Summary

To find the total distance travelled given a velocity function $v(t) = s'(t)$ on $a \leqslant t \leqslant b$:

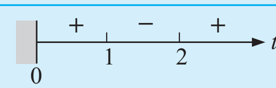
- Draw a sign diagram for $v(t)$ so that we can determine directional changes, if they exist.
- Determine $s(t)$ by integration, with integrating constant c, say.
- Find $s(a)$ and $s(b)$. Also find $s(t)$ at every point where there is a direction reversal.
- Draw a motion diagram.
- Determine the total distance travelled from the motion diagram.

Example 15

A particle P moves in a straight line with velocity function
$v(t) = t^2 - 3t + 2$ ms^{-1}. How far does P travel in the first 4 seconds of motion?

$$v(t) = s'(t) = t^2 - 3t + 2 = (t-1)(t-2)$$

$\therefore$ sign diagram of v is:



Since the signs change, P reverses direction at $t = 1$ and $t = 2$ secs.

Now $s(t) = \int (t^2 - 3t + 2)\, dt$

$$= \frac{t^3}{3} - \frac{3t^2}{2} + 2t + c$$

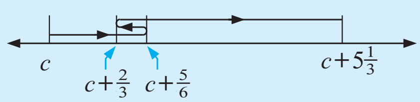
Now $s(0) = c$

$s(1) = \frac{1}{3} - \frac{3}{2} + 2 + c = c + \frac{5}{6}$

$s(2) = \frac{8}{3} - 6 + 4 + c = c + \frac{2}{3}$

$s(4) = \frac{64}{3} - 24 + 8 + c = c + 5\frac{1}{3}$

Motion diagram:



$\therefore$ total distance $= \left(c + \frac{5}{6} - c\right) + \left(c + \frac{5}{6} - [c + \frac{2}{3}]\right) + \left(c + 5\frac{1}{3} - [c + \frac{2}{3}]\right)$

$= \frac{5}{6} + \frac{5}{6} - \frac{2}{3} + 5\frac{1}{3} - \frac{2}{3}$

$= 5\frac{2}{3}$ m

EXERCISE 26H

1 A particle has velocity function $v(t) = 1 - 2t$ cms^{-1} as it moves in a straight line. Find the total distance travelled in the first second of motion.

2 Particle P has velocity $v(t) = t^2 - t - 2$ cms^{-1}. Find the total distance travelled in the first 3 seconds of motion.

3 A particle moves along the x-axis with velocity function $x'(t) = 16t - 4t^3$ units/s. Find the total distance travelled in the time interval:

a $0 \leqslant t \leqslant 3$ seconds **b** $1 \leqslant t \leqslant 3$ seconds.

4 The velocity of a particle travelling in a straight line is given by $v(t) = 50 - 10e^{-0.5t}$ ms^{-1}, where $t \geqslant 0$, t in seconds.

- **a** State the initial velocity of the particle.
- **b** Find the velocity of the particle after 3 seconds.
- **c** How long would it take for the particle's velocity to increase to 45 ms^{-1}?
- **d** Discuss $v(t)$ as $t \to \infty$.
- **e** Show that the particle's acceleration is always positive.
- **f** Draw the graph of $v(t)$ against t.
- **g** Find the total distance travelled by the particle in the first 3 seconds of motion.

5 A train moves along a straight track with acceleration $\frac{t}{10} - 3$ ms^{-2}. If the initial velocity of the train is 45 ms^{-1}, determine the total distance travelled in the first minute.

6 A body has initial velocity 20 ms^{-1} as it moves in a straight line with acceleration function $4e^{-\frac{t}{20}}$ ms^{-2}.

- **a** Show that as t increases the body approaches a limiting velocity.
- **b** Find the total distance travelled in the first 10 seconds of motion.

I DEFINITE INTEGRALS

If $F(x)$ is the antiderivative of $f(x)$ where $f(x)$ is continuous on the interval $a \leqslant x \leqslant b$ then the **definite integral** of $f(x)$ on this interval is

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Note: $\int_a^b f(x)\,dx$ reads "the integral of $f(x)$ from $x = a$ to $x = b$, with respect to x".

Notation: We write $F(b) - F(a) = [F(x)]_a^b$.

Example 16

Find $\int_1^3 (x^2 + 2)\,dx$

$\int_1^3 (x^2 + 2)\,dx$

$= \left[\frac{x^3}{3} + 2x\right]_1^3$

$= \left(\frac{3^3}{3} + 2(3)\right) - \left(\frac{1^3}{3} + 2(1)\right)$

$= (9 + 6) - (\frac{1}{3} + 2)$

$= 12\frac{2}{3}$

Check:

```
fnInt(X²+2,X,1,3
)
        12.66666667
```

g $\frac{4}{3}x^{\frac{3}{2}} - 2\sqrt{x} + c$ **h** $-\dfrac{2}{\sqrt{x}} - 4\ln|x| + c$

i $\frac{1}{4}(x+1)^4 + c$

2 **a** $y = -\frac{1}{6}(1-2x)^3 + c$ **b** $y = \frac{2}{3}x^{\frac{3}{2}} - 4\sqrt{x} + c$

c $y = x + 2\ln|x| + \dfrac{5}{x} + c$

3 **a** $f(x) = \frac{1}{4}x^4 - \frac{5}{2}x^2 + 3x + c$ **b** $f(x) = \frac{4}{3}x^{\frac{3}{2}} - \frac{12}{5}x^{\frac{5}{2}} + c$

c $f(x) = 3e^x - 4\ln|x| + c$

4 **a** $f(x) = x^2 - x + 3$ **b** $f(x) = x^3 + x^2 - 7$

c $f(x) = e^x + 2\sqrt{x} - 1 - e$ **d** $f(x) = \frac{1}{2}x^2 - 4\sqrt{x} + \frac{11}{2}$

5 **a** $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \frac{1}{3}$

b $f(x) = 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + 5$

c $f(x) = \frac{1}{3}x^3 - \frac{16}{3}x + 5$

EXERCISE 26F

1 **a** $\frac{1}{8}(2x+5)^4 + c$ **b** $\dfrac{1}{2(3-2x)} + c$ **c** $\dfrac{-2}{3(2x-1)^3} + c$

d $\frac{1}{32}(4x-3)^8 + c$ **e** $\frac{2}{9}(3x-4)^{\frac{3}{2}} + c$ **f** $-4\sqrt{1-5x} + c$

g $-\frac{3}{5}(1-x)^5 + c$ **h** $-2\sqrt{3-4x} + c$ **i** $\frac{3}{8}(2x-1)^{\frac{4}{3}} + c$

2 **a** $y = \frac{1}{3}(2x-7)^{\frac{3}{2}} + 2$ **b** $(-8, -19)$

3 **a** $\frac{1}{2}(2x-1)^3 + c$ **b** $\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 + c$

c $-\frac{1}{12}(1-3x)^4 + c$ **d** $x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + c$

e $-\frac{8}{3}(5-x)^{\frac{3}{2}} + c$ **f** $\frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 + x + c$

4 **a** $2e^x + \frac{5}{2}e^{2x} + c$ **b** $\frac{1}{3}x^3 + \frac{2}{3}e^{-3x} + c$

c $\frac{2}{3}x^{\frac{3}{2}} + 2e^{2x} + e^{-x} + c$ **d** $\frac{1}{2}\ln|2x-1| + c$

e $-\frac{5}{3}\ln|1-3x| + c$ **f** $-e^{-x} - 2\ln|2x+1| + c$

g $\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c$ **h** $-\frac{1}{2}e^{-2x} - 4e^{-x} + 4x + c$

i $\frac{1}{2}x^2 + 5\ln|1-x| + c$

5 **a** $y = x - 2e^x + \frac{1}{2}e^{2x} + c$ **b** $y = x - x^2 + 3\ln|x+2| + c$

c $y = -\frac{1}{2}e^{-2x} + 2\ln|2x-1| + c$

6 Both are correct. Recall that:

$$\frac{d}{dx}(\ln|Ax|) = \frac{d}{dx}(\ln|A| + \ln|x|) = \frac{1}{x}$$

7 **a** $f(x) = -e^{-2x} + 4$

b $f(x) = x^2 + 2\ln|1-x| + 2 - 2\ln 2$

c $f(x) = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{8}e^{-4x} + \frac{1}{8}e^{-4} - \frac{2}{3}$

8 $\displaystyle\int \frac{2x-8}{x^2-4}\,dx = 3\ln|x+2| - \ln|x-2| + c$

9 $\displaystyle\int \frac{2}{4x^2-1}\,dx = \frac{1}{2}\ln|2x-1| - \frac{1}{2}\ln|2x+1| + c$

EXERCISE 26G

1 **a** $\frac{1}{5}(x^3+1)^5 + c$ **b** $2\sqrt{x^2+3} + c$

c $\frac{2}{3}(x^3+x)^{\frac{3}{2}} + c$ **d** $\frac{1}{4}(2+x^4)^4 + c$

e $\frac{1}{5}(x^3+2x+1)^5 + c$ **f** $-\dfrac{1}{27(3x^3-1)^3} + c$

g $\dfrac{1}{8(1-x^2)^4} + c$ **h** $-\dfrac{1}{2(x^2+4x-3)} + c$

i $\frac{1}{5}(x^2+x)^5 + c$

2 **a** $e^{1-2x} + c$ **b** $e^{x^2} + c$ **c** $\frac{1}{3}e^{x^3+1} + c$

d $2e^{\sqrt{x}} + c$ **e** $-e^{x-x^2} + c$ **f** $e^{1-\frac{1}{x}} + c$

3 **a** $\ln\left|x^2+1\right| + c$ **b** $-\frac{1}{2}\ln\left|2-x^2\right| + c$

c $\ln\left|x^2-3x\right| + c$ **d** $2\ln\left|x^3-x\right| + c$

e $-2\ln\left|5x-x^2\right| + c$ **f** $-\frac{1}{3}\ln\left|x^3-3x\right| + c$

4 **a** $f(x) = -\frac{1}{9}(3-x^3)^3 + c$ **b** $f(x) = \frac{3}{2}\ln\left|x^2-2\right| + c$

c $f(x) = -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + c$ **d** $f(x) = -\frac{1}{2}e^{1-x^2} + c$

e $f(x) = -\ln\left|x^3-x\right| + c$ **f** $f(x) = \frac{1}{4}(\ln x)^4 + c$

g $f(x) = \ln\left|x^3+2x^2-1\right| + c$ **h** $f(x) = 4\ln|\ln x| + c$

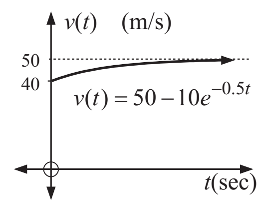
i $f(x) = \dfrac{-1}{\ln x} + c$

EXERCISE 26H

1 $\frac{1}{2}$ cm **2** $5\frac{1}{6}$ cm **3** **a** 41 units **b** 34 units

4 **a** 40 m/s **b** 47.77 m/s **c** 1.386 seconds

d as $t \to \infty$, $v(t) \to 50$ **f**



e $a(t) = 5e^{-0.5t}$ and as $e^x > 0$ for all x, $a(t) > 0$ for all t.

g 134.5 m

5 900 m

6 **a** Show that $v(t) = 100 - 80e^{-\frac{1}{20}t}$ m/s and as $t \to \infty$, $v(t) \to 100$ m/s **b** 370.4 m

EXERCISE 26I

1 **a** $\frac{1}{4}$ **b** $\frac{2}{3}$ **c** $e-1$ ($\doteqdot$ 1.718) **d** $1\frac{1}{2}$ **e** $6\frac{2}{3}$

f $\ln 3$ ($\doteqdot$ 1.099) **g** 1.524 **h** 2 **i** $e-1$ ($\doteqdot$ 1.718)

2 **a** $\frac{1}{12}$ **b** 1.557 **c** $20\frac{1}{3}$ **d** 0.0337

e $\frac{1}{2}\ln(\frac{2}{7})$ ($\doteqdot$ −0.6264) **f** $\frac{1}{2}(\ln 2)^2$ ($\doteqdot$ 0.2402)

g 0 **h** $2\ln 7$ ($\doteqdot$ 3.892) **i** $\dfrac{3^{n+1}}{2n+2}$, $n \neq -1$

3 **Hint**: $\ln A - \ln B = \ln\dfrac{A}{B}$

EXERCISE 26J

1 **a** $\frac{1}{3}$ units^2 **b** $3\frac{3}{4}$ units^2 **c** $e-1$ ($\doteqdot$ 1.718) units^2

d $20\frac{5}{6}$ units^2 **e** 18 units^2 **f** $\ln 4$ ($\doteqdot$ 1.386) units^2

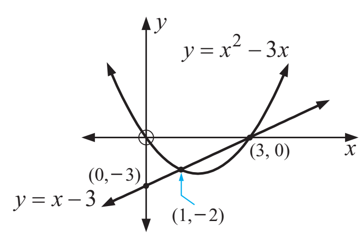
g $\ln 3$ ($\doteqdot$ 1.099) u^2 **h** $4\frac{1}{2}$ u^2 **i** $2e - \dfrac{2}{e}$ ($\doteqdot$ 4.701) u^2

2 **a** $4\frac{1}{2}$ u^2 **b** $1 + e^{-2}$ ($\doteqdot$ 1.135) u^2 **c** $1\frac{5}{27}$ u^2 **d** $2\frac{1}{4}$ u^2

3 **a** $40\frac{1}{2}$ units^2 **b** 8 units^2 **c** 8 units^2

4 **a** $10\frac{2}{3}$ units^2

b **i, ii**



iii $1\frac{1}{3}$ units^2

c $\frac{1}{3}$ units^2