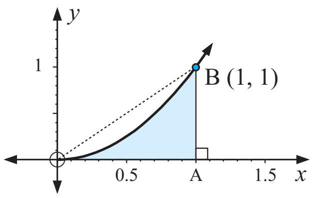
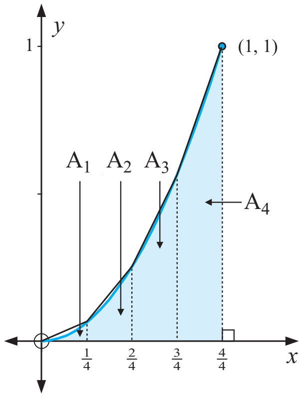
A AREAS WHERE BOUNDARIES ARE CURVED



We will begin with trying to find the area between the x-axis and $y = x^2$ from $x = 0$ to $x = 1$. The function $y = x^2$ provides us with a curved boundary.

It is clear that the required area is less than the area of triangle OAB, i.e., area $< \frac{1}{2}$ unit^2.



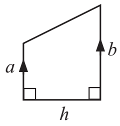
We cannot exactly subdivide the region into triangles, rectangles, trapezia, etc. so existing area formulae cannot be used.

However, we could find better approximations to the exact area by using them.

For example, if the region is subdivided into strips of width $\frac{1}{4}$, an over-estimate of the required area could be made by joining the points on the curve with straight lines as shown.

Now area $< A_1 + A_2 + A_3 + A_4$

i.e., area $< \left(\dfrac{\frac{0}{16}+\frac{1}{16}}{2}\right)\frac{1}{4} + \left(\dfrac{\frac{1}{16}+\frac{4}{16}}{2}\right)\frac{1}{4} + \left(\dfrac{\frac{4}{16}+\frac{9}{16}}{2}\right)\frac{1}{4} + \left(\dfrac{\frac{9}{16}+\frac{16}{16}}{2}\right)\frac{1}{4}$

{using area of trapezium = $\left(\frac{a+b}{2}\right)h$  }

$\therefore$ area $< \frac{1}{4}\left[\frac{1}{32} + \frac{5}{32} + \frac{13}{32} + \frac{25}{32}\right]$

i.e., area $< \frac{11}{32}$ where $\frac{11}{32} = 0.343\,75$

An estimate of the area is $\doteqdot 0.34$ units^2, which is a better estimate than that which would have been obtained if we had used two strips.

DISCUSSION

How could the estimate of the area under $y = x^2$ be improved?

What factors must be considered when trying to improve area estimates?

EXERCISE 25A.1

1 Use the method outlined above to find an estimate of the area:

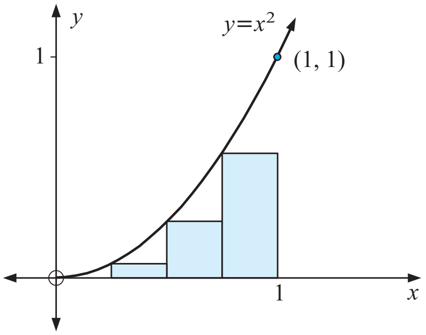
- **a** between $y = x^2$ and the x-axis from $x = 0$ to $x = 1$ if five vertical strips of equal width are drawn
- **b** between $y = \dfrac{1}{x}$ and the x-axis from $x = 1$ to $x = 2$ if five vertical strips of equal width are drawn.

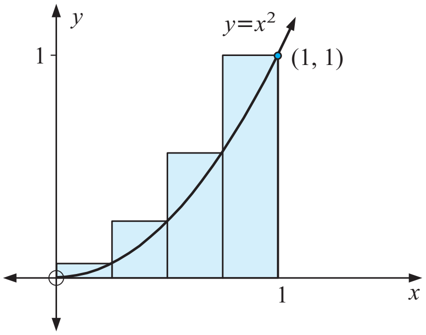
2 Repeat **1** but this time use ten vertical strips.

UPPER AND LOWER RECTANGLES

Another way of finding areas is to use only rectangles and find lower and upper sums of their areas. This gives us a lower and an upper bound for the actual area.

For example, consider $y = x^2$ again with four vertical strips.





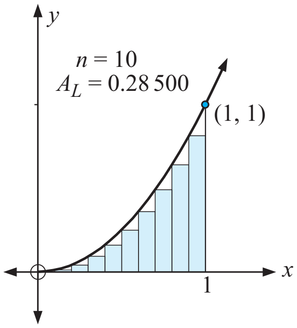
If A_L represents the lower area sum and A_U represents the upper sum, then

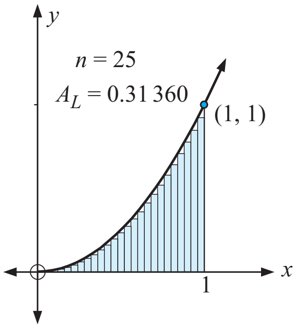
$$A_L = \tfrac{1}{4}(0)^2 + \tfrac{1}{4}\left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4}\left(\tfrac{3}{4}\right)^2 = 0.218\,75$$

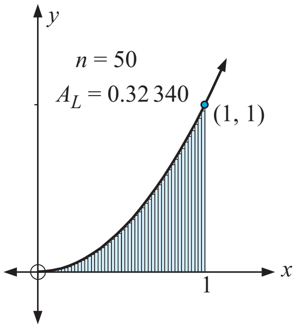
and $$A_U = \tfrac{1}{4}\left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4}\left(\tfrac{3}{4}\right)^2 + \tfrac{1}{4}(1)^2 = 0.468\,75$$

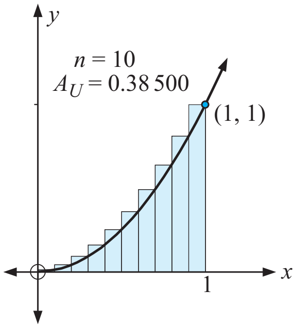
So, if A is the actual area then $$\underset{\substack{\uparrow \\ \text{a lower} \\ \text{bound}}}{0.218\,75} < A < \underset{\substack{\uparrow \\ \text{an upper} \\ \text{bound}}}{0.468\,75}$$

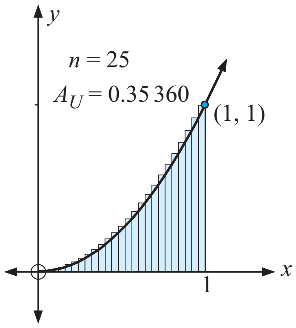
The following diagrams show lower and upper rectangles for n subdivisions where $n = 10$, 25 and 50.

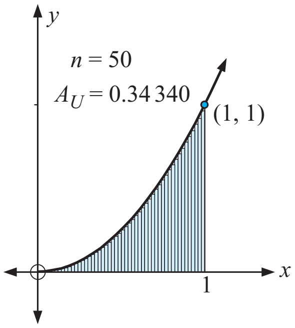












A summary of these results together with average of A_L and A_U are worth considering:

n	A_L	A_U	*Average*
4	0.218 75	0.468 75	0.343 75
10	0.285 00	0.385 00	0.335 00
25	0.313 60	0.353 60	0.333 60
50	0.323 40	0.343 40	0.333 40

Now click on the icon to examine cases $n = 4,\ 10,\ 25,\ 50,\ 100,\ 1000$ and 10 000.

AREA FINDER

From the table it seems that as n gets larger A_L and A_U both approach or converge to the same number 0.333 333 3..... which we recognise as the decimal expansion of $\frac{1}{3}$.

INVESTIGATION 1 FINDING AREAS USING RECTANGLES

This investigation is about finding estimates of areas under simple curves by finding upper and lower rectangle sums by direct calculation and by using technology. **Note:** $[a,\ b]$ is interval notation for $a \leqslant x \leqslant b$.

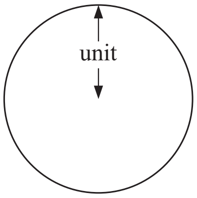
What to do:

1 Consider finding the area between $y = x^3$ and the x-axis from $x = 0$ to $x = 1$.
  - **a** First graph the curve and shade the required area.
  - **b** Subdivide the interval [0, 1] into five equal intervals and construct upper and lower rectangles. This case is $n = 5$.
  - **c** Find the upper and lower area sums.
  - **d** Click on the icon and use the technology to find upper and lower area sums when $n = 5,\ 10,\ 50,\ 200,\ 1000$ and 10 000. Display your answers in table form.
  - **e** What do you suspect the actual area to be?

AREA FINDER

2 Repeat **1 a** to **d** for the function $y = \dfrac{1}{x}$ and the x-axis from $x = 1$ to $x = 2$.

RATIONAL APPROXIMATIONS FOR π

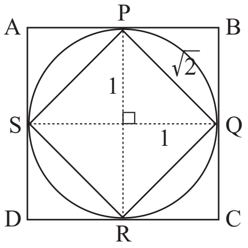


A circle of radius 1 unit has area $A = \pi \times 1^2 = \pi$ units^2.

Notice in the figure that the length of PQ is $\sqrt{2}$.

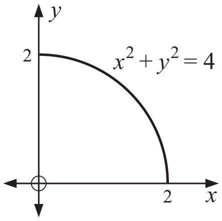
We can see that

$$\begin{aligned} &\text{area PQRS} < \pi < \text{area ABCD} \\ \therefore\quad &\sqrt{2} \times \sqrt{2} < \pi < 2 \times 2 \\ \therefore\quad &\underset{\substack{\uparrow \\ \text{lower bound}}}{2} < \pi < \underset{\substack{\uparrow \\ \text{upper bound}}}{4} \end{aligned}$$



It is clear that 2 and 4 are not good estimates of π, however they do provide us with lower and upper bounds between which the true value of π lies.

Consider the quarter circle of centre (0, 0) and radius 2 units as illustrated.



Its area is $\frac{1}{4}$ (full circle of radius 2)

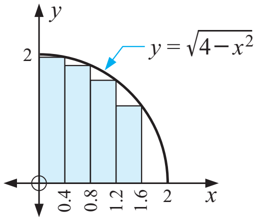
$$= \frac{1}{4} \times \pi \times 2^2$$

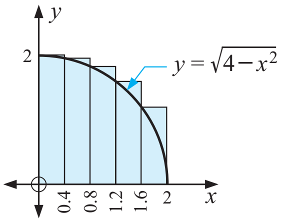
$$= \pi$$

Recall that π is an irrational number whose decimal expansion neither terminates nor recurs.

We will now employ the lower and upper rectangle technique to find **rational bounds** for π.

Consider the case $n = 5$.





$$\begin{aligned} A_L &= (0.4)\sqrt{4-(0.4)^2} + (0.4)\sqrt{4-(0.8)^2} + (0.4)\sqrt{4-(1.2)^2} + (0.4)\sqrt{4-(1.6)^2} \\ &= 2.637\,04...... \end{aligned}$$

and
$$\begin{aligned} A_U &= (0.4)\sqrt{4-0^2} + (0.4)\sqrt{4-(0.4)^2} + (0.4)\sqrt{4-(0.8)^2} + (0.4)\sqrt{4-(1.2)^2} + \\ &\qquad (0.4)\sqrt{4-(1.6)^2} \\ &= 3.437\,04...... \end{aligned}$$

From this, $2.637 < \pi < 3.437$ provides us with rational bounds for π which are *better* than the earlier ones obtained geometrically.

As in the case of previous area finding, using lower and upper sums, we should be able to obtain even better bounds by increasing the number of vertical strips from $n = 5$ to 10 to 50 to 100, etc.

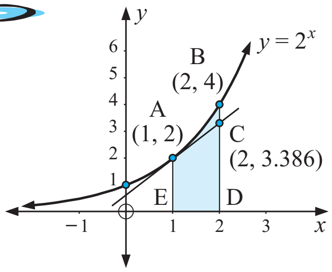
Click on the icon and experiment with increasing values of n to get better rational bounds for the actual value of π.

AREA FINDER

EXERCISE 25A.2

1 Alongside is the graph of $y = 2^x$. The tangent at (1, 2) is drawn.

AREA FINDER



- **a** Use the figure to find rational lower and upper bounds for the shaded area.
- **b** Now consider the original graph without the tangent.

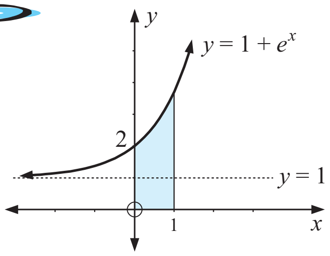
Use five subdivisions of the interval [1, 2] to obtain lower and upper sums of rectangular areas.

- **c** Click on the icon to check your answer to **b** ($n = 5$). Now obtain 'better' rational bounds for the area by considering $n = 10,\ 50,\ 100,\ 500,\ 5000$. Display your answers in table form.

2 Consider the graph of $y = 1 + e^x$ where $e = 2.718\,281\,82.....$

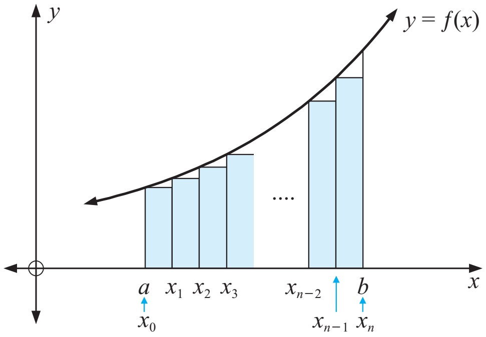
- **a** Use five subdivisions of [0, 1] and find the lower and upper area sums.
- **b** Now use the area finder to find the lower and upper sums for $n = 100$, 1000, $10\,000$, $100\,000$.
- **c** What do you notice about your answers to **b**?

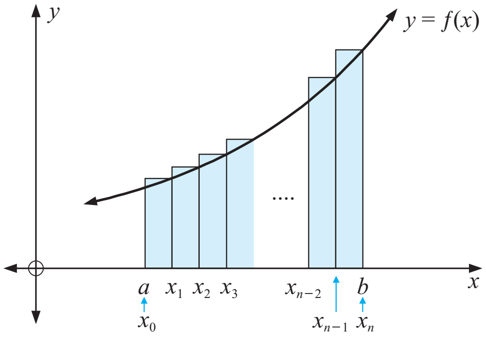
AREA FINDER



B DEFINITE INTEGRALS

We will now have a closer look at lower and upper rectangle sums for a function which is above the x-axis on the interval $[a, b]$, and is increasing.





Notice that the **lower sum** is

$$\begin{aligned} A_L &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + ... + f(x_{n-2})\Delta x + f(x_{n-1})\Delta x \\ &= \sum_{i=0}^{n-1} f(x_i)\Delta x \quad \text{where} \quad \Delta x = \frac{b-a}{n}. \end{aligned}$$

Likewise the **upper sum** is

$$\begin{aligned} A_U &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + ... + f(x_{n-1})\Delta x + f(x_n)\Delta x \\ &= \sum_{i=1}^{n} f(x_i)\Delta x \quad \text{where} \quad \Delta x = \frac{b-a}{n}. \end{aligned}$$

From the work of the previous two sections the following has been observed:

- As n gets larger, as $b - a$ is fixed, Δx gets smaller and closer to 0.
- There exists a unique number A, say, such that for any value of n $A_L < A < A_U$ and both A_L and A_U approach A as n gets very large.
- If $f(x) \geqslant 0$ on $[a, b]$, then A is the area between $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$.

Notation:

We talk about n getting very large and write $n \to \infty$.

$n \to \infty$ could be read as n *approaches* infinity or n *tends to* infinity.

Using this notation, as $n \to \infty$, $A_L \to A$ and $A_U \to A$.

c At $t=0$, $x(0)=-1$ cm, at $t=\pi$, $x(\pi)=3$ cm, at $t=2\pi$, $x(2\pi)=-1$ cm

d for $0 \leqslant t \leqslant \frac{\pi}{2}$ and $\pi \leqslant t \leqslant \frac{3\pi}{2}$

EXERCISE 24B

1 $\doteqdot 109.5^o$ **2** **c** $\theta = 30^o$

3 1 hour 34 min 53 sec when $\theta = 36.9^o$

4 **c** $4\sqrt{2}$ m **5** 9.866 m **6** 1.340 m from A

7 **e** AP + PB is a minimum when $\theta = \phi$

REVIEW SET 24

1 **a** $10-10\cos(10x)$ **b** $3\cos(3x)\cos(2x)-2\sin(3x)\sin(2x)$

c $-2e^{-2x}\sin x + e^{-2x}\cos x$ **d** $\tan x$

e $5\cos(5x)\ln(2x) + \dfrac{\sin(5x)}{x}$

3 **a** $f'(x) = 3\cos x + 8\sin(2x)$, $f''(x) = -3\sin x + 16\cos(2x)$

b $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}\cos(4x) - 4x^{\frac{1}{2}}\sin(4x)$,
$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}\cos(4x) - 4x^{-\frac{1}{2}}\sin(4x) - 16x^{\frac{1}{2}}\cos(4x)$

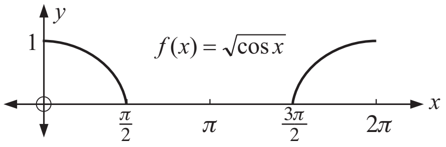
4 **a** $x(0)=3$ cm, $x'(0)=2$ cm/s, $x''(0)=0$ cm/s^2

b $t=\frac{\pi}{4}$ sec and $\frac{3\pi}{4}$ sec **c** 4 cm

5 **a** for $0 \leqslant x \leqslant \frac{\pi}{2}$ and $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$

b increasing for $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$, decreasing for $0 \leqslant x \leqslant \frac{\pi}{2}$

c



6 **a** $\ln(\sin x) + \dfrac{x\cos x}{\sin x}$ **b** $\frac{1}{2}(e^{\tan x})^{-\frac{1}{2}} \times \dfrac{e^{\tan x}}{\cos^2 x}$

c $\dfrac{-3\sqrt{x}\sin(3x) - \dfrac{\frac{1}{2}\cos(3x)}{\sqrt{x}}}{x}$

7 **a** $a^2+b^2-2ab\cos\theta = c^2+d^2-2cd\cos\phi$

8 **a** **i** 5 km **ii** $2\sqrt{10}$ km **9** **b** $\frac{1}{\sqrt{2}}$ m above the floor

10 **a** 0 cm/s, $-\pi$ cm/s, 0 cm/s, π cm/s, 0 cm/s

b $0 \leqslant t \leqslant 1$, $2 \leqslant t \leqslant 3$, $4 \leqslant t \leqslant 5$, etc

EXERCISE 25A.1

1 **a** 0.34 **b** 0.70 **2** **a** 0.335 **b** 0.6938

EXERCISE 25A.2

1 **a** lower = 2.693, upper = 3

b lower = 2.69, upper = 3.09

c

n	*Lower*	*Upper*
10	2.79	2.99
50	2.87	2.91
100	2.88	2.90
500	2.883	2.887
5000	2.8852	2.8856

2 **a** lower = 2.55, upper = 2.90

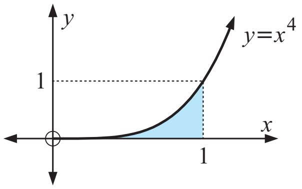
b

n	*Lower*	*Upper*
100	2.709 70	2.726 89
1000	2.717 42	2.719 14
10 000	2.718 20	2.718 37
100 000	2.718 27	2.718 29

c Upper and lower sums converge.

EXERCISE 25B.1

1 **a**

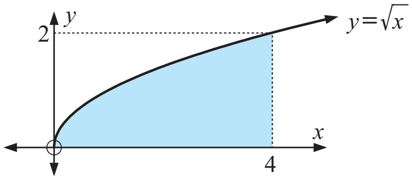


b

n	*Lower*	*Upper*
10	0.15	0.25
100	0.195	0.205
1000	0.1995	0.2005
10 000	0.199 95	0.20005

c 0.2

2 **a**

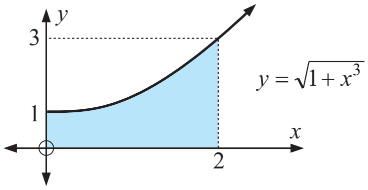


b

n	*Lower*	*Upper*
100	5.29	5.37
10 000	5.333	5.334

c $5\frac{1}{3}$

3 **a**

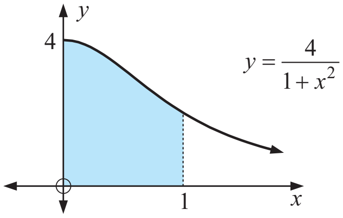


b

n	*Lower*	*Upper*
100	3.22	3.26
10 000	3.2411	3.2415

c 3.2413

4 **a**



b

n	*Lower*	*Upper*
100	3.13	3.15
1000	3.141	3.143
10 000	3.1415	3.1417

c 3.1416

5 10.203 22 **6** **a** 18 **b** 4.5 **c** 1 **d** 1 **e** 2π **f** $2+\pi$

EXERCISE 25B.2

1 **a** $\displaystyle\int_1^4 \sqrt{x}\,dx = 4.667$ $\displaystyle\int_1^4 (-\sqrt{x})\,dx = -4.667$

b $\displaystyle\int_0^1 x^7\,dx = \frac{1}{8}$ $\displaystyle\int_0^1 (-x^7)dx = -\frac{1}{8}$

2 **a** $\frac{1}{3}$ **b** $\frac{7}{3}$ **c** $\frac{8}{3}$ **d** 1 **3** **a** -4 **b** 6.25 **c** 2.25