

[08-12-09-VII]

14.3 The Cross Product

The dot product of two vectors is a scalar. We have explored some of its uses in earlier sections. Now we introduce the **cross product** (or vector product); it will also have many uses. The cross product $\mathbf{u} \times \mathbf{v}$ of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is defined by

$$\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$$

In this form, the formula is hard to remember and its significance is not obvious. Note the one thing that is obvious. The cross product of two vectors is a vector.

To help us remember the formula for the cross product, we recall a subject from an earlier mathematics course, namely, *determinants*. First, the value of a 2×2 determinant is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

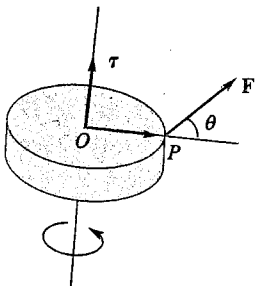
Then the value of a 3×3 determinant is (expanding along to the top row)

torque

The cross product plays an important role in mechanics. Let O be a fixed point in a body, and suppose that a force \mathbf{F} is applied at another point P of the body. Then \mathbf{F} tends to rotate the body about an axis through O and perpendicular to the plane of OP and \mathbf{F} . The vector

$$\boldsymbol{\tau} = \overrightarrow{OP} \times \mathbf{F}$$

is called the **torque**. It points in the direction of the axis and has magnitude $|\overrightarrow{OP}||\mathbf{F}|\sin\theta$, which is just the moment of force about the axis due to \mathbf{F} .



$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Using determinants, we may write the definition of $\mathbf{u} \times \mathbf{v}$ as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

Note that the components of the left vector \mathbf{u} go in the second row, and those of the right vector \mathbf{v} go in the third row. This is important, because if we interchange the positions of \mathbf{u} and \mathbf{v} , we interchange the second and third rows of the determinant, and this changes the sign of the determinant's value, as you may check. Thus,

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$

which is sometimes called the *anticommutative law*.

EXAMPLE 1 Let $\mathbf{u} = \langle 1, -2, -1 \rangle$ and $\mathbf{v} = \langle -2, 4, 1 \rangle$. Calculate $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ using the determinant definition.

Solution

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -1 \\ -2 & 4 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -2 & -1 \\ 4 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} \\ = 2\mathbf{i} + \mathbf{j} + 0\mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & 1 \\ 1 & -2 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 4 & 1 \\ -2 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & 4 \\ 1 & -2 \end{vmatrix} \\ = -2\mathbf{i} - \mathbf{j} + 0\mathbf{k}$$

Geometric Interpretation of $\mathbf{u} \times \mathbf{v}$ Like the dot product, the cross product gains significance from its geometric interpretation.

Theorem A

Let \mathbf{u} and \mathbf{v} be vectors in three-space and θ be the angle between them. Then:

1. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0 = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$, that is, $\mathbf{u} \times \mathbf{v}$ is perpendicular to both \mathbf{u} and \mathbf{v} ;
2. \mathbf{u} , \mathbf{v} , and $\mathbf{u} \times \mathbf{v}$ form a right-handed triple;
3. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$.

Proof Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

1. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1)$. When we remove parentheses, the six terms cancel in pairs. A similar event occurs when we expand $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$.
2. The meaning of right-handedness for the triple $\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}$ is illustrated in Figure 1. There θ is the angle between \mathbf{u} and \mathbf{v} , and the fingers of the right hand are curled in the direction of the rotation through θ that makes \mathbf{u} coincide with \mathbf{v} . It is difficult to establish analytically that the indicated triple is right-handed, but you might check it with a few examples. Note in particular that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, and by definition we know that the triple $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is right-handed.
3. We need Lagrange's Identity,

$$|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

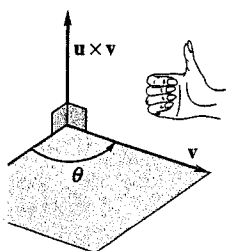


FIGURE 1

whose proof is a simple algebraic exercise (Problem 25). Using this identity, we may write

$$\begin{aligned} |\mathbf{u} \times \mathbf{v}|^2 &= |\mathbf{u}|^2|\mathbf{v}|^2 - (|\mathbf{u}||\mathbf{v}|\cos\theta)^2 \\ &= |\mathbf{u}|^2|\mathbf{v}|^2(1 - \cos^2\theta) \\ &= |\mathbf{u}|^2|\mathbf{v}|^2\sin^2\theta \end{aligned}$$

Since $0 \leq \theta \leq \pi$, $\sin\theta \geq 0$. Taking principal square roots yields

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta \quad \blacklozenge$$

It is important that we have geometric interpretations of both $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$. While both products were originally defined in terms of components that depend on a choice of coordinate system, they are actually independent of coordinate systems. They are intrinsic geometric quantities, and you will get the same results for $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$ no matter how you introduce the coordinates used to compute them.

Here is a simple consequence of Theorem A (part 3) and the fact that vectors are parallel if and only if the angle θ between them is either 0° or 180° .

Theorem B

Two vectors \mathbf{u} and \mathbf{v} in three-space are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Applications Our first application is to find the equation of the plane through three noncollinear points.

EXAMPLE 2 Find the equation of the plane (Figure 2) through the three points $P_1(1, -2, 3)$, $P_2(4, 1, -2)$, and $P_3(-2, -3, 0)$.

Solution Let $\mathbf{u} = \overrightarrow{P_2P_1} = \langle -3, -3, 5 \rangle$ and $\mathbf{v} = \overrightarrow{P_2P_3} = \langle -6, -4, 2 \rangle$. From the first part of Theorem A we know that

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -3 & 5 \\ -6 & -4 & 2 \end{vmatrix} = 14\mathbf{i} - 24\mathbf{j} - 6\mathbf{k}$$

is perpendicular to both \mathbf{u} and \mathbf{v} and thus to the plane containing them. The plane through $(4, 1, -2)$ with normal $14\mathbf{i} - 24\mathbf{j} - 6\mathbf{k}$ has equation (see Section 14.2)

$$14(x - 4) - 24(y - 1) - 6(z + 2) = 0$$

or

$$14x - 24y - 6z = 44 \quad \blacksquare$$

EXAMPLE 3 Show that the area of a parallelogram with \mathbf{a} and \mathbf{b} as adjacent sides is $|\mathbf{a} \times \mathbf{b}|$.

Solution Recall that the area of a parallelogram is the product of the base times the height. Now look at Figure 3 and use the fact that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$. \blacksquare

EXAMPLE 4 Show that the volume of the parallelepiped determined by the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} is

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Solution Refer to Figure 4 and regard the parallelogram determined by \mathbf{b} and \mathbf{c} as the base of the parallelepiped. The area of this base is $|\mathbf{b} \times \mathbf{c}|$ by Example 3; the height h of the parallelepiped is the absolute value of the scalar projection of \mathbf{a} on $\mathbf{b} \times \mathbf{c}$. Thus,

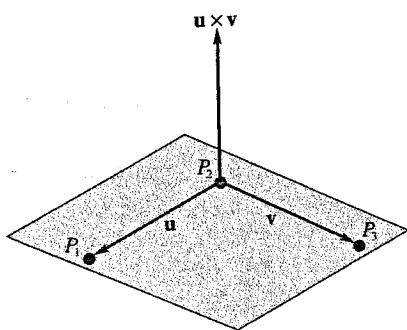


Figure 2

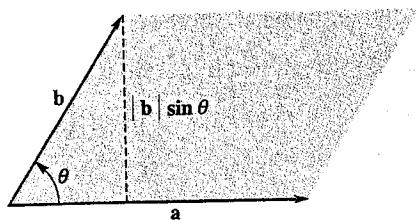


Figure 3

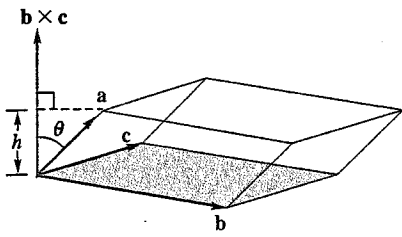


Figure 4

Check Extreme Cases

Never read a mathematics book passively; rather, ask questions as you go. In particular, you should look at extreme cases whenever possible. Here we look at the case where the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} are in the *same* plane. The volume of the parallelepiped should be zero, and indeed the formula does give zero. What happens in Example 3 if the vectors \mathbf{a} and \mathbf{b} are parallel?

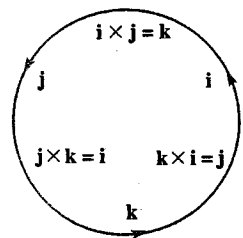


Figure 5

$$h = |\mathbf{a}| |\cos \theta| = \frac{|\mathbf{a}| |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a}| |\mathbf{b} \times \mathbf{c}|} = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{b} \times \mathbf{c}|}$$

and

$$V = h |\mathbf{b} \times \mathbf{c}| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

That V can also be expressed as a determinant is established by expanding $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ in terms of components and then comparing it with the value of the indicated determinant. ■

Suppose that the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} from the previous example are in the *same* plane. In this case, the parallelepiped has height zero, so the volume should be zero. Does the formula for the volume yield $V = 0$? If \mathbf{a} is in the plane determined by \mathbf{b} and \mathbf{c} , then any vector perpendicular to \mathbf{b} and \mathbf{c} will be perpendicular to \mathbf{a} as well. The vector $\mathbf{b} \times \mathbf{c}$ is perpendicular to both \mathbf{b} and \mathbf{c} ; hence $\mathbf{b} \times \mathbf{c}$ is perpendicular to \mathbf{a} . Thus, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$.

Algebraic Properties The rules for calculating with cross products are summarized in the following theorem. Proving this theorem is a matter of writing everything out in terms of components and will be left as an exercise.

Theorem C

If \mathbf{u} , \mathbf{v} , and \mathbf{w} are vectors in three-space and k is a scalar, then:

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ (anticommutative law);
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ (left distributive law);
3. $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$;
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$, $\mathbf{u} \times \mathbf{u} = \mathbf{0}$;
5. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$;
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

Once the rules in Theorem C are mastered, complicated calculations with vectors can be done with ease. We illustrate by calculating a cross product in a new way. We will need the following simple but important products.

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

These results have a cyclic order, which can be remembered by appealing to Figure 5.

EXAMPLE 5 Calculate $\mathbf{u} \times \mathbf{v}$ if $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

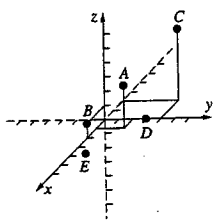
Solution We appeal to Theorem C, especially the distributive law and the anticommutative law.

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \times (4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \\ &= 12(\mathbf{i} \times \mathbf{i}) + 6(\mathbf{i} \times \mathbf{j}) - 9(\mathbf{i} \times \mathbf{k}) - 8(\mathbf{j} \times \mathbf{i}) - 4(\mathbf{j} \times \mathbf{j}) \\ &\quad + 6(\mathbf{j} \times \mathbf{k}) + 4(\mathbf{k} \times \mathbf{i}) + 2(\mathbf{k} \times \mathbf{j}) - 3(\mathbf{k} \times \mathbf{k}) \\ &= 12(\mathbf{0}) + 6(\mathbf{k}) - 9(-\mathbf{j}) - 8(-\mathbf{k}) - 4(\mathbf{0}) \\ &\quad + 6(\mathbf{i}) + 4(\mathbf{j}) + 2(-\mathbf{i}) - 3(\mathbf{0}) \\ &= 4\mathbf{i} + 13\mathbf{j} + 14\mathbf{k} \end{aligned}$$

Experts would do most of this in their heads; novices might find the determinant method easier. ■

Problem Set 14.1

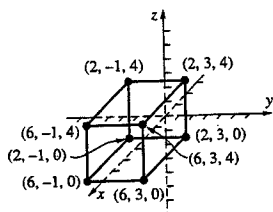
1. $A(1, 2, 3), B(2, 0, 1), C(-2, 4, 5), D(0, 3, 0), E(-1, -2, -3)$



3. $x = 0; x = 0, y = 0$

5. (a) $\sqrt{43}$; (b) 5; (c) $\sqrt{(e + \pi)^2 + (\pi + 4)^2 + 3}$

9.



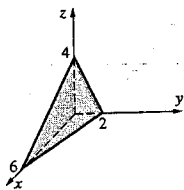
11. (a) $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 25$;

(b) $(x + 2)^2 + (y + 3)^2 + (z + 6)^2 = 5$;

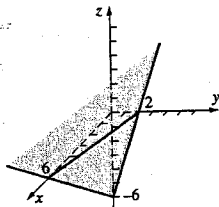
(c) $(x - \pi)^2 + (y - e)^2 + (z - \sqrt{2})^2 = \pi$

13. $(6, -7, 4); 10$ 15. $(\frac{1}{2}, -1, -2); \sqrt{\frac{17}{2}}$

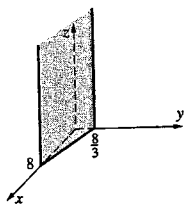
17.



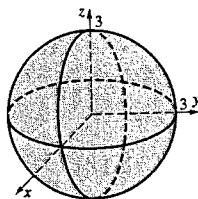
19.



21.



23.



25. $(x - 1)^2 + (y - 1)^2 + (z - \frac{11}{2})^2 = \frac{53}{4}$

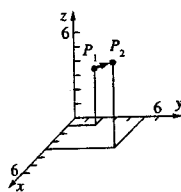
27. $(x - 6)^2 + (y - 6)^2 + (z - 6)^2 = 36$

29. (a) Plane parallel to and 2 units above the xy -plane;
 (b) Plane perpendicular to the xy -plane, whose trace in the xy -plane is the line $x = y$;
 (c) Union of the yz -plane ($x = 0$) and the xz -plane ($y = 0$);
 (d) Union of the three coordinate planes;
 (e) Cylinder of radius 2, parallel to the z -axis;
 (f) Top half of the sphere with center $(0, 0, 0)$ and radius 3

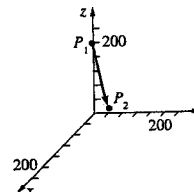
33. $\frac{11\pi}{12}$

Problem Set 14.2

1. (a)



(b)



$3i + 3j + 2k$

$-13i + 55j - 178k$

3. (a) length = $\sqrt{21}$; $\cos \alpha = \frac{4}{\sqrt{21}}$, $\cos \beta = \frac{1}{\sqrt{21}}$,

$\cos \gamma = \frac{2}{\sqrt{21}}$

(b) length = $\sqrt{62}$; $\cos \alpha = -\frac{2}{\sqrt{62}}$, $\cos \beta = -\frac{3}{\sqrt{62}}$,

$\cos \gamma = \frac{7}{\sqrt{62}}$

5. $\left\langle \frac{3}{5\sqrt{2}}, -\frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle; \left\langle -\frac{3}{\sqrt{2}}, \frac{4}{\sqrt{2}}, -\frac{5}{\sqrt{2}} \right\rangle$

7. $\cos^{-1}\left(-\frac{2}{\sqrt{247}}\right)$

9. $\frac{10}{\sqrt{593}}i - \frac{40}{\sqrt{593}}j + \frac{240}{\sqrt{593}}k$;

$-\frac{10}{\sqrt{593}}i + \frac{40}{\sqrt{593}}j - \frac{240}{\sqrt{593}}k$

11. $\cos^{-1} \frac{11}{\sqrt{129}}$ 13. $\sqrt{3}$

15. $m = -\frac{48}{43}i + \frac{80}{43}j - \frac{48}{43}k$; $n = -\frac{81}{43}i + \frac{6}{43}j + \frac{91}{43}k$

17. (a) $\alpha \approx 143.30^\circ, \beta \approx 57.69^\circ, \gamma \approx 74.50^\circ$

(b) $\alpha \approx 63.75^\circ, \beta \approx 27.79^\circ, \gamma \approx 98.48^\circ$

19. $2\sqrt{3}$ 23. (a), (b), (f)

25. (a) $2x - 4y + 3z = -15$; (b) $3x - 2y - z = -4$

27. 56.91° 29. (a) $z = 2$; (b) $2x - 3y - 4z = -13$

31. 0 33. $\frac{12}{\sqrt{38}}$ 37. 35.26° 39. 37.86°

41. 32 joules 43. 15 joules 45. $(3, 2, 1)$ 47. 0

49. $\left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2}, \frac{a_3 + b_3}{2}\right); \frac{1}{2}|a - b|$ 51. $(4, 2, 3)$

Problem Set 14.3

1. (a) $-4i - 10j - 4k$; (b) $-6i - 36j - 27k$; (c) 8;
 (d) $-98i - 59j + 88k$

3. $c(-14i - 2j + 6k), c \in \mathbb{R}$ 5. $\pm \left\langle \frac{7}{\sqrt{86}}, -\frac{1}{\sqrt{86}}, \frac{6}{\sqrt{86}} \right\rangle$

7. $2\sqrt{74}$ 9. $4\sqrt{6}$ 11. $2x - y - z = -3$

13. $7x + 5y + 4z = -5$ 15. $-x + 10y + 17z = -3$

17. 69 19. (a) 9; (b) $\sqrt{35}$; (c) 40.01° 21. (c), (d)

29. $\frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}$