see Figure 9).

EXAMPLE 7

(a) Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.

(b) Find symmetric equations for the line of intersection L of these two planes.

Figure 10 shows the planes in Example 7 and their line of intersection L.

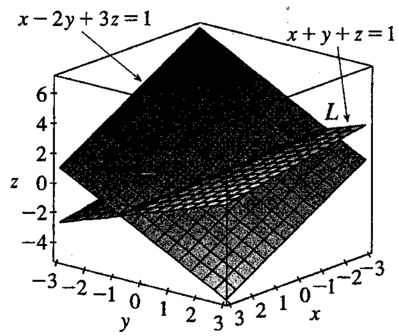


FIGURE 10

SOLUTION

(a) The normal vectors of these planes are

$$\mathbf{n}_1 = \langle 1, 1, 1 \rangle \qquad \mathbf{n}_2 = \langle 1, -2, 3 \rangle$$

and so if θ is the angle between the planes, Corollary 11.3.6 gives

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{1(1) + 1(-2) + 1(3)}{\sqrt{1+1+1}\sqrt{1+4+9}} = \frac{2}{\sqrt{42}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ$$

(b) We first need to find a point on L. For instance, we can find the point where the line intersects the xy-plane by setting $z = 0$ in the equations of both planes. This gives the equations $x + y = 1$ and $x - 2y = 1$, whose solution is $x = 1$, $y = 0$. So the point $(1, 0, 0)$ lies on L.

Now we observe that, since L lies in both planes, it is perpendicular to both of the normal vectors. Thus a vector $\mathbf{v}$ parallel to L is given by the cross product

Another way to find the line of intersection is to solve the equations of the planes for two of the variables in terms of the third, which can be taken as the parameter.

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$$

and so the symmetric equations of L can be written as

$$\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}$$

■

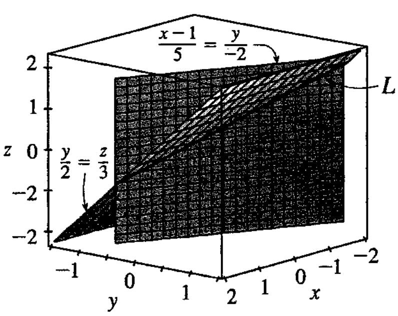


FIGURE 11

Figure 11 shows how the line L in Example 7 can also be regarded as the line of intersection of planes derived from its symmetric equations.

NOTE: Since a linear equation in x, y, and z represents a plane and two nonparallel planes intersect in a line, it follows that two linear equations can represent a line. The points (x, y, z) that satisfy both $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ lie on both of these planes and so the pair of linear equations represents the line of intersection of the planes (if they are not parallel). For instance, in Example 7 the line L was given as the line of intersection of the planes $x + y + z = 1$ and $x - 2y + 3z = 1$. The symmetric equations that we found for L could be written as

$$\frac{x-1}{5} = \frac{y}{-2} \qquad \text{and} \qquad \frac{y}{-2} = \frac{z}{-3}$$

which is again a pair of linear equations. They exhibit L as the line of intersection of the planes $(x - 1)/5 = y/(-2)$ and $y/(-2) = z/(-3)$. (See Figure 11.)

In general, when we write the equations of a line in the symmetric form

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

we can regard the line as the line of intersection of the two planes

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} \quad \text{and} \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

EXAMPLE 8 Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

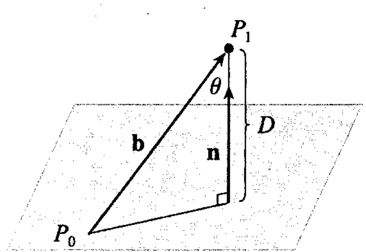


FIGURE 12

SOLUTION Let $P_0(x_0, y_0, z_0)$ be any point in the given plane and let $\mathbf{b}$ be the vector corresponding to $\overrightarrow{P_0P_1}$. Then

$$\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

From Figure 12 you can see that the distance D from P_1 to the plane is equal to the absolute value of the scalar projection of $\mathbf{b}$ onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$. (See Section 11.3.) Thus

$$\begin{aligned} D = \text{comp}_{\mathbf{n}}\,\mathbf{b} &= \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

Since P_0 lies in the plane, its coordinates satisfy the equation of the plane and so we have $ax_0 + by_0 + cz_0 + d = 0$. Thus the formula for D can be written as

(8)
$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

■

EXAMPLE 9 Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

SOLUTION First we note that the planes are parallel because their normal vectors $\langle 10, 2, -2 \rangle$ and $\langle 5, 1, -1 \rangle$ are parallel. To find the distance D between the planes, we choose any point on one plane and calculate its distance to the other plane. In particular, if we put $y = z = 0$ in the equation of the first plane, we get $10x = 5$ and so $(\frac{1}{2}, 0, 0)$ is a point in this plane. By Formula 8, the distance between $(\frac{1}{2}, 0, 0)$ and the plane $5x + y - z - 1 = 0$ is

$$D = \frac{|5(\frac{1}{2}) + 1(0) - 1(0) - 1|}{\sqrt{5^2 + 1^2 + (-1)^2}} = \frac{\frac{3}{2}}{3\sqrt{3}} = \frac{\sqrt{3}}{6}$$

So the distance between the planes is $\sqrt{3}/6$. ■

EXAMPLE 10 In Example 3 we showed that the lines

$$L_1: \quad x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$

$$L_2: \quad x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

are skew. Find the distance between them.

SOLUTION Since the two lines L_1 and L_2 are skew, they can be viewed as lying on two parallel planes P_1 and P_2. The distance between L_1 and L_2 is the same as the distance between P_1 and P_2, which can be computed as in Example 9. The common normal vector to both planes must be orthogonal to both $\mathbf{v}_1 = \langle 1, 3, -1 \rangle$ (the direction of L_1) and $\mathbf{v}_2 = \langle 0, 1, 4 \rangle$ (the direction of L_2). So a normal vector is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 0 & 1 & 4 \end{vmatrix} = 13\mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

If we put $s = 0$ in the equations of L_2, we get the point $(0, 3, -3)$ on L_2 and so an equation for P_2 is

$$13(x - 0) - 4(y - 3) + 1(z + 3) = 0 \qquad \text{or} \qquad 13x - 4y + z + 15 = 0$$

If we now set $t = 0$ in the equations for L_1, we get the point $(1, -2, 4)$ on P_1. So the distance between L_1 and L_2 is the same as the distance from $(1, -2, 4)$ to $13x - 4y + z + 15 = 0$. By Formula 8, this distance is

$$D = \frac{|13(1) - 4(-2) + 1(4) + 15|}{\sqrt{13^2 + (-4)^2 + 1^2}} = \frac{40}{\sqrt{186}} \approx 2.9$$

■

EXERCISES 11.5

1–4 ■ Find the vector equation and parametric equations for the line passing through the given point and parallel to the vector **a**.

1. $(3, -1, 8)$, $\mathbf{a} = \langle 2, 3, 5 \rangle$
2. $(-2, 4, 5)$, $\mathbf{a} = \langle 3, -1, 6 \rangle$
3. $(0, 1, 2)$, $\mathbf{a} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$
4. $(1, -1, -2)$, $\mathbf{a} = 2\mathbf{i} - 7\mathbf{k}$

5–10 ■ Find parametric equations and symmetric equations for the line passing through the given points.

5. $(2, 1, 8)$, $(6, 0, 3)$
6. $(-1, 0, 5)$, $(4, -3, 3)$
7. $(3, 1, -1)$, $(3, 2, -6)$
8. $(3, 1, \frac{1}{2})$, $(-1, 4, 1)$
9. $(-\frac{1}{3}, 1, 1)$, $(0, 5, -8)$
10. $(2, -7, 5)$, $(-4, 2, 5)$

11. Show that the line through the points $(2, -1, -5)$ and $(8, 8, 7)$ is parallel to the line through the points $(4, 2, -6)$ and $(8, 8, 2)$.

12. Show that the line through the points $(0, 1, 1)$ and $(1, -1, 6)$ is perpendicular to the line through the points $(-4, 2, 1)$ and $(-1, 6, 2)$.

13. (a) Find symmetric equations for the line that passes through the point $(0, 2, -1)$ and is parallel to the line with parametric equations $x = 1 + 2t$, $y = 3t$, and $z = 5 - 7t$.
(b) Find the points in which the required line in part (a) intersects the coordinate planes.

14. (a) Find parametric equations for the line through $(5, 1, 0)$ that is perpendicular to the plane $2x - y + z = 1$.
(b) In what points does this line intersect the coordinate planes?

15–18 ■ Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

15. L_1: $\dfrac{x - 4}{2} = \dfrac{y + 5}{4} = \dfrac{z - 1}{-3}$, L_2: $\dfrac{x - 2}{1} = \dfrac{y + 1}{3} = \dfrac{z}{2}$

16. L_1: $\dfrac{x - 1}{2} = \dfrac{y}{1} = \dfrac{z - 1}{4}$, L_2: $\dfrac{x}{1} = \dfrac{y + 2}{2} = \dfrac{z + 2}{3}$

17. L_1: $x = -6t$, $y = 1 + 9t$, $z = -3t$
L_2: $x = 1 + 2s$, $y = 4 - 3s$, $z = s$

18. L_1: $x = 1 + t, y = 2 - t, z = 3t$
L_2: $x = 2 - s, y = 1 + 2s, z = 4 + s$

19–22 ■ Find an equation of the plane through the given point and with the specified normal vector.

19. $(1, 4, 5)$, $\mathbf{n} = \langle 7, 1, 4 \rangle$

20. $(-5, 1, 2)$, $\mathbf{n} = \langle 3, -5, 2 \rangle$

21. $(1, 2, 3)$, $\mathbf{n} = 15\mathbf{i} + 9\mathbf{j} - 12\mathbf{k}$

22. $(-1, -6, -4)$, $\mathbf{n} = -5\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

23–26 ■ Find an equation of the plane passing through the given point and parallel to the indicated plane.

23. $(6, 5, -2)$, $x + y - z + 1 = 0$

24. $(3, 0, 8)$, $2x + 5y + 8z = 17$

25. $(-1, 3, -8)$, $3x - 4y - 6z = 9$

26. $(2, -4, 5)$, $z = 2x + 3y$

27–30 ■ Find an equation of the plane passing through the three given points.

27. $(0, 0, 0)$, $(1, 1, 1)$, $(1, 2, 3)$

28. $(-1, 1, -1)$, $(1, -1, 2)$, $(4, 0, 3)$

29. $(1, 0, -3)$, $(0, -2, -4)$, $(4, 1, 6)$

30. $(2, 1, -3)$, $(5, -1, 4)$, $(2, -2, 4)$

31–34 ■ Find an equation of the plane that passes through the given point and contains the indicated line.

31. $(1, 6, -4)$; $x = 1 + 2t, y = 2 - 3t, z = 3 - t$

32. $(-1, -3, 2)$; $x = -1 - 2t, y = 4t, z = 2 + t$

33. $(0, 1, 2)$; $x = y = z$

34. $(-1, 0, 1)$; $x = 5t, y = 1 + t, z = -t$

35–38 ■ Find the point at which the given line intersects the specified plane.

35. $x = 1 + t, y = 2t, z = 3t$; $x + y + z = 1$

36. $x = 5, y = 4 - t, z = 2t$; $2x - y + z = 5$

37. $x = 1 + 2t, y = -1, z = t$; $2x + y - z + 5 = 0$

38. $x = 1 - t, y = t, z = 1 + t$; $z = 1 - 2x + y$

39. Find direction numbers for the line of intersection of the planes $x + y + z = 1$ and $x + z = 0$.

40. Find the cosine of the angle between the planes $x + y + z = 0$ and $x + 2y + 3z = 1$.

41–46 ■ Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them.

41. $x + z = 1$, $y + z = 1$

42. $-8x - 6y + 2z = 1$, $z = 4x + 3y$

43. $x + 4y - 3z = 1$, $-3x + 6y + 7z = 0$

44. $2x + 2y - z = 4$, $6x - 3y + 2z = 5$

45. $2x + 4y - 2z = 1$, $-3x - 6y + 3z = 10$

46. $2x - 5y + z = 3$, $4x + 2y + 2z = 1$

47–48 ■ (a) Find symmetric equations for the line of intersection of the planes and (b) find the angle between the planes.

47. $x + y - z = 2$, $3x - 4y + 5z = 6$

48. $x - 2y + z = 1$, $2x + y + z = 1$

49–50 ■ Find parametric equations for the line of intersection of the planes.

49. $z = x + y$, $2x - 5y - z = 1$

50. $2x + 5z + 3 = 0$, $x - 3y + z + 2 = 0$

51. Find an equation for the plane consisting of all points that are equidistant from the points $(1, 1, 0)$ and $(0, 1, 1)$.

52. Find an equation for the plane consisting of all points that are equidistant from the points $(-4, 2, 1)$ and $(2, -4, 3)$.

53. Find an equation of the plane that passes through the line of intersection of the planes $x + y - z = 2$ and $2x - y + 3z = 1$ and passes through the point $(-1, 2, 1)$.

54. Find an equation of the plane that passes through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and is perpendicular to the plane $x + y - 2z = 1$.

55. Find an equation of the plane with x-intercept a, y-intercept b, and z-intercept c.

56. (a) Find the point at which the lines $\mathbf{r} = \langle 1, 1, 0 \rangle + t\langle 1, -1, 2 \rangle$ and $\mathbf{r} = \langle 2, 0, 2 \rangle + s\langle -1, 1, 0 \rangle$ intersect.
(b) Find an equation of the plane that contains these lines.

57. Find parametric equations for the line through the point $(0, 1, 2)$ that is parallel to the plane $x + y + z = 2$ and perpendicular to the line $x = 1 + t, y = 1 - t, z = 2t$.

58. Find parametric equations for the line through the point $(0, 1, 2)$ that is perpendicular to the line $x = 1 + t$, $y = 1 - t, z = 2t$ and intersects this line.

59. Which of the following four planes are parallel? Are any of them identical?

P_1: $4x - 2y + 6z = 3$ P_2: $4x - 2y - 2z = 6$
P_3: $-6x + 3y - 9z = 5$ P_4: $z = 2x - y - 3$

60. Which of the following four lines are parallel? Are any of them identical?

L_1: $x = 1 + t, \quad y = t, \quad z = 2 - 5t$
L_2: $x + 1 = y - 2 = 1 - z$
L_3: $x = 1 + t, \quad y = 4 + t, \quad z = 1 - t$
L_4: $\mathbf{r} = \langle 2, 1, -3 \rangle + t\langle 2, 2, -10 \rangle$

61–62 ■ Use the formula in Exercise 33 in Section 11.4 to find the distance from the point to the given line.

61. $(1,2,3)$; $x = 2 + t, y = 2 - 3t, z = 5t$

62. $(1,0,-1)$; $x = 5 - t, y = 3t, z = 1 + 2t$

63–64 ■ Find the distance from the point to the given plane.

63. $(2,8,5)$, $x - 2y - 2z = 1$

64. $(3,-2,7)$, $4x - 6y + z = 5$

65–66 ■ Find the distance between the given parallel planes.

65. $z = x + 2y + 1$, $3x + 6y - 3z = 4$

66. $3x + 6y - 9z = 4$, $x + 2y - 3z = 1$

67. Show that the distance between the parallel planes $ax + by + cz = d_1$ and $ax + by + cz = d_2$ is

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

68. Find equations of the planes that are parallel to the plane $x + 2y - 2z = 1$ and two units away from it.

69. Show that the lines with symmetric equations $x = y = z$ and $x + 1 = y/2 = z/3$ are skew lines, and find the distance between these lines.

70. Find the distance between the skew lines with parametric equations $x = 1 + t, y = 1 + 6t, z = 2t$ and $x = 1 + 2s$, $y = 5 + 15s, z = -2 + 6s$.

71. If a, b, and c are not all 0, show that the equation $ax + by + cz = d$ represents a plane and $\langle a, b, c\rangle$ is a normal vector to the plane. *Hint:* Suppose $a \neq 0$ and rewrite the equation in the form

$$a\left(x - \frac{d}{a}\right) + b(y - 0) + c(z - 0) = 0$$

72. Give a geometric description of each family of planes.
(a) $x + y + z = c$
(b) $x + y + cz = 1$
(c) $y\cos\theta + z\sin\theta = 1$

11.6 QUADRIC SURFACES

A **quadric surface** is the graph of a second-degree equation in three variables x, y, and z. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where $A, B, C, \ldots, J$ are constants, but by translation and rotation it can be brought into one of the two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0$$

Quadric surfaces are the analogues in three dimensions of the conic sections in the plane. (See Section 9.6 for a review of conic sections.)

In order to sketch the graph of a quadric surface (or any surface), it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called **traces** (or cross-sections) of the surface.

ELLIPSOIDS The quadric surface with equation

$$\text{(1)} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

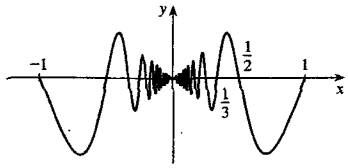
is called an **ellipsoid** because its traces are ellipses. For instance, the horizontal plane $z = k$ (where $-c < k < c$) intersects the surface in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2} \qquad z = k$$

and, in particular, the trace in the xy-plane is just the ellipse $x^2/a^2 + y^2/b^2 = 1, z = 0$. Similarly, the traces in the other coordinate planes are the ellipses with equations

PROBLEMS PLUS ■ page 658

1. $15!/5! = 10{,}897{,}286{,}400$
3. (b) 0 if $x = 0$, $(1/x) - \cot x$ if $x \neq n\pi$, n an integer
5. (a) $s_n = 3 \cdot 4^n$, $l_n = 1/3^n$, $p_n = 4^n/3^{n-1}$ (c) $2\sqrt{3}/5$
7. $2\pi/3 - \sqrt{3}/2$ **11.** $(-1, 1)$, $(x^3 + 4x^2 + x)/(1 - x)^4$
25. (b)

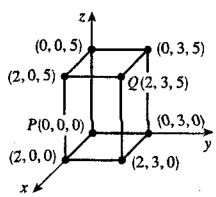


29. 100 **31.** $\frac{7}{4}$ **33.** (b) 1 (c) $\sqrt{3}$

CHAPTER 11

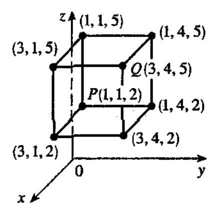
Exercises 11.1 ■ page 668

1. (a)



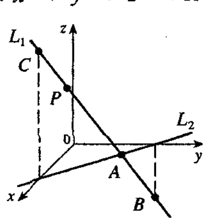
(b) $|PQ| = \sqrt{38}$

3. (a)



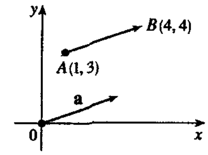
(b) $|PQ| = \sqrt{22}$

5. $|AB| = \sqrt{21}$, $|BC| = \sqrt{6}$, $|CA| = 3\sqrt{3}$; right triangle
7. $|AB| = \sqrt{69}$, $|BC| = \sqrt{158}$, $|CA| = 13$; neither
9. Not collinear **11.** $x^2 + (y - 1)^2 + (z + 1)^2 = 16$
13. $(x + 6)^2 + (y + 1)^2 + (z - 2)^2 = 12$ **15.** $(-1, -4, 2)$, 7
17. $(-\frac{1}{2}, 1, -3)$, $\frac{7}{2}$ **19.** $(\frac{1}{2}, 0, 0)$, $\frac{1}{2}$ **23.** $\sqrt{85}/2$, $\frac{5}{2}$, $\sqrt{94}/2$
25. (a) $(x - 2)^2 + (y + 3)^2 + (z - 6)^2 = 36$
(b) $(x - 2)^2 + (y + 3)^2 + (z - 6)^2 = 4$
(c) $(x - 2)^2 + (y + 3)^2 + (z - 6)^2 = 9$
27. $14x - 6y - 10z = 9$, a plane perpendicular to AB
29. A plane parallel to the yz-plane and 9 units in front of it
31. A half-space consisting of all points to the right of the plane $y = 2$
33. A plane perpendicular to the xz-plane and intersecting it in the line $x = z$, $y = 0$
35. Circular cylinder, radius 1, axis the z-axis
37. All points outside the sphere with radius 1 and center O
39. All points inside the sphere with radius 2 and center $(0, 0, 1)$ **41.** Hyperbolic cylinder
43. All points on and between the horizontal planes $z = 2$ and $z = -2$
45. $y < 0$ **47.** $r^2 < x^2 + y^2 + z^2 < R^2$
49. (a) $(2, 1, 4)$ (b)

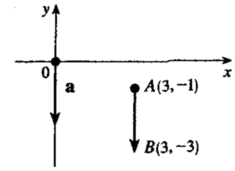


Exercises 11.2 ■ page 675

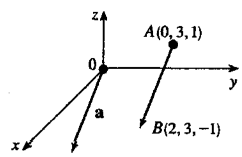
1. $\mathbf{a} = \langle 3, 1 \rangle$



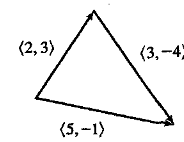
3. $\mathbf{a} = \langle 0, -2 \rangle$



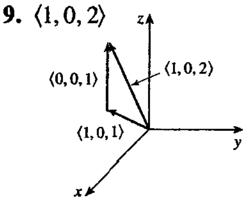
5. $\mathbf{a} = \langle 2, 0, -2 \rangle$



7. $\langle 5, -1 \rangle$

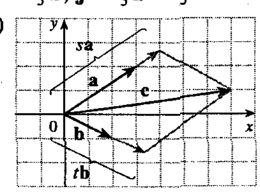


9. $\langle 1, 0, 2 \rangle$



11. 13, $\langle 3, -4 \rangle$, $\langle 7, -20 \rangle$, $\langle 10, -24 \rangle$, $\langle 7, -4 \rangle$

13. 7, $\langle 3, -2, 10 \rangle$, $\langle 1, -4, 2 \rangle$, $\langle 4, -6, 12 \rangle$, $\langle 10, -5, 34 \rangle$
15. $\sqrt{2}$, $2\mathbf{i}$, $-2\mathbf{j}$, $2\mathbf{i} - 2\mathbf{j}$, $7\mathbf{i} + \mathbf{j}$
17. $\sqrt{3}$, $3\mathbf{i} + 4\mathbf{k}$, $-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, $2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $11\mathbf{i} - \mathbf{j} + 15\mathbf{k}$ **19.** $\langle 1/\sqrt{5}, 2/\sqrt{5} \rangle$
21. $\langle -2/\sqrt{29}, 4/\sqrt{29}, 3/\sqrt{29} \rangle$ **23.** $(\mathbf{i} + \mathbf{j})/\sqrt{2}$
25. $\mathbf{i} = \frac{1}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}$, $\mathbf{j} = \frac{1}{5}\mathbf{a} - \frac{2}{5}\mathbf{b}$ **27.** **0**
29. (a), (b)



(d) $s = \frac{9}{7}$, $t = \frac{11}{7}$

31. $\mathbf{F} = (6\sqrt{3} - 5\sqrt{2})\mathbf{i} + (6 + 5\sqrt{2})\mathbf{j} \approx 3.32\mathbf{i} + 13.07\mathbf{j}$, $|\mathbf{F}| \approx 13.5$ lb, $\theta \approx 76°$ **33.** $\sqrt{493} \approx 22.2$ mi/h N8°W
35. A sphere with radius 1, centered at (x_0, y_0, z_0)

Exercises 11.3 ■ page 680

1. -1 **3.** -5 **5.** -11 **7.** 3 **11.** $\cos^{-1}\frac{11}{15} \approx 43°$
13. $\cos^{-1}(2/(13\sqrt{5})) \approx 86°$ **15.** $\cos^{-1}(1/(7\sqrt{3})) \approx 85°$
17. 114°, 33°, 33° **19.** Parallel **21.** Neither
23. Orthogonal **25.** ± 4 **27.** $-\frac{4}{5}$
29. $(\mathbf{i} - \mathbf{j} - \mathbf{k})/\sqrt{3}$ [or $(-\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$]
31. $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$; 71°, 48°, 48°
33. $-8/\sqrt{77}$, $3/\sqrt{77}$, $2/\sqrt{77}$; 156°, 70°, 77°
35. $5/\sqrt{38}$, $3/\sqrt{38}$, $2/\sqrt{38}$; 36°, 61°, 71°
37. $11/\sqrt{13}$, $\langle 22/13, 33/13 \rangle$ **39.** $3/\sqrt{5}$, $\langle 6/5, 3/5, 0 \rangle$
41. $1/\sqrt{2}$, $(\mathbf{i} + \mathbf{k})/2$
45. $\langle 0, 0, -2\sqrt{10} \rangle$ or any vector of the form $\langle s, t, 3s - 2\sqrt{10} \rangle$, $s, t \in \mathbb{R}$ **47.** 38 J **49.** $250\cos 20° \approx 235$ ft-lb
51. (a), (e), (f) **53.** $\frac{13}{5}$ **55.** $\cos^{-1}(1/\sqrt{3}) \approx 55°$

Exercises 11.4 ■ page 687

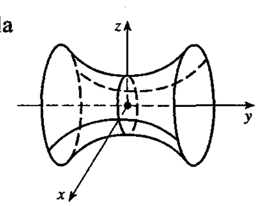
1. $\langle -1, 0, 1\rangle$ **3.** $\langle 3, 14, -9\rangle$ **5.** $-2\mathbf{i} + 2\mathbf{j}$
7. $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ **9.** $\langle -2, 6, -3\rangle$, $\langle 2, -6, 3\rangle$
11. $\langle -2/\sqrt{6}, -1/\sqrt{6}, 1/\sqrt{6}\rangle$, $\langle 2/\sqrt{6}, 1/\sqrt{6}, -1/\sqrt{6}\rangle$ **19.** 4
21. (a) $\langle 6, 3, 2\rangle$ (b) $\frac{7}{2}$ **23.** (a) $\langle -10, -3, 7\rangle$ (b) $\sqrt{158}/2$
25. 226 **27.** 21 **31.** $10.8 \sin 100° \approx 10.6$ J
33. (b) $\sqrt{97/3}$ **39.** (a) No (b) No (c) Yes

Exercises 11.5 ■ page 696

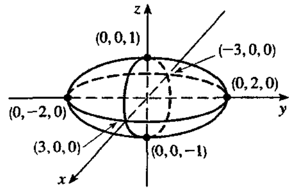
1. $\mathbf{r} = \langle 3, -1, 8\rangle + t\langle 2, 3, 5\rangle$,
$x = 3 + 2t$, $y = -1 + 3t$, $z = 8 + 5t$
3. $\mathbf{r} = (\mathbf{j} + 2\mathbf{k}) + t(6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$,
$x = 6t$, $y = 1 + 3t$, $z = 2 + 2t$
5. $x = 2 + 4t$, $y = 1 - t$, $z = 8 - 5t$;
$(x - 2)/4 = (y - 1)/(-1) = (z - 8)/(-5)$
7. $x = 3$, $y = 1 + t$, $z = -1 - 5t$;
$x = 3$, $y - 1 = (z + 1)/(-5)$
9. $x = (-1 + t)/3$, $y = 1 + 4t$, $z = 1 - 9t$;
$(x + \frac{1}{3})/(\frac{1}{3}) = (y - 1)/4 = (z - 1)/(-9)$
13. (a) $x/2 = (y - 2)/3 = (z + 1)/(-7)$
(b) $(-\frac{2}{7}, \frac{11}{7}, 0)$, $(-\frac{4}{3}, 0, \frac{11}{3})$, $(0, 2, -1)$
15. Skew **17.** Parallel
19. $7x + y + 4z = 31$ **21.** $5x + 3y - 4z + 1 = 0$
23. $x + y - z = 13$ **25.** $3x - 4y - 6z = 33$
27. $x - 2y + z = 0$ **29.** $17x - 6y - 5z = 32$
31. $25x + 14y + 8z = 77$ **33.** $x - 2y + z = 0$
35. $(1, 0, 0)$ **37.** $(-3, -1, -2)$ **39.** $1, 0, -1$
41. Neither, 60° **43.** Perpendicular **45.** Parallel
47. (a) $x - 2 = y/(-8) = z/(-7)$
(b) $\cos^{-1}(-\sqrt{6}/5) \approx 119°$ (or 61°)
49. $x = 6t$, $y = -\frac{1}{6} + t$, $z = -\frac{1}{6} + 7t$ **51.** $x = z$
53. $x - 2y + 4z + 1 = 0$ **55.** $(x/a) + (y/b) + (z/c) = 1$
57. $x = 3t$, $y = 1 - t$, $z = 2 - 2t$
59. P_1 and P_3 are parallel, P_2 and P_4 are identical
61. $\sqrt{22/5}$ **63.** $\frac{25}{3}$ **65.** $7\sqrt{6}/18$ **69.** $1/\sqrt{6}$

Exercises 11.6 ■ page 702

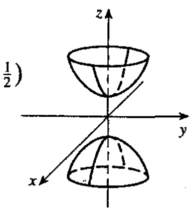
1. $x = k$, $z^2 - y^2 = 1 - k^2$, hyperbola
$y = k$, $x^2 + z^2 = 1 + k^2$, circle
$z = k$, $x^2 - y^2 = 1 - k^2$,
hyperbola
Hyperboloid of one sheet
with axis the y-axis



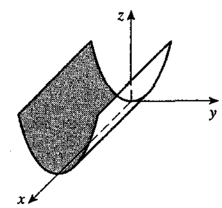
3. $x = k$, $y^2 + 4z^2 = 4 - 4k^2/9$, ellipse ($|k| < 3$)
$y = k$, $x^2 + 9z^2 = 9 - 9k^2/4$, ellipse ($|k| < 2$)
$z = k$, $4x^2 + 9y^2 = 36(1 - k^2)$, ellipse ($|k| < 1$)
Ellipsoid



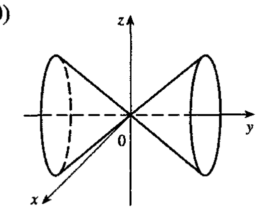
5. $x = k$, $4z^2 - y^2 = 1 + k^2$, hyperbola
$y = k$, $4z^2 - x^2 = 1 + k^2$, hyperbola
$z = k$, $x^2 + y^2 = 4k^2 - 1$, circle ($|k| > \frac{1}{2}$)
Hyperboloid of two sheets



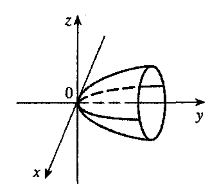
7. $x = k$, $z = y^2$, parabola
$y = k$, $z = k^2$, line
$z = k$, $y = \pm\sqrt{k}$, lines ($k > 0$)
Parabolic cylinder



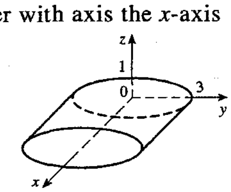
9. $x = k$, $y^2 - z^2 = k^2$, hyperbola ($k \neq 0$)
$y = k$, $x^2 + z^2 = k^2$, circle ($k \neq 0$)
$z = k$, $y^2 - x^2 = k^2$,
hyperbola ($k \neq 0$)
$x = 0$, $y = \pm z$, lines
Cone



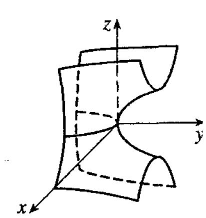
11. $x = k$, $y = 4z^2 + k^2$, parabola
$y = k$, $x^2 + 4z^2 = k$, ellipse ($k > 0$)
$z = k$, $y = x^2 + 4k^2$, parabola
Elliptic paraboloid with
axis the y-axis



13. $x = k$, $y^2 + 9z^2 = 9$, ellipse
$y = k$, $z = \pm\sqrt{1 - (k^2/9)}$, pairs of lines ($|k| < 3$)
$z = k$, $y = \pm 3\sqrt{1 - k^2}$, pairs of lines ($|k| < 1$)
Elliptic cylinder with axis the x-axis



15. $x = k$, $y = z^2 - k^2$, parabola
$y = k$, $z^2 - x^2 = k$, hyperbola ($k \neq 0$)
$z = k$, $y = k^2 - x^2$, parabola
Hyperbolic paraboloid



17. VII **19.** II **21.** VI **23.** VIII
25. $(x^2/4) + (y^2/3) - (z^2/12) = 1$
Hyperboloid of one sheet with
axis the z-axis