Example 4

A particle moves in a straight line with displacement from O given by $s(t) = 3t - t^2$ metres at time t seconds. Find:

a the average velocity in the time interval from $t = 2$ to $t = 5$ seconds

b the average velocity in the time interval from $t = 2$ to $t = 2 + h$ seconds

c $\lim\limits_{h \to 0} \dfrac{s(2+h) - s(2)}{h}$ and comment on its significance.

a average velocity

$$= \frac{s(5) - s(2)}{5 - 2} \text{ ms}^{-1}$$
$$= \frac{(15 - 25) - (6 - 4)}{3} \text{ ms}^{-1}$$
$$= \frac{-10 - 2}{3} \text{ ms}^{-1}$$
$$= -4 \text{ ms}^{-1}$$

b average velocity

$$= \frac{s(2+h) - s(2)}{2 + h - 2}$$
$$= \frac{3(2+h) - (2+h)^2 - 2}{h}$$
$$= \frac{6 + 3h - 4 - 4h - h^2 - 2}{h}$$
$$= \frac{-h - h^2}{h}$$
$$= -1 - h \text{ ms}^{-1} \text{ as } h \neq 0$$

c $$\lim_{h \to 0} \frac{s(2+h) - s(2)}{h} = \lim_{h \to 0} (-1 - h)$$
$$= -1 \text{ ms}^{-1}$$

and this is the instantaneous velocity at time $t = 2$ seconds.

EXERCISE 22D.1

1 A particle P moves in a straight line with a displacement function of $s(t) = t^2 + 3t - 2$ metres, where $t \geqslant 0$, t in seconds.

a Find the average velocity from $t = 1$ to $t = 3$ seconds.

b Find the average velocity from $t = 1$ to $t = 1 + h$ seconds.

c Find the value of $\lim\limits_{h \to 0} \dfrac{s(1+h) - s(1)}{h}$ and comment on its significance.

d Find the average velocity from time t to time $t + h$ seconds and interpret

$$\lim_{h \to 0} \frac{s(t+h) - s(t)}{h}.$$

2 A particle P moves in a straight line with a displacement function of $s(t) = 5 - 2t^2$ cm, where $t \geqslant 0$, t in seconds.

a Find the average velocity from $t = 2$ to $t = 5$ seconds.

b Find the average velocity from $t = 2$ to $t = 2 + h$ seconds.

c Find the value of $\lim\limits_{h \to 0} \dfrac{s(2+h) - s(2)}{h}$ and state the meaning of this value.

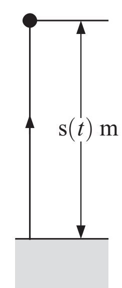
d Interpret $\lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$.

EXERCISE 22D.2

1 An object moves in a straight line with position given by $s(t) = t^2 - 4t + 3$ cm from an origin O, $t \geqslant 0$, t in seconds.

- **a** Find expressions for its velocity and acceleration at any instant and draw sign diagrams for each function.
- **b** Find the initial conditions and explain what is happening to the object at that instant.
- **c** Describe the motion of the object at time $t = 2$ seconds.
- **d** At what time(s) does the object reverse direction? Find the position of the object at these instants.
- **e** Draw a motion diagram of the object.
- **f** For what time intervals is the speed of the object decreasing?

2 A stone is projected vertically upwards so that its position above ground level after t seconds is given by $s(t) = 98t - 4.9t^2$ metres, $t \geqslant 0$.



- **a** Find the velocity and acceleration functions for the stone and draw sign diagrams for each function.
- **b** Find the initial position and velocity of the stone.
- **c** Describe the stone's motion at times $t = 5$ and $t = 12$ seconds.
- **d** Find the maximum height reached by the stone.
- **e** Find the time taken for the stone to hit the ground.

3 A particle moves in a straight line with displacement function

$$s(t) = 12t - 2t^3 - 1 \text{ centimetres}, \quad t \geqslant 0, \quad t \text{ in seconds.}$$

- **a** Find velocity and acceleration functions for the particle's motion.
- **b** Find the initial conditions and interpret their meaning.
- **c** Find the times and positions when the particle reverses direction.
- **d** At what times is the particle's: **i** speed increasing **ii** velocity increasing?

4 The position of a particle moving along the x-axis is given by $x(t) = t^3 - 9t^2 + 24t$ metres, $t \geqslant 0$, t in seconds.

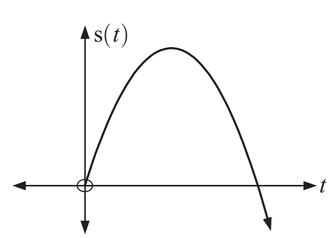
- **a** Draw sign diagrams for the particle's velocity and acceleration functions.
- **b** Find the position of the particle at the times when it reverses direction, and hence draw a motion diagram for the particle.
- **c** At what times is the particle's: **i** speed decreasing **ii** velocity decreasing?
- **d** Find the total distance travelled by the particle in the first 5 seconds of motion.

5 An experiment to determine the position of an object fired vertically upwards from the earth's surface was performed. From the results, a two dimensional graph of position above the earth's surface $s(t)$ metres, against time t seconds, was plotted.

It was noted that the graph was *parabolic*.
Assuming a constant gravitational acceleration g, show that if the initial velocity is $v(0)$ then:

a $v(t) = v(0) + gt$, and **b** $s(t) = v(0) \times t + \frac{1}{2}gt^2$.

[**Hint:** Assume $s(t) = at^2 + bt + c$.]



4 **a** $\dfrac{dV}{dt} = 1.2\ \text{m}^3/\text{min}$ **b** $\dfrac{dV}{dr} = 4\pi r^2$ **d** 0.007 368 m/min

5 **a** 1.2 m

b $\dfrac{ds}{dt} = 28.1 - 9.8t$ represents the instantaneous velocity of the ball

c $t = 2.867$ secs. The ball has stopped and reached its maximum height. **d** 41.49 m

e **i** 28.1 m/s **ii** 8.5 m/s **iii** -20.9 m/s

$s'(t) \geqslant 0$ ball travelling upwards

$s'(t) \leqslant 0$ ball travelling downwards

f 5.735 sec **g** $\dfrac{d^2s}{dt^2}$ is the rate of change of $\dfrac{ds}{dt}$, i.e., the instantaneous acceleration.

6 **b** 69.58 m/s

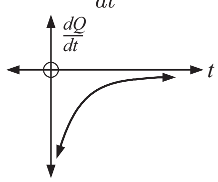
EXERCISE 22C.1

1 **a** **i** $Q = 100$ **ii** $Q = 50$ **iii** $Q = 0$

b **i** decr. 1 unit per year **ii** decr. $\frac{1}{\sqrt{2}}$ units per year

c **Hint**: Consider the graph of $\dfrac{dQ}{dt}$ against t.

$\dfrac{dQ}{dt} = \dfrac{-5}{\sqrt{t}} < 0$ for all $t > 0$



2 **a** 18.2 metres

b $t = 4$; 19 m, $t = 8$; 19.3 m, $t = 12$; 19.5 m

c $t = 0$: 0.36 m/year $t = 5$: 0.09 m/year $t = 10$: 0.04 m/year

d as $\dfrac{dH}{dt} = \dfrac{9}{(t+5)^2} > 0$, for all $t \geqslant 0$, the tree is always growing, and $\dfrac{dH}{dt} \to 0$ as t increases

3 **a** 0^oC; 20, 20^oC; 24, 40^oC; 32 **b** $\dfrac{dR}{dT} = \frac{1}{10} + \dfrac{T}{100}$

c $\dfrac{dR}{dT} > 0$ (i.e., inc) for all $T > -10^o$C

4 **a** **i** \$4500 **ii** \$8250

b **i** increase of \$100 per km/h

ii increase of \$188.89 per km/h

c $\dfrac{dC}{dt} = 0$ at $v = \sqrt{50}$ i.e., 7.1 km/h

5 **a** The near part of the lake is 2 km from the sea, the furthest part is 3 km.

b $\dfrac{dy}{dx} = \frac{3}{10}x^2 - x + \frac{3}{5}$ $x = \frac{1}{2}$; $\dfrac{dy}{dx} = 0.175$, height of hill is increasing as slope is positive

$x = 1\frac{1}{2}$; $\dfrac{dy}{dx} = -0.225$, height of hill is decreasing as slope is negative

$\therefore$ top of the hill is between $x = \frac{1}{2}$ and $x = 1\frac{1}{2}$.

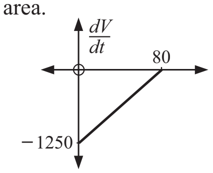
c 2.55 km from the sea, 63.1 m deep

6 **a** $\dfrac{dV}{dx} = 3x^2\ \text{mm}^3/\text{mm}$. This is the rate at which the volume increases as the length of the sides increase.

b $\dfrac{dV}{dx} = 12\ \text{mm}^3/\text{mm}$ at $x = 2$. For every millimetre the sides increase, the volume increases by 12 mm^3.

c For a small change in side length (Δx), the increase in volume is approx. $\Delta x \times$ surface area.

7 **a** $\dfrac{dV}{dt} = -1250\left(1 - \dfrac{t}{80}\right)$



b at $t = 0$ (when the tap was first opened)

c $\dfrac{d^2V}{dt^2} = \dfrac{125}{8}$ This shows that the rate of change of V is constantly increasing, i.e., the outflow is decreasing at a constant rate.

8 **a** When $\dfrac{dP}{dt} = 0$, the population is not changing over time, i.e., it is stable.

b 4000 fish **c** 8000 fish

EXERCISE 22C.2

1 **a** $C'(x) = 0.0009x^2 + 0.04x + 4$ dollars per pair

b $C'(220) = \$56.36$ per pair. This estimates the additional cost of making one more pair of jeans if 220 pairs are currently being made.

c \$56.58 This is the actual increase in cost to make an extra pair of jeans (221 rather than 220).

d $C''(x) = 0.0018x + 0.04$, $C''(x) = 0$ when $x = -22.2$ This is where the rate of change is a minimum, however it is out of the bounds of the model (you cannot make < 0 jeans!)

2 **a** $C'(x) = 0.000\,216x^2 - 0.001\,22x + 0.19$ dollars per item. $C'(x)$ is the instantaneous rate of change in cost with respect to the number of items made.

b $C'(300) = \$19.26$ per item. This estimates the increase in cost to make 301 items, rather than 300.

c \$19.33

d $C''(x) = 0.000\,432x - 0.001\,22$, $x = 2.8$ This is the minimum of the $C'(x)$ curve.

EXERCISE 22D.1

1 **a** 7 m/s **b** $(h+5)$ m/s **c** 5 m/s $= s'(1)$

d av. velocity $= (2t + h + 3)$ m/s,

$\lim\limits_{h \to 0}(2t + h + 3) = s'(t) \to 2t + 3$ as $h \to 0$

2 **a** -14 cm/s **b** $(-8 - 2h)$ cm/s

c -8 cm/s $= s'(2)$ i.e., velocity $= -8$ cm/s at $t = 2$

d $-4t = s'(t) = v(t)$

3 **a** $\frac{2}{3}$ m/s^2 **b** $\left(\dfrac{2}{\sqrt{1+h}+1}\right)$ m/s^2 **c** 1 m/s^2 $= v'(1)$

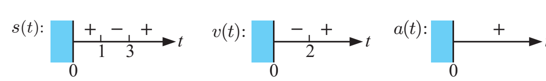
d $\dfrac{1}{\sqrt{t}}$ m/s^2 $= v'(t)$ i.e., the instantaneous accn. at time t.

4 **a** velocity at $t = 3$ **b** acceleration at $t = 5$

c velocity at t, i.e., $v(t)$ **d** acceleration at t, i.e., $a(t)$

EXERCISE 22D.2

1 **a** $v(t) = 2t - 4$, $a(t) = 2$

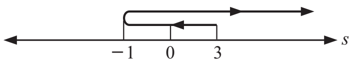


b The object is initially 3 cm to the right of the origin and is moving to the left at 4 cm/s. It is accelerating at 2 m/s^2 to the right.

c The object is instantaneously stationary, 1 cm to the left of the origin and is accelerating to the right at 2 m/s^2.

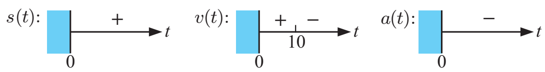
d At $t = 2$, $s(2) = 1$ cm to the left of the origin.

e



f $0 \leqslant t \leqslant 2$

2 **a** $v(t) = 98 - 9.8t$, $a(t) = -9.8$



b $s(0) = 0$ m above the ground
$v(0) = 98$ m/s skyward

c $t = 5$ Stone is 367.5 m above the ground and moving skyward at 49 m/s. Its speed is decreasing. $t = 12$ Stone is 470.4 m above the ground and moving groundward at 19.6 m/s. Its speed is increasing.

d 490 m **e** 20 seconds

3 **a** $v(t) = 12 - 6t^2$, $a(t) = -12t$

b $s(0) = -1$, $v(0) = 12$, $a(0) = 0$

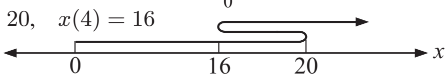
Particle started 1 cm to the left of the origin and was travelling to the right at a constant speed of 12 cm/s.

c $t = \sqrt{2}$, $s(\sqrt{2}) = 8\sqrt{2} - 1 \doteqdot 10.3$

d **i** $t \geqslant \sqrt{2}$ **ii** never

4 **a** $v(t) = 3t^2 - 18t + 24$ $\quad a(t) = 6t - 18$

b $x(2) = 20$, $x(4) = 16$



c **i** $0 \leqslant t \leqslant 2$ and $3 \leqslant t \leqslant 4$ **ii** $0 \leqslant t \leqslant 3$ **d** 28 m

5 **Hint**: $s'(t) = v(t)$ and $s''(t) = a(t) = g$

Show that $a = \frac{1}{2}g$ $\quad b = v(0)$ $\quad c = 0$

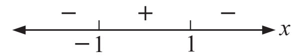
EXERCISE 22E.1

1 **a** **i** $x \geqslant 0$ **ii** never **b** **i** never **ii** $-2 < x \leqslant 3$
c **i** $x \leqslant 2$ **ii** $x \geqslant 2$ **d** **i** all real x **ii** never
e **i** $1 \leqslant x \leqslant 5$ **ii** $x \leqslant 1, x \geqslant 5$
f **i** $2 \leqslant x < 4$, $x > 4$ **ii** $x < 0$, $0 < x \leqslant 2$

EXERCISE 22E.2

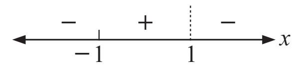
1 **a** increasing for $x \geqslant 0$, decreasing for $x \leqslant 0$
b decreasing for all x
c increasing for $x \geqslant -\frac{3}{4}$, decreasing for $x \leqslant -\frac{3}{4}$
d increasing for $x \geqslant 0$, never decreasing
e decreasing for $x > 0$, never increasing
f incr. for $x \leqslant 0$ and $x \geqslant 4$, decr. for $0 \leqslant x \leqslant 4$
g increasing for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$,
decreasing for $x \leqslant -\sqrt{\frac{2}{3}}$, $x \geqslant \sqrt{\frac{2}{3}}$
h decr. for $x \leqslant -\frac{1}{2}$, $x \geqslant 3$, incr. for $-\frac{1}{2} \leqslant x \leqslant 3$
i increasing for $x \geqslant 0$, decreasing for $x \leqslant 0$
j increasing for $x \geqslant -\frac{3}{2} + \frac{\sqrt{5}}{2}$ and $x \leqslant -\frac{3}{2} - \frac{\sqrt{5}}{2}$
decreasing for $-\frac{3}{2} - \frac{\sqrt{5}}{2} \leqslant x \leqslant -\frac{3}{2} + \frac{\sqrt{5}}{2}$
k increasing for $x \leqslant 2 - \sqrt{3}$, $x \geqslant 2 + \sqrt{3}$
decreasing for $2 - \sqrt{3} \leqslant x \leqslant 2 + \sqrt{3}$
l increasing for $x \geqslant 1$, decreasing for $0 \leqslant x \leqslant 1$
m increasing for $-1 \leqslant x \leqslant 1$, $x \geqslant 2$
decreasing for $x \leqslant -1$, $1 \leqslant x \leqslant 2$
n increasing for $1 - \sqrt{2} \leqslant x \leqslant 1$, $x \geqslant 1 + \sqrt{2}$
decreasing for $x \leqslant 1 - \sqrt{2}$, $1 \leqslant x \leqslant 1 + \sqrt{2}$

2 **a** **i**



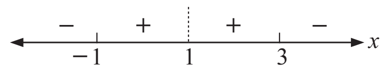
ii increasing for $-1 \leqslant x \leqslant 1$
decreasing for $x \leqslant -1$, $x \geqslant 1$

b **i**



ii increasing for $-1 \leqslant x < 1$
decreasing for $x \leqslant -1$, $x > 1$

c **i**



ii increasing for $-1 \leqslant x < 1$, $1 < x \leqslant 3$
decreasing for $x \leqslant -1$, $x \geqslant 3$

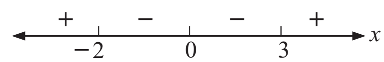
3 **a** increasing for $x \geqslant \sqrt{3}$ and $x \leqslant -\sqrt{3}$
decreasing for $-\sqrt{3} \leqslant x < -1$, $-1 < x \leqslant 0$,
$0 \leqslant x < 1$, $1 < x \leqslant \sqrt{3}$

b increasing for $x \geqslant 2$ decreasing for $x < 1$, $1 < x \leqslant 2$

EXERCISE 22E.3

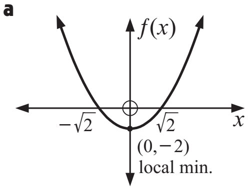
1 **a** A - local min B - local max C - horiz. inflection

b

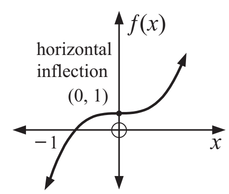


c **i** $x \leqslant -2, x \geqslant 3$ **ii** $-2 \leqslant x \leqslant 3$

2 **a**



b



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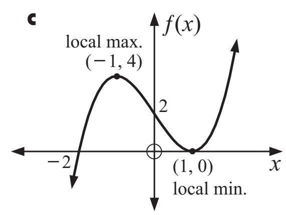

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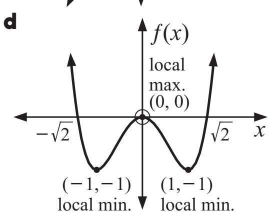

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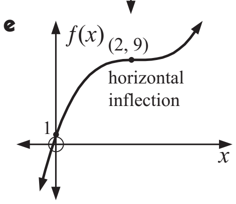

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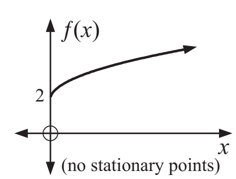

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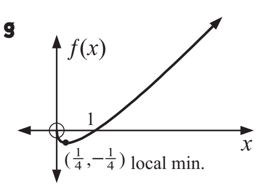

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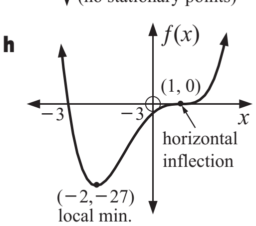

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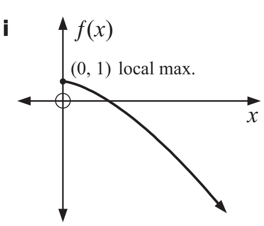

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