Demonstration 2 Find the following sum, provided that $x \neq 1$.

$$1 + 2x + 3x^2 + \dots + nx^{n-1}$$

[Solution] Set

$$S_n = 1 + 2x + 3x^2 + \dots + (n-1)x^{n-2} + nx^{x-1}. \quad (1)$$

Multiplying both sides by x, we obtain

$$xS_n = x + 2x^2 + 3x^3 + \dots + (n-1)x^{n-1} + nx^n. \quad (2)$$

Subtracting (2) from (1), we get

$$(1-x)S_n = 1 + x + x^2 + \dots + x^{n-1} - nx^n.$$

Since $x \neq 1$, we can rearrange the right side to get

$$(1-x)S_n = \frac{1-x^n}{1-x} - nx^n = \frac{1-(n-1)x^n + nx^{n+1}}{1-x}.$$

Thus,

$$S_n = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}.$$

Problem 8 Find the sum of the first n terms of the following progression:

$$1 \cdot 2,\ 4 \cdot 2^2,\ 7 \cdot 2^3,\ 10 \cdot 2^4,\ 13 \cdot 2^5,\ \dots .$$

④ Various Types of Progressions

There are progressions besides arithmetic and geometric progressions in which the general term or the sum of the first n terms can easily be found. Let's consider some of these other progressions.

Demonstration 1 Find the following sum:

$$1^2 + 2^2 + 3^2 + \dots + n^2.$$

[Solution] In the identity $(k+1)^3 - k^3 = 3k^2 + 3k + 1$,

if $k = 1$, then $2^3 - 1^3 = 3 \cdot 1^2 + 3 \cdot 1 + 1$;

if $k = 2$, then $3^3 - 2^3 = 3 \cdot 2^2 + 3 \cdot 2 + 1$;

...

if $k = n$, then $(n+1)^3 - n^3 = 3 \cdot n^2 + 3 \cdot n + 1$.

Adding the corresponding sides of these equalities, we obtain

$$\begin{aligned}(n+1)^3 - 1^3 &= 3(1^2 + 2^2 + \dots + n^2) + 3(1 + 2 + \dots + n) + n \\ &= 3(1^2 + 2^2 + \dots + n^2) + \frac{3n(n+1)}{2} + n.\end{aligned}$$

Therefore,

$$\begin{aligned}3(1^2 + 2^2 + \dots + n^2) &= (n+1)^3 - \frac{3n(n+1)}{2} - (n+1) \\ &= \frac{n(n+1)(2n+1)}{2}.\end{aligned}$$

Thus,

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Problem 1 Find the following values using the result of Demonstration 1:

(1) $1^2 + 2^2 + 3^2 + ... + 10^2$

(2) $8^2 + 9^2 + 10^2 + ... + 15^2$

Problem 2 Prove the following equality, using the identity $(k + 1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$ as in Demonstration 1.

$$1^3 + 2^3 + 3^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Use this equality to find the following sum:

$$6^3 + 7^3 + 8^3 + ... + 13^3.$$

The Symbol $\sum$ for the Sum of a Progression

We can write the sum of the first n terms of a progression $a_1, a_2, a_3, ..., a_n ...$ using the symbol $\sum$* as

$$\sum_{k=1}^{n} a_k.$$

Thus,

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + ... + a_n.$$

Using this symbol, $1^2 + 2^2 + 3^2 + ... + n^2$ can be represented as $\sum_{k=1}^{n} k^2$. This expression can also be written using letters other than k such as $\sum_{i=1}^{n} i^2$ or $\sum_{j=1}^{n} j^2$.

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*$\sum$ is the Greek letter corresponding to S, the first letter of *Sum*, and is read "sigma" or, if you prefer, as "sum".