CHECKLIST OF INFORMATION FOR SKETCHING A CURVE $y = f(x)$

A. **Domain** The first step is to determine the domain D of f, that is, the set of values of x for which $f(x)$ is defined.

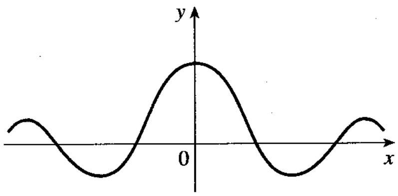
B. **Intercepts** The y-intercept is $f(0)$ and tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (If this is not easily done, the x-intercepts could be estimated.)

C. **Symmetry**

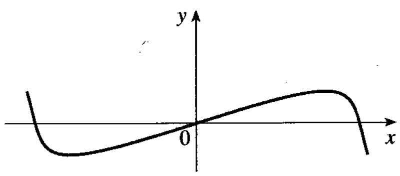
(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an **even function** and the curve is symmetric about the y-axis. This means that our work is cut in half. If we know what the curve looks like for $x \geqslant 0$, then we need only reflect about the y-axis to obtain the complete curve [see Figure 6(a)]. Here are some examples: $y = x^2$, $y = x^4$, $y = |x|$, and $y = \cos x$.

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geqslant 0$ [see Figure 6(b)]. Some simple examples of odd functions are $y = x$, $y = x^3$, $y = x^5$, and $y = \sin x$.

(iii) If $f(x + p) = f(x)$ for all x in D, where p is a positive constant, then f is called a **periodic function** and the smallest such number p is called the **period.** For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph (see Figure 7).



(a) Even function: reflectional symmetry



(b) Odd function: rotational symmetry

FIGURE 6

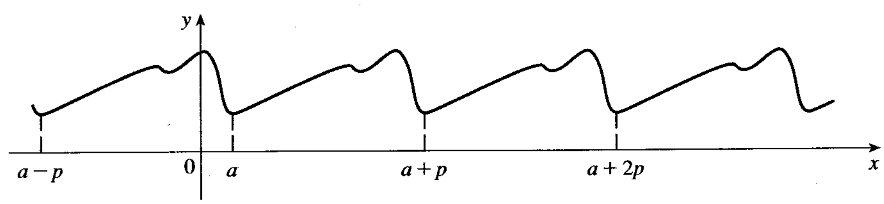


FIGURE 7 Periodic function: translational symmetry

D. **Asymptotes**

(i) *Horizontal Asymptotes.* Recall from Section 1.6 that if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$. If it turns out that $\lim_{x\to\infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but that is still useful information for sketching the curve.

(ii) *Vertical Asymptotes.* Recall from Section 1.2 that the line $x = a$ is a vertical asymptote if at least one of the following statements is true:

$$\textbf{(1)} \quad \lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.) Furthermore, in sketching the curve it is very useful to know exactly which of the statements in (1) is true. If $f(a)$ is not defined but a is an endpoint of the domain of f, then you should compute $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$, whether or not this limit is infinite.

(iii) *Slant Asymptotes.* These are discussed at the end of this section.

E. **Intervals of Increase or Decrease** Use the Test for Monotonic Functions. Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

F. **Local Maximum and Minimum Values** Find the critical numbers of f [the numbers c where $f'(c) = 0$ or $f'(c)$ does not exist]. Then use the First Derivative Test. If f' changes from positive to negative at a critical number c, then $f(c)$ is a local maximum. If f' changes from negative to positive at c, then $f(c)$ is a local minimum. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if c is a critical number such that $f''(c) \neq 0$. Then $f''(c) > 0$ implies that $f(c)$ is a local minimum, whereas $f''(c) < 0$ implies that $f(c)$ is a local maximum.

G. **Concavity and Points of Inflection** Compute $f''(x)$ and use the Test for Concavity. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Inflection points occur where the direction of concavity changes.

H. **Sketch the Curve** Using the information in items A–G, draw the graph. Draw in the asymptotes as broken lines. Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes. If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

EXAMPLE 1 Discuss the curve $y = \dfrac{2x^2}{x^2 - 1}$ under the headings A–H.

SOLUTION

A. The domain is

$$\{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

B. The x- and y-intercepts are both 0.

C. Since $f(-x) = f(x)$, f is even. The curve is symmetric about the y-axis.

D. $$\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - 1/x^2} = 2$$

Therefore, the line $y = 2$ is a horizontal asymptote. Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \infty \qquad \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty \qquad \lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = \infty$$

Therefore, the lines $x = 1$ and $x = -1$ are vertical asymptotes.

E. $$f'(x) = \frac{4x(x^2 - 1) - 2x^2 \cdot 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

Since $f'(x) > 0$ when $x < 0$ $(x \neq -1)$ and $f'(x) < 0$ when $x > 0$ $(x \neq 1)$, f is increasing on $(-\infty, -1)$ and $(-1, 0]$ and decreasing on $[0, 1)$ and $(1, \infty)$.

F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = 0$ is a local maximum by the First Derivative Test.

G. $$f''(x) = \frac{-4(x^2 - 1)^2 + 4x \cdot 2(x^2 - 1)2x}{(x^2 - 1)^4} = \frac{12x^2 + 4}{(x^2 - 1)^3}$$

Since $12x^2 + 4 > 0$ for all x, we have

$$f''(x) > 0 \iff x^2 - 1 > 0 \iff |x| > 1$$

and $f''(x) < 0 \iff |x| < 1$. Thus the curve is concave upward on the intervals $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 1)$. It has no point of inflection since 1 and -1 are not in the domain of f.

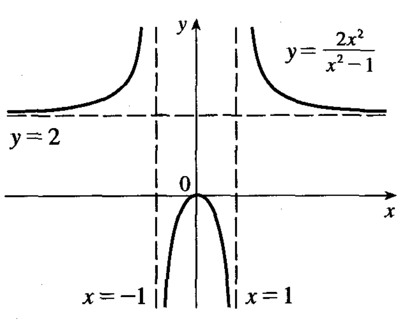


FIGURE 8

H. Using the information in A–G, we sketch the curve in Figure 8. ■

EXERCISES 4.5

1–44 ■ Discuss the curves under the headings A–H given in this section.

1. $y = 1 - 3x + 5x^2 - x^3$
2. $y = 2x^3 - 6x^2 - 18x + 7$
3. $y = x^4 - 6x^2$
4. $y = 4x^3 - x^4$
5. $y = \dfrac{x}{x - 1}$
6. $y = \dfrac{x}{(x - 1)^2}$
7. $y = \dfrac{1}{x^2 - 9}$
8. $y = \dfrac{x}{x^2 - 9}$
9. $y = \dfrac{1}{(x - 1)(x + 2)}$
10. $y = \dfrac{1}{x^2(x + 3)}$
11. $y = \dfrac{1 + x^2}{1 - x^2}$
12. $y = \dfrac{x^3 - 1}{x^3 + 1}$
13. $y = \dfrac{1}{x^3 - x}$
14. $y = \dfrac{1 - x^2}{x^3}$
15. $y = x\sqrt{x + 3}$
16. $y = \sqrt{x} - \sqrt{x - 1}$
17. $y = \sqrt{x^2 + 1} - x$
18. $y = \sqrt{\dfrac{x}{x - 5}}$
19. $y = \sqrt[4]{x^2 - 25}$
20. $y = x\sqrt{x^2 - 9}$
21. $y = \dfrac{\sqrt{1 - x^2}}{x}$
22. $y = \dfrac{x + 1}{\sqrt{x^2 + 1}}$
23. $y = x + 3x^{2/3}$
24. $y = x^{5/3} - 5x^{2/3}$
25. $y = x + \sqrt{|x|}$
26. $y = \sqrt[3]{(x^2 - 1)^2}$
27. $y = \cos x - \sin x$
28. $y = \sin x - \tan x$
29. $y = x \tan x, \quad -\pi/2 < x < \pi/2$
30. $y = 2x + \cot x, \quad 0 < x < \pi$
31. $y = \dfrac{x}{2} - \sin x, \quad 0 < x < 3\pi$
32. $y = 2\sin x + \sin^2 x$
33. $y = 2\cos x + \sin^2 x$
34. $y = \sin x - x$
35. $y = \sin 2x - 2\sin x$
36. $y = \dfrac{\cos x}{2 + \sin x}$
37. $y = e^{-1/(x+1)}$
38. $y = xe^{x^2}$
39. $y = 1/(1 + e^{-x})$
40. $y = \ln(\cos x)$
41. $y = \ln(1 + x^2)$
42. $y = \ln(\tan^2 x)$
43. $y = \ln(x^2 - x)$
44. $y = x^{-\ln x}$

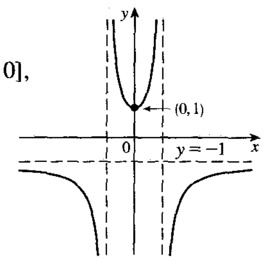
45–54 ■ Sketch the curve under the headings A–H using l'Hospital's Rule where appropriate.

45. $y = xe^{-x}$
46. $y = x^2 e^{-x}$
47. $y = x \ln x$
48. $y = (\ln x)/x$
49. $y = x^2 \ln x$
50. $y = x(\ln x)^2$
51. $y = xe^{-x^2}$
52. $y = e^x/x$
53. $y = xe^{1/x}$
54. $y = e^x - 3e^{-x} - 4x$

55–60 ■ Discuss each curve under the headings A–H. In D find an equation of the slant asymptote.

55. $y = \dfrac{x^3}{x^2 - 1}$
56. $y = x - \dfrac{1}{x}$
57. $xy = x^2 + 4$
58. $y = e^x - x$
59. $y = \dfrac{1}{x - 1} - x$
60. $y = \dfrac{x^2}{2x + 5}$
61. Show that the lines $y = (b/a)x$ and $y = -(b/a)x$ are slant asymptotes of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$.
62. Let $f(x) = (x^3 + 1)/x$. Show that
$$\lim_{x \to \pm\infty} [f(x) - x^2] = 0$$
This shows that the graph of f approaches the graph of $y = x^2$, and we say that the curve $y = f(x)$ is *asymptotic* to the parabola $y = x^2$. Use this fact to help sketch the graph of f.
63. Discuss the asymptotic behavior of $f(x) = (x^4 + 1)/x$ in the same manner as in Exercise 62. Then use your results to help sketch the graph of f.
64. Use the asymptotic behavior of $f(x) = \cos x + 1/x^2$ to sketch its graph without going through the curve-sketching procedure of this section.

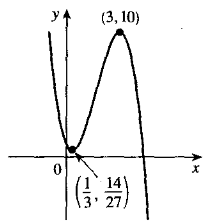
11. (a) $\{x \mid x \neq \pm 1\}$ (b) y-intercept 1
(c) About y-axis
(d) VA $x = \pm 1$, HA $y = -1$
(e) Decreasing on $(-\infty, -1)$ and $(-1, 0]$, increasing on $[0, 1)$ and $(1, \infty)$
(f) Local minimum $f(0) = 1$
(g) CD on $(-\infty, -1)$ and $(1, \infty)$, CU on $(-1, 1)$, no IP
(h) See graph at right.



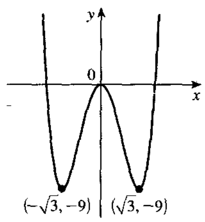
Exercises 4.5 ■ page 287

Abbreviations: VA, vertical asymptote; HA, horizontal asymptote; IP, inflection point

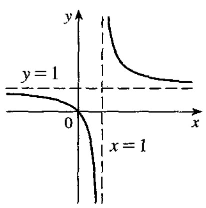
1. (a) $\mathbb{R}$ (b) y-intercept 1 (c) None
(d) None (e) Increasing on $[\frac{1}{3}, 3]$, decreasing on $(-\infty, \frac{1}{3}]$ and $[3, \infty)$
(f) Local minimum $f(\frac{1}{3}) = \frac{14}{27}$, local maximum $f(3) = 10$
(g) CD on $(\frac{5}{3}, \infty)$, CU on $(-\infty, \frac{5}{3})$, IP $(\frac{5}{3}, \frac{142}{27})$
(h) See graph at right.



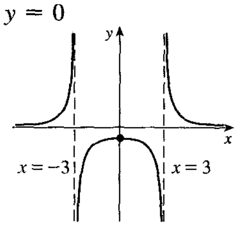
3. (a) $\mathbb{R}$ (b) y-intercept 0; x-intercepts 0, $\pm\sqrt{6}$
(c) About y-axis (d) None
(e) Increasing on $[-\sqrt{3}, 0]$ and $[\sqrt{3}, \infty)$, decreasing on $(-\infty, -\sqrt{3}]$ and $[0, \sqrt{3}]$
(f) Local minima $f(\pm\sqrt{3}) = -9$, maximum $f(0) = 0$
(g) CU on $(-\infty, -1)$ and $(1, \infty)$, CD on $(-1, 1)$, IP $(1, -5)$ and $(-1, -5)$
(h) See graph at right.



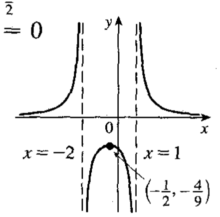
5. (a) $\{x \mid x \neq 1\}$
(b) x-intercept 0, y-intercept 0
(c) None (d) VA $x = 1$, HA $y = 1$
(e) Decreasing on $(-\infty, 1)$ and $(1, \infty)$
(f) No maximum or minimum
(g) CD on $(-\infty, 1)$, CU on $(1, \infty)$, no IP
(h) See graph at right.



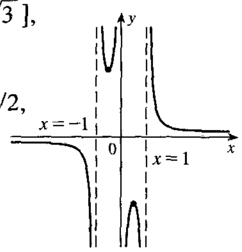
7. (a) $\{x \mid x \neq \pm 3\}$ (b) y-intercept $-\frac{1}{9}$
(c) About y-axis (d) VA $x = \pm 3$, HA $y = 0$
(e) Increasing on $(-\infty, -3)$ and $(-3, 0]$, decreasing on $[0, 3)$ and $(3, \infty)$
(f) Local maximum $f(0) = -\frac{1}{9}$
(g) CU on $(-\infty, -3)$ and $(3, \infty)$, CD on $(-3, 3)$, no IP
(h) See graph at right.



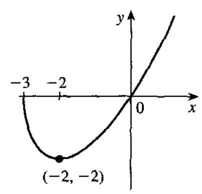
9. (a) $\{x \mid x \neq 1, -2\}$ (b) y-intercept $-\frac{1}{2}$
(c) None (d) VA $x = 1$, $x = -2$, HA $y = 0$
(e) Increasing on $(-\infty, -2)$ and $(-2, -\frac{1}{2}]$, decreasing on $[-\frac{1}{2}, 1)$ and $(1, \infty)$
(f) Local maximum $f(-\frac{1}{2}) = -\frac{4}{9}$
(g) CU on $(-\infty, -2)$ and $(1, \infty)$, CD on $(-2, 1)$, no IP
(h) See graph at right.



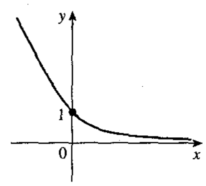
13. $\{x \mid x \neq 0, \pm 1\}$ (b) None (c) About $(0, 0)$
(d) VA $x = -1$, $x = 0$, $x = 1$; HA $y = 0$
(e) Decreasing on $(-\infty, -1)$, $(-1, -1/\sqrt{3}]$, $[1/\sqrt{3}, 1)$, and $(1, \infty)$, increasing on $[-1/\sqrt{3}, 0)$ and $(0, 1/\sqrt{3}]$
(f) Local minimum $f(-1/\sqrt{3}) = 3\sqrt{3}/2$, maximum $f(1/\sqrt{3}) = -3\sqrt{3}/2$
(g) CD on $(-\infty, -1)$, $(0, 1)$, CU on $(-1, 0)$, $(1, \infty)$
(h) See graph at right.



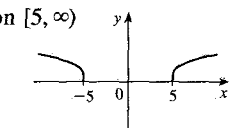
15. (a) $[-3, \infty)$
(b) x-intercepts 0, -3; y-intercept 0
(c) None (d) None
(e) Increasing on $[-2, \infty)$, decreasing on $[-3, -2]$
(f) Local minimum $f(-2) = -2$
(g) CU on $(-3, \infty)$
(h) See graph at right.



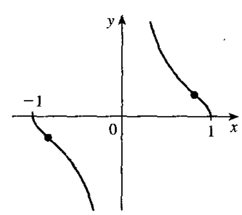
17. (a) $\mathbb{R}$ (b) y-intercept 1
(c) None (d) HA $y = 0$
(e) Decreasing on $(-\infty, \infty)$
(f) No local maximum or minimum
(g) CU on $(-\infty, \infty)$, no IP
(h) See graph at right.



19. (a) $\{x \mid |x| \geqslant 5\} = (-\infty, -5] \cup [5, \infty)$
(b) x-intercepts ± 5 (c) About y-axis (d) None
(e) Decreasing on $(-\infty, -5]$, increasing on $[5, \infty)$
(f) No local maximum or minimum
(g) CD on $(-\infty, -5)$ and $(5, \infty)$, no IP
(h) See graph at right.



21. (a) $\{x \mid |x| \leqslant 1, x \neq 0\} = [-1, 0) \cup (0, 1]$
(b) x-intercepts ± 1 (c) About $(0, 0)$
(d) VA $x = 0$
(e) Decreasing on $[-1, 0)$ and $(0, 1]$
(f) No local maximum or minimum
(g) CU on $(-1, -\sqrt{2/3})$ and $(0, \sqrt{2/3})$, CD on $(-\sqrt{2/3}, 0)$ and $(\sqrt{2/3}, 1)$, IP $(\pm\sqrt{2/3}, \pm 1/\sqrt{2})$
(h) See graph at right.



23. (a) $(-\infty, \infty)$ (b) x-intercepts 0, -27; y-intercept 0
(c) None (d) None