CHAPTER

14

Geometry in Space, Vectors

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14.1

Cartesian Coordinates in Three-Space

We have reached an important transition point in our study of calculus. Until now, we have been traveling across that broad flat expanse known as the Euclidean plane, or two-space. The concepts of calculus have been applied to functions of a single variable, functions whose graphs can be drawn in the plane.

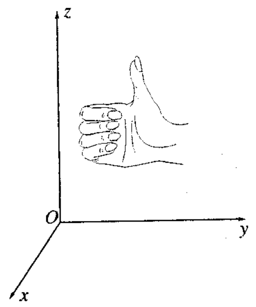
The mountains lie ahead. Our charted course winds through three-space and occasionally into n-space. We are going to study *multiple variable calculus*, the calculus that applies to functions of two or more variables. All the familiar ideas (such as limit, derivative, integral) are to be explored again from a loftier perspective.

To begin, consider three mutually perpendicular coordinate lines (the x-, y-, and z-axes) with their zero points at a common point O, called the *origin*. Although these lines can be oriented in any way one pleases, we follow a custom in thinking of the y- and z-axes as lying in the plane of the paper with their positive directions to the right and upward, respectively. The x-axis is then perpendicular to the paper, and we suppose its positive end to point toward us, thus forming a **right-handed system**. We call it right-handed because, if the fingers of the right hand are curled so that they curve from the positive x-axis toward the positive y-axis, the thumb points in the direction of the positive z-axis (Figure 1).

The three axes determine three planes, the yz-, xz-, and xy-planes, which divide space into eight octants (Figure 2). To each point P in space corresponds an ordered triple of numbers (x, y, z), its **Cartesian coordinates**, which measure its directed distances from the three planes (Figure 3).

Plotting points in the first octant (the octant where all three coordinates are positive) is relatively easy. In Figures 4 and 5, we illustrate something more difficult by plotting two points from other octants, the points $P(2, -3, 4)$ and $Q(-3, 2, -5)$.

Right-handed system



Left-handed system

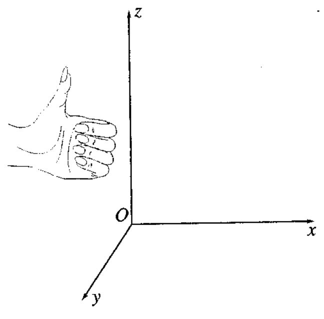


Figure 1

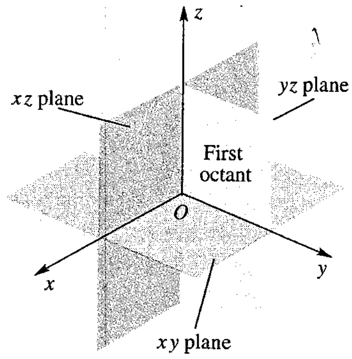


Figure 2

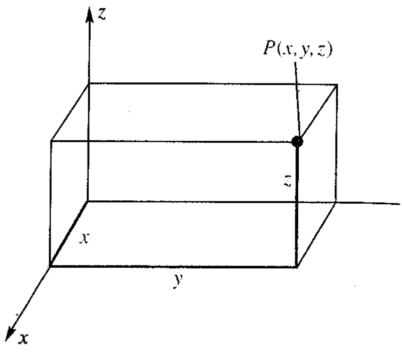


Figure 3

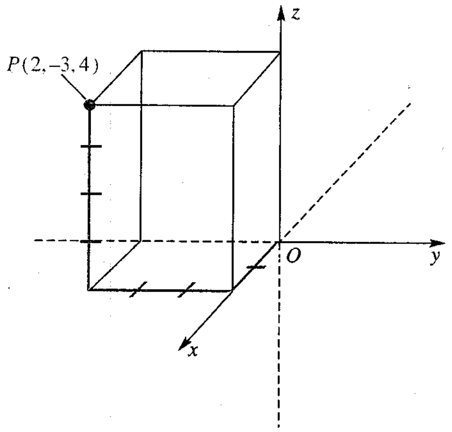


Figure 4

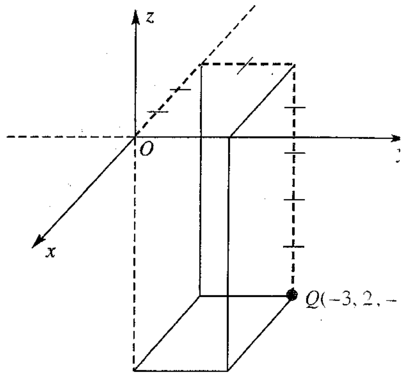


Figure 5

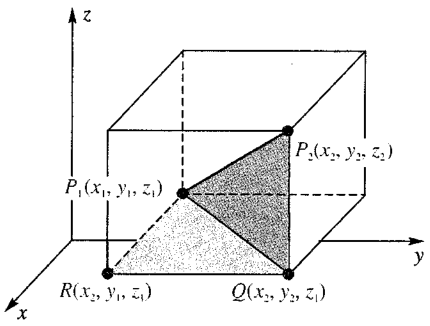


Figure 6

The Distance Formula Consider two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ three-space $(x_1 \neq x_2, y_1 \neq y_2, z_1 \neq z_2)$. They determine a **parallelepiped** (i.e., a re tangular box) with P_1 and P_2 as opposite vertices and with edges parallel to the c ordinate axes (Figure 6). The triangles P_1QP_2 and P_1RQ are right triangles and, the Pythagorean Theorem,

$$|P_1P_2|^2 = |P_1Q|^2 + |QP_2|^2$$

and

$$|P_1Q|^2 = |P_1R|^2 + |RQ|^2$$

Thus,

$$\begin{aligned} |P_1P_2|^2 &= |P_1R|^2 + |RQ|^2 + |QP_2|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \end{aligned}$$

This gives us the **Distance Formula** in three-space.

$$\boxed{|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

The formula is correct even if some of the coordinates are identical.

EXAMPLE 1 Find the distance between the points $P(2, -3, 4)$ and $Q(-3, 2, -$ which were plotted in Figures 4 and 5.

Solution

$$|PQ| = \sqrt{(-3 - 2)^2 + (2 + 3)^2 + (-5 - 4)^2} = \sqrt{131} \approx 11.45$$

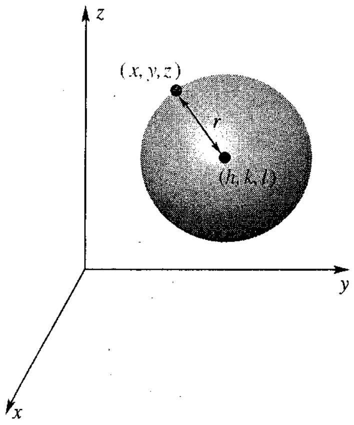


Figure 7

Spheres and Their Equations It is a small step from the Distance Formula to the equation of a sphere. By a **sphere**, we mean the set of all points in three-dimensional space that are a constant distance (the radius) from a fixed point (the center). (Recall that a circle is defined as the set of points *in a plane* that are a constant distance from a fixed point.) In fact, if (x, y, z) is a point on the sphere of radius r centered at (h, k, l), then (see Figure 7).

$$\boxed{(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2}$$

We call this the **standard equation of a sphere.**

In expanded form, the boxed equation may be written as

$$x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$$

Conversely, the graph of any equation of this form is either a sphere, a point (a degenerate sphere), or the empty set. To see why, consider the following example.

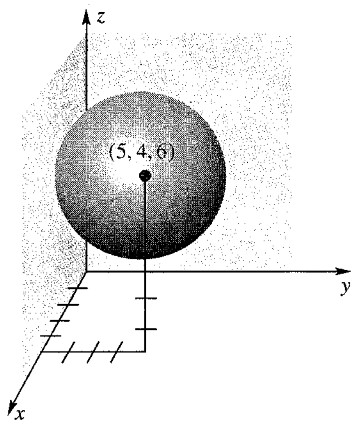


Figure 8

EXAMPLE 2 Find the center and radius of the sphere with equation

$$x^2 + y^2 + z^2 - 10x - 8y - 12z + 68 = 0$$

and sketch its graph.

Solution We use the process of completing the square.

$$(x^2 - 10x + \quad) + (y^2 - 8y + \quad) + (z^2 - 12z + \quad) = -68$$
$$(x^2 - 10x + 25) + (y^2 - 8y + 16) + (z^2 - 12z + 36) = -68 + 25 + 16 + 36$$
$$(x - 5)^2 + (y - 4)^2 + (z - 6)^2 = 9$$

Thus, the equation represents a sphere with center at (5, 4, 6) and radius 3. Its graph is shown in Figure 8. ■

If, after completing the square in Example 2, the equation had been

$$(x - 5)^2 + (y - 4)^2 + (z - 6)^2 = 0$$

then the graph would be the single point (5, 4, 6); if the right side were negative, the graph would be the empty set.

Another simple result that follows from the Distance Formula is the **Midpoint Formula.** If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are end points of a line segment, then the midpoint $M(m_1, m_2, m_3)$ has coordinates

$$\boxed{m_1 = \frac{x_1 + x_2}{2}, \qquad m_2 = \frac{y_1 + y_2}{2}, \qquad m_3 = \frac{z_1 + z_2}{2}}$$

In other words, to find the coordinates of the midpoint of a segment, simply take the average of corresponding coordinates of the end points.

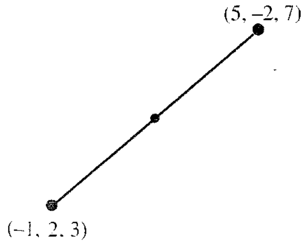


Figure 9

EXAMPLE 3 Find the equation of the sphere that has the line segment joining (−1, 2, 3) and (5, −2, 7) as a diameter (Figure 9).

Solution The center of this sphere is at the midpoint of the segment, that is, at (2, 0, 5); the radius r satisfies

$$r^2 = (5 - 2)^2 + (-2 - 0)^2 + (7 - 5)^2 = 17$$

We conclude that the equation of the sphere is

$$(x - 2)^2 + y^2 + (z - 5)^2 = 17$$ ■

Graphs in Three-Space It was natural to consider a quadratic equation first because of its relation to the Distance Formula. But, presumably, a **linear equation** in x, y, and z, that is, an equation of the form

$$Ax + By + Cz = D, \qquad A^2 + B^2 + C^2 \neq 0$$

should be even easier to analyze. As a matter of fact, we will show in the next section that the graph of a linear equation is a plane. Taking this for granted for now, let's consider how we might graph such an equation.

If, as will often be the case, the plane intersects the three axes, we begin by finding these intersection points; that is, we find the x-, y-, and z-intercepts. These three points determine the plane and allow us to draw the (coordinate-plane) **traces**, which are the lines of intersection of that plane with the coordinate planes. Then, with just a bit of artistry, we can shade in the plane.

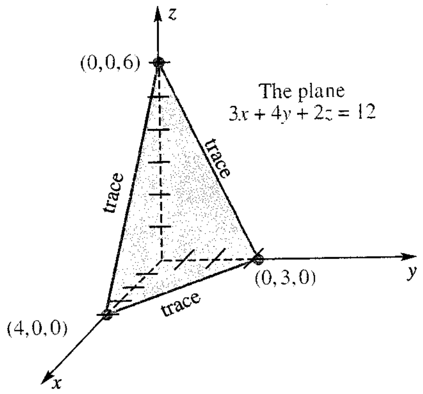


Figure 10

EXAMPLE 4 Sketch the graph of $3x + 4y + 2z = 12$.

Solution To find the x-intercept, set y and z equal to zero and solve for x, obtaining $x = 4$. The corresponding point is $(4, 0, 0)$. Similarly, the y- and z-intercepts are $(0, 3, 0)$ and $(0, 0, 6)$. Next, connect these points by line segments to get the traces. Then shade in (the first octant part of) the plane, thereby obtaining the result shown in Figure 10. ■

What if the plane does not intersect all three axes? This will happen, for example, if one of the variables in the equation of the plane is missing (i.e., has a zero coefficient).

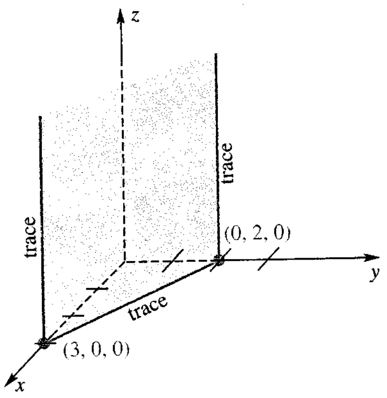


Figure 11

EXAMPLE 5 Sketch the graph of the linear equation

$$2x + 3y = 6$$

in three-space.

Solution The x- and y-intercepts are $(3, 0, 0)$ and $(0, 2, 0)$, respectively, and these points determine the trace in the xy-plane. The plane never crosses the z-axis (x and y cannot both be 0), and so the plane is parallel to the z-axis. We have sketched the graph in Figure 11. ■

Notice that in each of our examples the graph of an equation in three-space was a *surface*. This contrasts with the two-space case, where the graph of an equation was usually a *curve*. We will have a good deal more to say about graphing equations and the corresponding surfaces in Section 14.6.

Concepts Review

1. The numbers x, y, and z in (x, y, z) are called the ________ of a point in three-space.

2. The distance between the points $(-1, 3, 5)$ and (x, y, z) is ________.

3. The equation $(x + 1)^2 + (y - 3)^2 + (z - 5)^2 = 16$ determines a sphere with center ________ and radius ________.

4. The graph of $3x - 2y + 4z = 12$ is a ________ with x-intercept ________, y-intercept ________, and z-intercept ________.

Problem Set 14.1

1. Plot the points whose coordinates are $(1, 2, 3)$, $(2, 0, 1)$, $(-2, 4, 5)$, $(0, 3, 0)$, and $(-1, -2, -3)$. If appropriate, show the "box" as in Figures 4 and 5.

2. Follow the directions of Problem 1 for $(\sqrt{3}, -3, 3)$, $(0, \pi, -3)$, $(-2, \frac{1}{3}, 2)$, and $(0, 0, e)$.

3. What is peculiar to the coordinates of all points in the yz-plane? On the z-axis?

4. What is peculiar to the coordinates of all points in the xz-plane? On the y-axis?

5. Find the distance between the following pairs of points.

(a) $(6, -1, 0)$ and $(1, 2, 3)$ (b) $(-2, -2, 0)$ and $(2, -2, -3)$

(c) $(e, \pi, 0)$ and $(-\pi, -4, \sqrt{3})$

6. Show that $(4, 5, 3)$, $(1, 7, 4)$, and $(2, 4, 6)$ are vertices of an equilateral triangle.

7. Show that $(2, 1, 6)$, $(4, 7, 9)$, and $(8, 5, -6)$ are vertices of a right triangle. *Hint*: Only right triangles satisfy the Pythagorean Theorem.

8. Find the distance from $(2, 3, -1)$ to
(a) the xy-plane, (b) the y-axis, and
(c) the origin.

9. A rectangular box has its faces parallel to the coordinate planes and has $(2, 3, 4)$ and $(6, -1, 0)$ as the end points of a main diagonal. Sketch the box and find the coordinates of all eight vertices.

10. $P(x, 5, z)$ is on a line through $Q(2, -4, 3)$ that is parallel to one of the coordinate axes. Which axis must it be and what are x and z?

11. Write the equation of the sphere with the given center and radius.
(a) $(1, 2, 3); 5$ (b) $(-2, -3, -6); \sqrt{5}$
(c) $(\pi, e, \sqrt{2}); \sqrt{\pi}$

12. Find the equation of the sphere whose center is $(2, 4, 5)$ and that is tangent to the xy-plane.

In Problems 13–16, complete the squares to find the center and radius of the sphere whose equation is given (see Example 2).

13. $x^2 + y^2 + z^2 - 12x + 14y - 8z + 1 = 0$

14. $x^2 + y^2 + z^2 + 2x - 6y - 10z + 34 = 0$

15. $4x^2 + 4y^2 + 4z^2 - 4x + 8y + 16z - 13 = 0$

16. $x^2 + y^2 + z^2 + 8x - 4y - 22z + 77 = 0$

In Problems 17–24, sketch the graphs of the given equations. Begin by sketching the traces in the coordinate planes (see Examples 4 and 5).

17. $2x + 6y + 3z = 12$ **18.** $3x - 4y + 2z = 24$

19. $x + 3y - z = 6$ **20.** $-3x + 2y + z = 6$

21. $x + 3y = 8$ **22.** $3x + 4z = 12$

23. $x^2 + y^2 + z^2 = 9$ **24.** $(x - 2)^2 + y^2 + z^2 = 4$

25. Find the equation of the sphere that has the line segment joining $(-2, 3, 6)$ and $(4, -1, 5)$ as a diameter (see Example 3).

26. Find the equations of the tangent spheres of equal radii whose centers are $(-3, 1, 2)$ and $(5, -3, 6)$.

27. Find the equation of the sphere that is tangent to the three coordinate planes if its radius is 6 and its center is in the first octant.

28. Find the equation of the sphere with center $(1, 1, 4)$ that is tangent to the plane $x + y = 12$.

29. Describe the graph in three-space of each equation.
(a) $z = 2$ (b) $x = y$
(c) $xy = 0$ (d) $xyz = 0$
(e) $x^2 + y^2 = 4$ (f) $z = \sqrt{9 - x^2 - y^2}$

30. The sphere $(x - 1)^2 + (y + 2)^2 + (z + 1)^2 = 10$ intersects the plane $z = 2$ in a circle. Find the circle's center and radius.

31. A point P moves so that its distance from $(1, 2, -3)$ is twice its distance from $(1, 2, 3)$. Show that P is on a sphere and find its center and radius.

32. A point P moves so that its distance from $(1, 2, -3)$ equals its distance from $(2, 3, 2)$. Find the equation of the plane on which P lies.

33. The balls $(x - 1)^2 + (y - 2)^2 + (z - 1)^2 \le 4$ and $(x - 2)^2 + (y - 4)^2 + (z - 3)^2 \le 4$ intersect in a solid. Find its volume.

34. Do Problem 33 assuming that the second ball is $(x - 2)^2 + (y - 4)^2 + (z - 3)^2 \le 9$.

Answers to Concepts Review: 1. coordinates
2. $\sqrt{(x + 1)^2 + (y - 3)^2 + (z - 5)^2}$ 3. $(-1, 3, 5); 4$
4. plane; 4; −6; 3

14.2
Vectors in Three-Space

The material from Sections 13.2 and 13.3 on vectors in the plane can be repeated almost word for word for vectors in space. About the only difference is that a vector **u** now has three components; that is,

$$\mathbf{u} = \langle u_1, u_2, u_3 \rangle = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$

Here, $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$, are the standard unit vectors, called **basis vectors,** in the directions of the three positive coordinate axes (Figure 1). From the Distance Formula, the **length** of **u**, denoted by $|\mathbf{u}|$, is given by

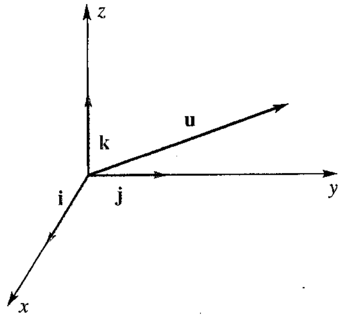


Figure 1

$$\boxed{|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}}$$

Vectors in space are added, multiplied by scalars, and subtracted just as in the plane, and the algebraic laws that are satisfied agree with those studied earlier. The **dot product** of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is defined by

$$\boxed{\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3}$$

and it has the geometric interpretation noted in the previous chapter,

$$\boxed{\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta}$$

where θ is the angle between **u** and **v**. Consequently, it continues to be true that two vectors are perpendicular if and only if their dot product is zero.

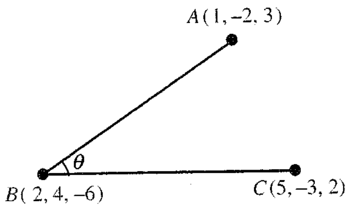


Figure 2

EXAMPLE 1 Find the angle ABC if $A = (1, -2, 3)$, $B = (2, 4, -6)$, and $C = (5, -3, 2)$ (Figure 2).

Solution First we determine vectors **u** and **v** (emanating from the origin) equivalent to $\overrightarrow{BA}$ and $\overrightarrow{BC}$. This is done by subtracting the coordinates of the initial points from those of the terminal points, that is,

$$\mathbf{u} = \langle 1 - 2, -2 - 4, 3 + 6\rangle = \langle -1, -6, 9\rangle$$

$$\mathbf{v} = \langle 5 - 2, -3 - 4, 2 + 6\rangle = \langle 3, -7, 8\rangle$$

Thus,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{(-1)(3) + (-6)(-7) + (9)(8)}{\sqrt{1 + 36 + 81}\sqrt{9 + 49 + 64}} \approx 0.9251$$

$$\theta = 0.3894 \quad (\text{about } 22.31°)$$

■

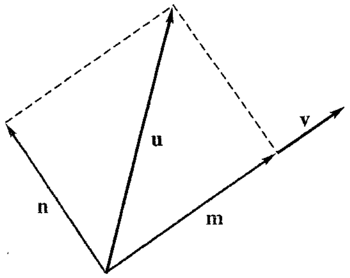


Figure 3

EXAMPLE 2 Express $\mathbf{u} = \langle 2, 4, 5\rangle$ as the sum of a vector **m** parallel to $\mathbf{v} = \langle 2, -1, -2\rangle$ and a vector **n** perpendicular to **v**.

Solution Figure 3 tells the story. First, we find $\mathbf{m} = \text{pr}_{\mathbf{v}}\mathbf{u}$, the projection of **u** on **v** (see Section 13.3).

$$\begin{aligned}\mathbf{m} &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\\ &= \frac{\langle 2, 4, 5\rangle \cdot \langle 2, -1, -2\rangle}{|\langle 2, -1, -2\rangle|^2}\langle 2, -1, -2\rangle\\ &= \frac{(2)(2) + (4)(-1) + (5)(-2)}{4 + 1 + 4}\langle 2, -1, -2\rangle\\ &= \left\langle \frac{-20}{9}, \frac{10}{9}, \frac{20}{9}\right\rangle\end{aligned}$$

Then

$$\mathbf{n} = \mathbf{u} - \mathbf{m} = \left\langle \frac{38}{9}, \frac{26}{9}, \frac{25}{9}\right\rangle$$

If you doubt that **m** and **n** are perpendicular, compute their dot product. You will get zero. ■

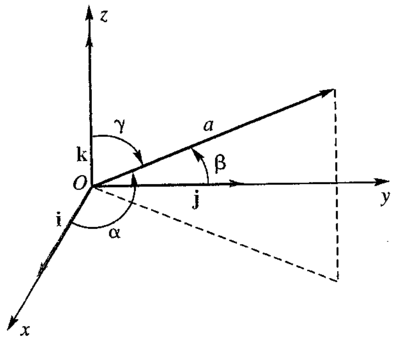


Figure 4

Direction Angles and Cosines The (smallest nonnegative) angles between a nonzero vector **a** and the basis vectors **i**, **j**, and **k** are the **direction angles** of **a**; they are designated by α, β, and γ, respectively (Figure 4). It is generally more convenient to work with the **direction cosines** $\cos\alpha$, $\cos\beta$, and $\cos\gamma$. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, then

$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}$$

and similarly

$$\cos\beta = \frac{a_2}{|\mathbf{a}|}, \qquad \cos\gamma = \frac{a_3}{|\mathbf{a}|}$$

Notice that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

In fact, the vector $\langle \cos\alpha, \cos\beta, \cos\gamma\rangle$ is a unit vector with the same direction as the original vector **a**.

EXAMPLE 3 Find the direction angles for the vector $\mathbf{a} = 4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$.

Solution Since $|\mathbf{a}| = \sqrt{4^2 + (-5)^2 + 3^2} = 5\sqrt{2}$,

$$\cos\alpha = \frac{4}{5\sqrt{2}} = \frac{2\sqrt{2}}{5}, \qquad \cos\beta = \frac{-\sqrt{2}}{2}, \qquad \cos\gamma = \frac{3\sqrt{2}}{10}$$

and

$$\alpha \approx 55.55^\circ, \qquad \beta = 135^\circ, \qquad \gamma \approx 64.90^\circ$$

■

EXAMPLE 4 Find a vector 5 units long that has $\alpha = 32^\circ$ and $\beta = 100^\circ$ as two of its direction angles.

Solution First, we note that the third direction angle γ must satisfy

$$\cos^2\gamma = 1 - \cos^2 32^\circ - \cos^2 100^\circ \approx 0.25066$$

Thus,

$$\cos\gamma \approx \pm 0.50066$$

Two vectors meet the requirements of the problem. They are

$$\begin{aligned} 5\langle\cos\alpha, \cos\beta, \cos\gamma\rangle &\approx 5\langle 0.84805, -0.17365, 0.50066\rangle \\ &= \langle 4.2403, -0.8683, 2.5033\rangle \end{aligned}$$

and $\langle 4.2403, -0.8683, -2.5033\rangle$. ■

Planes One fruitful way to describe a plane is by using vector language. Let $\mathbf{n} = \langle A, B, C\rangle$ be a fixed nonzero vector and $P_1(x_1, y_1, z_1)$ be a fixed point. The set of points $P(x, y, z)$ satisfying $\overrightarrow{P_1P} \cdot \mathbf{n} = 0$ is the **plane** through P_1 perpendicular to $\mathbf{n}$. Since every plane contains a point and is perpendicular to some vector, a plane can be characterized in this way.

To get the Cartesian equation of the plane, write the vector $\overrightarrow{P_1P}$ in component form; that is,

$$\overrightarrow{P_1P} = \langle x - x_1, y - y_1, z - z_1\rangle$$

Then $\overrightarrow{P_1P} \cdot \mathbf{n} = 0$ is equivalent to

$$\boxed{A(x - x_1) + B(y - y_1) + C(z - z_1) = 0}$$

This equation (in which at least one of A, B, and C is different from zero) is called the **standard form for the equation of a plane.**

If we remove the parentheses and simplify, the boxed equation takes the form of the general linear equation

$$Ax + By + Cz = D, \qquad A^2 + B^2 + C^2 \neq 0$$

Thus, every plane has a linear equation. Conversely, the graph of a linear equation in three-space is always a plane. To see the latter, let (x_1, y_1, z_1) satisfy the equation; that is,

$$Ax_1 + By_1 + Cz_1 = D$$

When we subtract this equation from the one above, we have the boxed equation, which we know represents a plane.

EXAMPLE 5 Find the equation of the plane through $(5, 1, -2)$ perpendicular to $\mathbf{n} = \langle 2, 4, 3\rangle$. Then find the angle between this plane and the one with equation $3x - 4y + 7z = 5$.

Solution To perform the first task, simply apply the standard form for the equation of a plane to the problem at hand, which gives

$$2(x-5)+4(y-1)+3(z+2)=0$$

or, equivalently,

$$2x+4y+3z=8$$

A vector **m** perpendicular to the second plane is $\mathbf{m}=\langle 3,-4,7\rangle$. The angle θ between two planes is the angle between their normals (Figure 5). Thus,

$$\cos\theta=\frac{\mathbf{m}\cdot\mathbf{n}}{|\mathbf{m}||\mathbf{n}|}=\frac{(3)(2)+(-4)(4)+(7)(3)}{\sqrt{9+16+49}\sqrt{4+16+9}}\approx 0.2375$$

$$\theta\approx 76.26^\circ$$

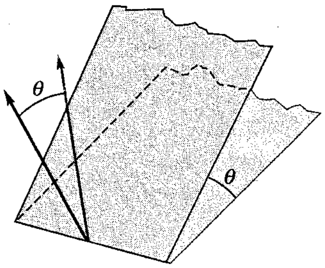


Figure 5

Actually, there are two angles between two planes, but they are *supplementary*. The process just described will lead to one of them. The other, if desired, is obtained by subtracting the first value from 180°. In our case, it would be 103.74°. ■

EXAMPLE 6 Show that the distance L from the point (x_0, y_0, z_0) to the plane $Ax+By+Cz=D$ is given by the formula

$$L=\frac{|Ax_0+By_0+Cz_0-D|}{\sqrt{A^2+B^2+C^2}}$$

Solution Let (x_1, y_1, z_1) be a point on the plane, and let $\mathbf{m}=\langle x_0-x_1, y_0-y_1, z_0-z_1\rangle$ be the vector from (x_1, y_1, z_1) to (x_0, y_0, z_0), as in Figure 6. Now $\mathbf{n}=\langle A, B, C\rangle$ is a vector perpendicular to the given plane, though it might point in the opposite direction of that in our figure. The number L that we seek is the length of the projection of **m** on **n**. Thus,

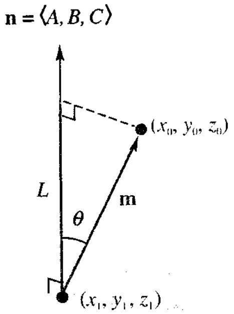


Figure 6

$$L=\big||\mathbf{m}|\cos\theta\big|=\frac{|\mathbf{m}\cdot\mathbf{n}|}{|\mathbf{n}|}$$

$$=\frac{|A(x_0-x_1)+B(y_0-y_1)+C(z_0-z_1)|}{\sqrt{A^2+B^2+C^2}}$$

$$=\frac{|Ax_0+By_0+Cz_0-(Ax_1+By_1+Cz_1)|}{\sqrt{A^2+B^2+C^2}}$$

But (x_1, y_1, z_1) is on the plane, and so

$$Ax_1+By_1+Cz_1=D$$

Substitution of this result in the expression for L yields the desired formula. ■

EXAMPLE 7 Find the distance between the parallel planes $3x-4y+5z=9$ and $3x-4y+5z=4$.

Solution The planes are parallel, since the vector $\langle 3,-4,5\rangle$ is perpendicular to both of them (Figure 7). The point $(1,1,2)$ is easily seen to be on the first plane. We find the distance L from $(1,1,2)$ to the second plane using the formula of Example 6.

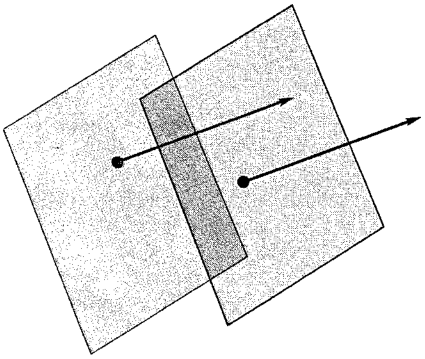


Figure 7

$$L=\frac{|3(1)-4(1)+5(2)-4|}{\sqrt{9+16+25}}=\frac{5}{5\sqrt{2}}\approx 0.7071$$ ■

Concepts Review

1. Let $\mathbf{u} = \langle 2, -3, \sqrt{3} \rangle$ and $\mathbf{v} = \langle 3, 2, -2\sqrt{3} \rangle$ be two vectors. The length of $\mathbf{u}$ is $|\mathbf{u}| =$ ________, and the dot product of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u} \cdot \mathbf{v} =$ ________.

2. Two vectors are perpendicular if and only if their ________ is ________.

3. $3(x - 2) - 2(y + 1) + 4z = 0$ is the equation of a plane through the point ________, with ________ being a vector perpendicular to the plane.

4. The (smallest nonnegative) angle θ between the vectors $\mathbf{u}$ and $\mathbf{v}$ can be found from the geometric formula for the dot product, $\mathbf{u} \cdot \mathbf{v} =$ ________. This gives $\theta =$ ________.

Problem Set 14.2

1. For each pair of points P_1 and P_2 given below, sketch the directed line segment $\overrightarrow{P_1 P_2}$ and then write the corresponding vector in the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

(a) $P_1(1, 2, 4), P_2(4, 5, 6)$

(b) $P_1(-1, -3, 204), P_2(-14, 52, 26)$

2. Follow the directions of Problem 1.

(a) $P_1(-2, -2, -2), P_2(-3, -4, 5)$

(b) $P_1(0, -1, e), P_2(-\sqrt{14}, -5, \pi)$

3. Find the length of and direction cosines for each of the following vectors:

(a) $4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ (b) $-2\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$

4. Follow the directions for Problem 3.

(a) $\langle 2, -1, -2 \rangle$ (b) $\langle -1, 2, -2 \rangle$

5. Find the unit vector with the same direction as $\langle 3, -4, 5 \rangle$. Also, find a vector of length 5 oriented in the opposite direction.

6. Find a vector of length 10 with direction opposite to $-4\mathbf{i} + 3\mathbf{j} + -2\mathbf{k}$.

7. Find the angle between $\langle 4, -3, -1 \rangle$ and $\langle -2, -3, 5 \rangle$.

8. Find the angle between $-4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$.

9. Find two vectors of length 10, each of which is perpendicular to both $-4\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ and $4\mathbf{i} + \mathbf{j}$.

10. Find all the vectors perpendicular to both $\langle 1, -2, -3 \rangle$ and $\langle -3, 2, 0 \rangle$.

11. Find the angle ABC if $A = (1, 2, 3)$, $B = (-4, 5, 6)$, and $C = (1, 0, 1)$ (see Example 1).

12. Show that the triangle ABC is a right triangle if $A = (6, 3, 3)$, $B = (3, 1, -1)$, and $C = (-1, 10, -2.5)$. *Hint*: Check the angle at B.

13. Find the *scalar projection* of $\mathbf{u} = -\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$ on $\mathbf{v} = -\mathbf{i} + \mathbf{j} - \mathbf{k}$. The scalar projection is the signed magnitude of the vector projection (Example 2); that is, it is $|\mathbf{u}|\cos\theta = \mathbf{u} \cdot \mathbf{v}/|\mathbf{v}|$.

14. Find the scalar projection of $\mathbf{u} = 5\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ on $\mathbf{v} = -\sqrt{5}\mathbf{i} + \sqrt{5}\mathbf{j} + \mathbf{k}$.

15. If $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -3\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, express $\mathbf{u}$ as the sum of a vector $\mathbf{m}$ parallel to $\mathbf{v}$ and a vector $\mathbf{n}$ perpendicular to $\mathbf{v}$ (see Example 2).

16. Follow the directions of Problem 15 for $\mathbf{u} = e\mathbf{i} + \pi\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

17. Find the direction angles for each vector.

(a) $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ (b) $\mathbf{u} = 3\mathbf{i} + 6\mathbf{j} - \mathbf{k}$

18. If $\alpha = 46°$ and $\beta = 108°$ are direction angles for a vector $\mathbf{u}$, find two possible values for the third angle (see Example 4).

19. A vector $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + z\mathbf{k}$ emanating from the origin points into the first octant. If $|\mathbf{u}| = 5$, find z.

20. If $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + z\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ are perpendicular, find z.

21. Find two perpendicular vectors $\mathbf{u}$ and $\mathbf{v}$ such that each is also perpendicular to $\mathbf{w} = \langle -4, 2, 5 \rangle$.

22. Find the vector emanating from the origin whose terminal point is the midpoint of the segment joining $(3, 2, -1)$ and $(5, -7, 2)$.

23. Which of the following do *not* make sense?

(a) $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$ (b) $(\mathbf{u} \cdot \mathbf{w}) + \mathbf{w}$

(c) $|\mathbf{u}|(\mathbf{v} \cdot \mathbf{w})$ (d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

(e) $(|\mathbf{u}|\mathbf{v}) \cdot \mathbf{w}$ (f) $|\mathbf{u}| \cdot \mathbf{v}$

24. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ be vectors emanating from the origin and terminating at A, B, C, and D, respectively. Use vector notation to express a necessary and sufficient condition that the figure $ABCD$ be a parallelogram.

25. Find the equation of the plane passing through P and perpendicular to $\mathbf{n}$ (see Example 5).

(a) $P(1, 2, -3), \mathbf{n} = 2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

(b) $P(-2, -3, 4), \mathbf{n} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

26. Find the smaller of the angles between the planes $3x - 2y + 5z = 7$ and $4x - 2y - 3z = 2$ (see Example 5).

27. Find the smaller of the angles between the two planes of Problem 25.

28. Find the equation of a plane through $(-1, 2, -3)$ and parallel to the plane $2x + 4y - z = 6$.

29. Find the equation of the plane through $(-4, -1, 2)$ and parallel

(a) to the xy-plane,

(b) to the plane $2x - 3y - 4z = 0$

30. Find the distance from $(1, -1, 2)$ to the plane

$$x + 3y + z = 7$$

(see Example 6).

31. Find the distance from $(2, 6, 3)$ to the plane

$$-3x + 2y + z = 9$$

32. Find the distance between the parallel planes

$$-3x + 2y + z = 9 \quad \text{and} \quad 6x - 4y - 2z = 19$$

(see Example 7).

33. Find the distance between the parallel planes

$$5x - 3y - 2z = 5 \quad \text{and} \quad -5x + 3y + 2z = 7$$

34. Find the equation of the plane each of whose points is equidistant from $(-2, 1, 4)$ and $(6, 1, -2)$.

35. Prove that $|\mathbf{u} + \mathbf{v}|^2 + |\mathbf{u} - \mathbf{v}|^2 = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2$. *Hint*: $|\mathbf{w}|^2 = \mathbf{w} \cdot \mathbf{w}$.

36. Prove that $\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}|\mathbf{u} + \mathbf{v}|^2 - \frac{1}{4}|\mathbf{u} - \mathbf{v}|^2$.

37. Find the angle between a main diagonal of a cube and one of its faces.

38. Find a unit vector whose direction angles are equal.

39. Find the smallest angle between the main diagonals of a rectangular box 4 feet by 6 feet by 10 feet.

40. Find the angles formed by the diagonals of a cube.

41. A constant force of $\mathbf{F} = -4\mathbf{k}$ newtons is applied to an object in moving it from $(0, 0, 8)$ to $(4, 4, 0)$, where coordinates are given in meters. Find the work done. (Recall that $W = \mathbf{F} \cdot \mathbf{D}$ (see Section 13.3).)

42. A constant force of $\mathbf{F} = 3\mathbf{i} - 6\mathbf{j} + 7\mathbf{k}$ pounds is applied to an object in moving it from $(2, 1, 3)$ to $(9, 4, 6)$, coordinates given in feet. Find the work done.

43. How much work is done by a force of 5 newtons acting in the direction $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ in moving an object from $(0, 1, 2)$ to $(3, 5, 7)$, distances being measured in meters? (See Problem 41.)

44. A weight of 30 pounds is suspended by three wires with resulting tensions $3\mathbf{i} + 4\mathbf{j} + 15\mathbf{k}$, $-8\mathbf{i} - 2\mathbf{j} + 10\mathbf{k}$, and $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Determine a, b, and c, assuming that $\mathbf{k}$ points straight up.

45. Find the point one-fifth of the way from $(2, 3, -1)$ to $(7, -2, 9)$.

46. Suppose that the three coordinate planes bounding the first octant are mirrors. A light ray with direction $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is reflected successively from the xy-plane, the xz-plane, and the yz-plane. Determine the direction of the ray after each reflection, and state a nice conclusion concerning the final reflected ray.

47. Find the distance from the sphere $x^2 + y^2 + z^2 + 2x + 6y - 8z = 0$ to the plane $3x + 4y + z = 15$.

48. Refine the method of Example 7 by showing that the distance L between the parallel planes $Ax + By + Cz = D$ and $Ax + By + Cz = E$ is

$$L = \frac{|D - E|}{\sqrt{A^2 + B^2 + C^2}}$$

49. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ be fixed vectors. Show that $(\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{b}) = 0$ is the equation of a sphere, and find its center and radius.

50. Show that the work done by a constant force $\mathbf{F}$ on an object that moves completely around a closed polygonal path is 0.

51. The medians of a triangle meet in a point P (the centroid by Problem 30 of Section 6.6) that is two-thirds of the way from a vertex to the midpoint of the opposite edge. Show that P is the head of the position vector $(\mathbf{a} + \mathbf{b} + \mathbf{c})/3$, where $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are the position vectors of the vertices, and use this to find P if the vertices are $(2, 6, 5)$, $(4, -1, 2)$, and $(6, 1, 2)$.

52. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and $\mathbf{d}$ be the position vectors of the vertices of a tetrahedron. Show that the lines joining the vertices to the centroids of the opposite faces meet in a point P, and give a nice vector formula for it, thus generalizing Problem 51.

Answers to Concepts Review: **1.** 4; -6 **2.** dot product; 0 **3.** $(2, -1, 0)$; $\langle 3, -2, 4 \rangle$ **4.** $|\mathbf{u}||\mathbf{v}|\cos\theta$; $\cos^{-1}(\mathbf{u} \cdot \mathbf{v}/|\mathbf{u}||\mathbf{v}|)$

14.3
The Cross Product

The dot product of two vectors is a scalar. We have explored some of its uses in earlier sections. Now we introduce the **cross product** (or vector product); it will also have many uses. The cross product $\mathbf{u} \times \mathbf{v}$ of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is defined by

$$\boxed{\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle}$$

In this form, the formula is hard to remember and its significance is not obvious. Note the one thing that is obvious. The cross product of two vectors is a vector.

To help us remember the formula for the cross product, we recall a subject from an earlier mathematics course, namely, *determinants*. First, the value of a 2×2 determinant is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Then the value of a 3×3 determinant is (expanding along to the top row)

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + a_3\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Using determinants, we may write the definition of $\mathbf{u} \times \mathbf{v}$ as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

Note that the components of the left vector $\mathbf{u}$ go in the second row, and those of the right vector $\mathbf{v}$ go in the third row. This is important, because if we interchange the positions of $\mathbf{u}$ and $\mathbf{v}$, we interchange the second and third rows of the determinant, and this changes the sign of the determinant's value, as you may check. Thus,

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$

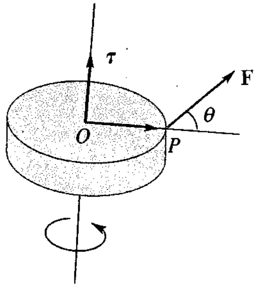
which is sometimes called the *anticommutative law*.

Torque

The cross product plays an important role in mechanics. Let O be a *fixed point* in a body, and suppose that a force $\mathbf{F}$ is applied at another point P of the body. Then $\mathbf{F}$ tends to rotate the body about an axis through O and perpendicular to the plane of OP and $\mathbf{F}$. The vector

$$\boldsymbol{\tau} = \overrightarrow{OP} \times \mathbf{F}$$

is called the **torque**. It points in the direction of the axis and has magnitude $|\overrightarrow{OP}||\mathbf{F}|\sin\theta$, which is just the moment of force about the axis due to $\mathbf{F}$.



EXAMPLE 1 Let $\mathbf{u} = \langle 1, -2, -1\rangle$ and $\mathbf{v} = \langle -2, 4, 1\rangle$. Calculate $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ using the determinant definition.

Solution

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -1 \\ -2 & 4 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -2 & -1 \\ 4 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix}$$

$$= 2\mathbf{i} + \mathbf{j} + 0\mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & 1 \\ 1 & -2 & -1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 4 & 1 \\ -2 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & 4 \\ 1 & -2 \end{vmatrix}$$

$$= -2\mathbf{i} - \mathbf{j} + 0\mathbf{k}$$

■

Geometric Interpretation of $\mathbf{u} \times \mathbf{v}$ Like the dot product, the cross product gains significance from its geometric interpretation.

Theorem A

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in three-space and θ be the angle between them. Then:

1. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0 = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$, that is, $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$;
2. $\mathbf{u}, \mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ form a right-handed triple;
3. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$.

Proof Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$.

1. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1)$. When we remove parentheses, the six terms cancel in pairs. A similar event occur's when we expand $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$.
2. The meaning of right-handedness for the triple $\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}$ is illustrated in Figure 1. There θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, and the fingers of the right hand are curled in the direction of the rotation through θ that makes $\mathbf{u}$ coincide with $\mathbf{v}$. It is difficult to establish analytically that the indicated triple is right-handed, but you might check it with a few examples. Note in particular that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, and by definition we know that the triple $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is right-handed.
3. We need Lagrange's Identity,

$$|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

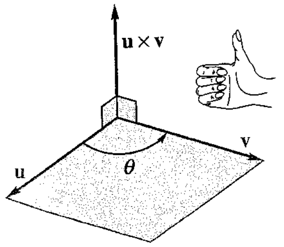


Figure 1

whose proof is a simple algebraic exercise (Problem 25). Using this identity, we may write

$$\begin{aligned}|\mathbf{u}\times\mathbf{v}|^2 &= |\mathbf{u}|^2|\mathbf{v}|^2 - (|\mathbf{u}||\mathbf{v}|\cos\theta)^2\\ &= |\mathbf{u}|^2|\mathbf{v}|^2(1-\cos^2\theta)\\ &= |\mathbf{u}|^2|\mathbf{v}|^2\sin^2\theta\end{aligned}$$

Since $0 \le \theta \le \pi$, $\sin\theta \ge 0$. Taking principal square roots yields

$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta \ \blacklozenge$$

It is important that we have geometric interpretations of both $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\times\mathbf{v}$. While both products were originally defined in terms of components that depend on a choice of coordinate system, they are actually independent of coordinate systems. They are intrinsic geometric quantities, and you will get the same results for $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\times\mathbf{v}$ no matter how you introduce the coordinates used to compute them.

Here is a simple consequence of Theorem A (part 3) and the fact that vectors are parallel if and only if the angle θ between them is either 0° or 180°.

Theorem B

Two vectors **u** and **v** in three-space are parallel if and only if $\mathbf{u}\times\mathbf{v} = \mathbf{0}$.

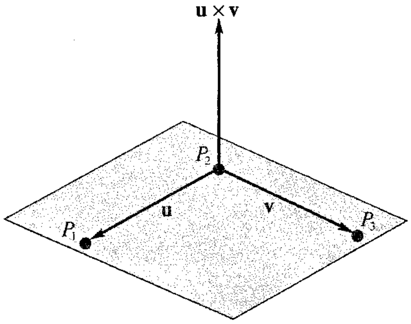


Figure 2

Applications

Our first application is to find the equation of the plane through three noncollinear points.

EXAMPLE 2 Find the equation of the plane (Figure 2) through the three points $P_1(1,-2,3)$, $P_2(4,1,-2)$, and $P_3(-2,-3,0)$.

Solution Let $\mathbf{u} = \overrightarrow{P_2P_1} = \langle -3,-3,5\rangle$ and $\mathbf{v} = \overrightarrow{P_2P_3} = \langle -6,-4,2\rangle$. From the first part of Theorem A we know that

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -3 & -3 & 5\\ -6 & -4 & 2\end{vmatrix} = 14\mathbf{i} - 24\mathbf{j} - 6\mathbf{k}$$

is perpendicular to both **u** and **v** and thus to the plane containing them. The plane through $(4,1,-2)$ with normal $14\mathbf{i} - 24\mathbf{j} - 6\mathbf{k}$ has equation (see Section 14.2)

$$14(x-4) - 24(y-1) - 6(z+2) = 0$$

or

$$14x - 24y - 6z = 44$$

■

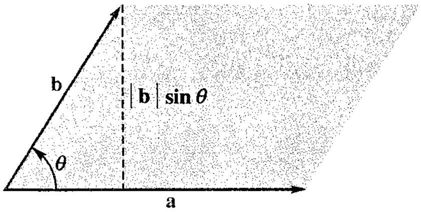


Figure 3

EXAMPLE 3 Show that the area of a parallelogram with **a** and **b** as adjacent sides is $|\mathbf{a}\times\mathbf{b}|$.

Solution Recall that the area of a parallelogram is the product of the base times the height. Now look at Figure 3 and use the fact that $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$. ■

EXAMPLE 4 Show that the volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is

$$V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = \left|\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}\right|$$

Solution Refer to Figure 4 and regard the parallelogram determined by **b** and **c** as the base of the parallelepiped. The area of this base is $|\mathbf{b}\times\mathbf{c}|$ by Example 3; the height h of the parallelepiped is the absolute value of the scalar projection of **a** on $\mathbf{b}\times\mathbf{c}$. Thus,

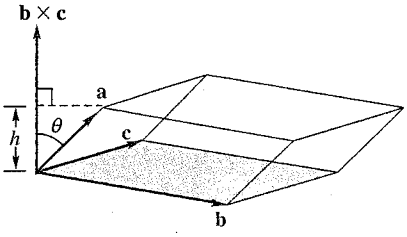


Figure 4

$$h = |\mathbf{a}||\cos\theta| = \frac{|\mathbf{a}||\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|}{|\mathbf{a}||\mathbf{b}\times\mathbf{c}|} = \frac{|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|}{|\mathbf{b}\times\mathbf{c}|}$$

and

$$V = h|\mathbf{b}\times\mathbf{c}| = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$

That V can also be expressed as a determinant is established by expanding $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$ in terms of components and then comparing it with the value of the indicated determinant. ■

Suppose that the vectors **a**, **b**, and **c** from the previous example are in the *same* plane. In this case, the parallelepiped has height zero, so the volume should be zero. Does the formula for the volume yield $V = 0$? If **a** is in the plane determined by **b** and **c**, then any vector perpendicular to **b** and **c** will be perpendicular to **a** as well. The vector $\mathbf{b}\times\mathbf{c}$ is perpendicular to both **b** and **c**; hence $\mathbf{b}\times\mathbf{c}$ is perpendicular to **a**. Thus, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$.

Check Extreme Cases

Never read a mathematics book passively; rather, ask questions as you go. In particular, you should look at extreme cases whenever possible. Here we look at the case where the vectors **a**, **b**, and **c** are in the *same* plane. The volume of the parallelepiped should be zero, and indeed the formula does give zero. What happens in Example 3 if the vectors **a** and **b** are parallel?

Algebraic Properties The rules for calculating with cross products are summarized in the following theorem. Proving this theorem is a matter of writing everything out in terms of components and will be left as an exercise.

Theorem C

If **u**, **v**, and **w** are vectors in three-space and k is a scalar, then:

1. $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$ (anticommutative law);
2. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = (\mathbf{u}\times\mathbf{v}) + (\mathbf{u}\times\mathbf{w})$ (left distributive law);
3. $k(\mathbf{u}\times\mathbf{v}) = (k\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(k\mathbf{v})$;
4. $\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \mathbf{0}, \mathbf{u}\times\mathbf{u} = \mathbf{0}$;
5. $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$;
6. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$.

Once the rules in Theorem C are mastered, complicated calculations with vectors can be done with ease. We illustrate by calculating a cross product in a new way. We will need the following simple but important products.

$$\boxed{\mathbf{i}\times\mathbf{j} = \mathbf{k}, \quad \mathbf{j}\times\mathbf{k} = \mathbf{i}, \quad \mathbf{k}\times\mathbf{i} = \mathbf{j}}$$

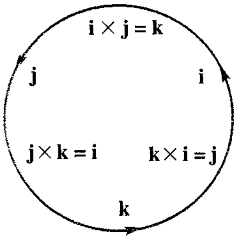


Figure 5

These results have a cyclic order, which can be remembered by appealing to Figure 5.

EXAMPLE 5 Calculate $\mathbf{u}\times\mathbf{v}$ if $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

Solution We appeal to Theorem C, especially the distributive law and the anticommutative law.

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= (3\mathbf{i} - 2\mathbf{j} + \mathbf{k})\times(4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})\\ &= 12(\mathbf{i}\times\mathbf{i}) + 6(\mathbf{i}\times\mathbf{j}) - 9(\mathbf{i}\times\mathbf{k}) - 8(\mathbf{j}\times\mathbf{i}) - 4(\mathbf{j}\times\mathbf{j})\\ &\quad + 6(\mathbf{j}\times\mathbf{k}) + 4(\mathbf{k}\times\mathbf{i}) + 2(\mathbf{k}\times\mathbf{j}) - 3(\mathbf{k}\times\mathbf{k})\\ &= 12(\mathbf{0}) + 6(\mathbf{k}) - 9(-\mathbf{j}) - 8(-\mathbf{k}) - 4(\mathbf{0})\\ &\quad + 6(\mathbf{i}) + 4(\mathbf{j}) + 2(-\mathbf{i}) - 3(\mathbf{0})\\ &= 4\mathbf{i} + 13\mathbf{j} + 14\mathbf{k}\end{aligned}$$

Experts would do most of this in their heads; novices might find the determinant method easier. ■

Concepts Review

1. The cross product of $\mathbf{u} = \langle -1, 2, 1\rangle$ and $\mathbf{v} = \langle 3, 1, -1\rangle$ is given by a specific determinant; evaluation of this determinant gives $\mathbf{u} \times \mathbf{v} =$ ________.

2. Geometrically, $\mathbf{u} \times \mathbf{v}$ is a vector perpendicular to the plane of $\mathbf{u}$ and $\mathbf{v}$ and has length $|\mathbf{u} \times \mathbf{v}| =$ ________.

3. The cross product is anticommutative; that is, $\mathbf{u} \times \mathbf{v} =$ ________.

4. Two vectors are ________ if and only if their cross product is $\mathbf{0}$.

Problem Set 14.3

1. Let $\mathbf{a} = -3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$, and $\mathbf{c} = 7\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$. Find each of the following:

(a) $\mathbf{a} \times \mathbf{b}$
(b) $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$
(c) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$
(d) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

2. If $\mathbf{a} = \langle 3, 3, 1\rangle$, $\mathbf{b} = \langle -2, -1, 0\rangle$, and $\mathbf{c} = \langle -2, -3, -1\rangle$, find each of the following:

(a) $\mathbf{a} \times \mathbf{b}$
(b) $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$
(c) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
(d) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

3. Find all vectors perpendicular to both of the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$.

4. Find all vectors perpendicular to both of the vectors $\mathbf{a} = -2\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$.

5. Find the unit vectors perpendicular to the plane determined by the three points $(1, 3, 5)$, $(3, -1, 2)$, and $(4, 0, 1)$.

6. Find the unit vectors perpendicular to the plane determined by the three points $(-1, 3, 0)$, $(5, 1, 2)$, and $(4, -3, -1)$.

7. Find the area of the parallelogram with $\mathbf{a} = -\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ as the adjacent sides.

8. Find the area of the parallelogram with $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ as the adjacent sides.

9. Find the area of the triangle with $(3, 2, 1)$, $(2, 4, 6)$, and $(-1, 2, 5)$ as vertices.

10. Find the area of the triangle with $(1, 2, 3)$, $(3, 1, 5)$, and $(4, 5, 6)$ as vertices.

11. Find the equation of the plane through $(1, 3, 2)$, $(0, 3, 0)$, and $(2, 4, 3)$ (see Example 2).

12. Find the equation of the plane through $(1, 1, 2)$, $(0, 0, 1)$, and $(-2, -3, 0)$.

13. Find the equation of the plane through $(-1, -2, 3)$ and perpendicular to both the planes $x - 3y + 2z = 7$ and $2x - 2y - z = -3$.

14. Find the equation of the plane through $(2, -3, 2)$ and parallel to the plane of the vectors $4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $2\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$.

15. Find the equation of the plane through $(6, 2, -1)$ and perpendicular to the line of intersection of the planes $4x - 3y + 2z + 5 = 0$ and $3x + 2y - z + 11 = 0$.

16. Let $\mathbf{a}$ and $\mathbf{b}$ be nonparallel vectors, and let $\mathbf{c}$ be any nonzero vector. Show that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is a vector in the plane of $\mathbf{a}$ and $\mathbf{b}$.

17. Find the volume of the parallelepiped with edges $\langle 2, 3, 4\rangle$, $\langle 0, 4, -1\rangle$, and $\langle 5, 1, 3\rangle$ (see Example 4).

18. Find the volume of the parallelepiped with edges $3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $-\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, and $3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$.

19. Let K be the parallelepiped determined by $\mathbf{u} = \langle 3, 2, 1\rangle$, $\mathbf{v} = \langle 1, 1, 2\rangle$, and $\mathbf{w} = \langle 1, 3, 3\rangle$.

(a) Find the volume of K.
(b) Find the area of the face determined by $\mathbf{u}$ and $\mathbf{v}$.
(c) Find the angle between $\mathbf{u}$ and the plane containing the face determined by $\mathbf{v}$ and $\mathbf{w}$.

20. The formula for the volume of a parallelepiped derived in Example 4 should not depend on the choice of which one of the three vectors we call $\mathbf{a}$, which one we call $\mathbf{b}$, and which one we call $\mathbf{c}$. Use this result to explain why $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})| = |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|$.

21. Which of the following do *not* make sense?

(a) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$
(b) $\mathbf{u} + (\mathbf{v} \times \mathbf{w})$
(c) $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$
(d) $(\mathbf{a} \times \mathbf{b}) + k$
(e) $(\mathbf{a} \cdot \mathbf{b}) + k$
(f) $(\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d})$
(g) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$
(h) $(k\mathbf{u}) \times \mathbf{v}$

22. Show that if $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and $\mathbf{d}$ all lie in the same plane then

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$$

23. The volume of a tetrahedron is known to be $\frac{1}{3}$(area of base)(height). From this, show that the volume of the tetrahedron with edges $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ is $\frac{1}{6}|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

24. Find the volume of the tetrahedron with vertices $(-1, 2, 3)$, $(4, -1, 2)$, $(5, 6, 3)$, and $(1, 1, -2)$ (see Problem 23).

25. Prove **Lagrange's Identity**,

$$|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

without using Theorem A.

26. Prove the left distributive law,

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$$

27. Use Problem 26 and the anticommutative law to prove the right distributive law.

28. If both $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{v} = 0$, what can you conclude about $\mathbf{u}$ or $\mathbf{v}$?

29. Use Example 3 to develop a formula for the area of the triangle with vertices $P(a, 0, 0)$, $Q(0, b, 0)$, and $R(0, 0, c)$ shown in the top half of Figure 6.

30. Show that the triangle in the plane with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) has area equal to one-half the absolute value of the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

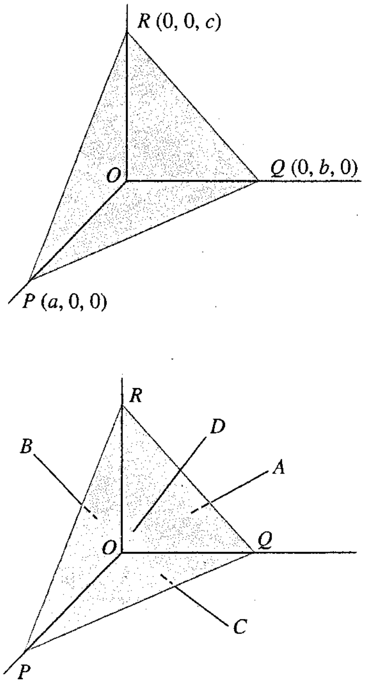


Figure 6

31. A Pythagorean Theorem in Three-Space As in Figure 6, let P, Q, R, and O be the vertices of a (right-angled) tetrahedron, and let A, B, C, and D be the areas of the opposite faces, respectively. Show that $A^2 + B^2 + C^2 = D^2$.

32. Let vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ with common initial point determine a tetrahedron, and let $\mathbf{m}, \mathbf{n}, \mathbf{p}$, and $\mathbf{q}$ be vectors perpendicular to the four faces, pointing outward, and having length equal to the area of the corresponding face. Show that $\mathbf{m} + \mathbf{n} + \mathbf{p} + \mathbf{q} = \mathbf{0}$.

33. Let $\mathbf{a}, \mathbf{b}$, and $\mathbf{a} - \mathbf{b}$ denote the three edges of a triangle with lengths a, b, and c, respectively. Use Lagrange's Identity together with $2\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2$ to prove **Heron's Formula** for the area A of a triangle,

$$A = \sqrt{s(s - a)(s - b)(s - c)}$$

where s is the semiperimeter $(a + b + c)/2$.

34. Use the method of Example 5 to show directly that, if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Answers to Concepts Review: **1.** $\langle -3, 2, -7\rangle$ or $-3\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$ **2.** $|\mathbf{u}||\mathbf{v}|\sin\theta$ **3.** $-(\mathbf{v} \times \mathbf{u})$ **4.** parallel

---

14.4
Lines and Curves in Three-Space

Our study of lines and curves in the plane extends easily to three-space. A **space curve** is determined by a triple of parametric equations

$$\boxed{x = f(t), \qquad y = g(t), \qquad z = h(t), \qquad t \in I}$$

with f, g, and h continuous on the interval I. In vector language, a curve is specified by giving the position vector $\mathbf{r} = \mathbf{r}(t)$ of a point $P = P(t)$; that is,

$$\mathbf{r} = \mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

The tip of $\mathbf{r}$ traces out the curve as t ranges over the interval I, as we see in Figure 1.

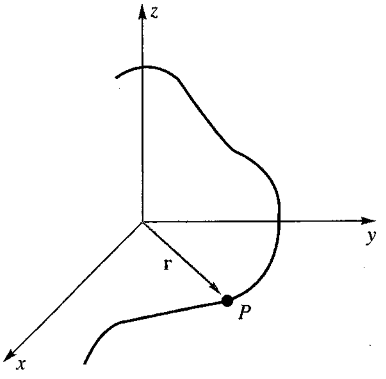


Figure 1

Lines The simplest of all curves is a line. A line is determined by a fixed point P_0 and a fixed vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. It is the set of all points P such that $\overrightarrow{P_0P}$ is parallel to $\mathbf{v}$, that is, that satisfy

$$\overrightarrow{P_0P} = t\mathbf{v}$$

for some real number t (Figure 2). If $\mathbf{r} = \overrightarrow{OP}$ and $\mathbf{r}_0 = \overrightarrow{OP}_0$ are the position vectors of P and P_0, respectively, then $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0$, and the equation of the line can thus be written

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

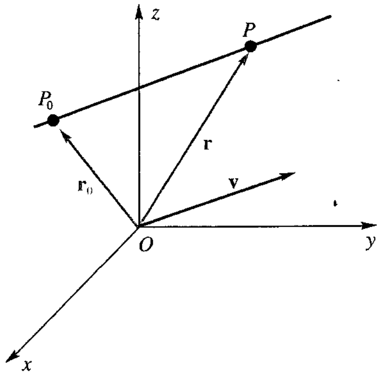


Figure 2

If we write $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ and equate components in the last equation above, we obtain

$$\boxed{x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct}$$

These are **parametric equations** of the line through (x_0, y_0, z_0) and parallel to $\mathbf{v} = \langle a, b, c\rangle$. The numbers a, b, and c are called **direction numbers** for the line. They are not unique; any nonzero constant multiples ka, kb, and kc are also direction numbers.

EXAMPLE 1 Find parametric equations for the line through $(3, -2, 4)$ and $(5, 6, -2)$ (see Figure 3).

Solution A vector parallel to the given line is

$$\mathbf{v} = \langle 5 - 3, 6 + 2, -2 - 4 \rangle = \langle 2, 8, -6 \rangle$$

If we choose (x_0, y_0, z_0) as $(3, -2, 4)$, we obtain the parametric equations

$$x = 3 + 2t, \qquad y = -2 + 8t, \qquad z = 4 - 6t$$

Note that $t = 0$ determines the point $(3, -2, 4)$, whereas $t = 1$ gives $(5, 6, -2)$. In fact, $0 \leq t \leq 1$ corresponds to the segment joining these two points. ■

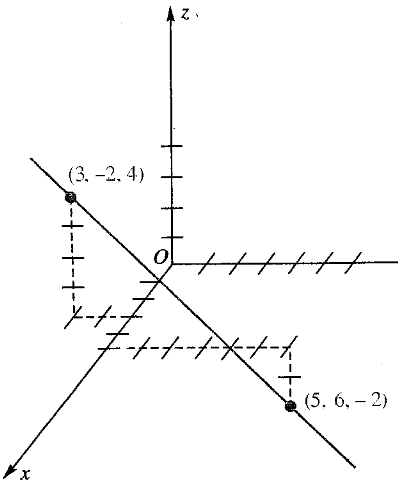


Figure 3

If we solve each of the parametric equations for t (assuming that a, b, and c are all different from zero) and equate the results, we obtain the **symmetric equations** for the line through (x_0, y_0, z_0) with direction numbers a, b, c; that is,

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

This is the conjunction of the two equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} \quad \text{and} \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

both of which are the equations of planes (Figure 4); and, of course, the intersection of two planes is a line.

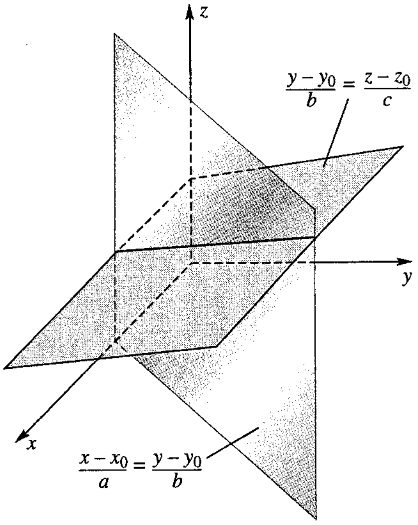


Figure 4

EXAMPLE 2 Find the symmetric equations of the line that is parallel to the vector $\langle 4, -3, 2 \rangle$ and goes through $(2, 5, -1)$.

Solution

$$\frac{x - 2}{4} = \frac{y - 5}{-3} = \frac{z + 1}{2}$$

■

EXAMPLE 3 Find the symmetric equations of the line of intersection of the planes

$$2x - y - 5z = -14 \quad \text{and} \quad 4x + 5y + 4z = 28$$

Solution We begin by finding two points on the line. Any two points would do, but we choose to find the points where the line pierces the yz-plane and the xz-plane (Figure 5). The former is obtained by setting $x = 0$ and solving the resulting equations $-y - 5z = -14$ and $5y + 4z = 28$ simultaneously. This yields the point $(0, 4, 2)$. A similar procedure with $y = 0$ gives the point $(3, 0, 4)$. Consequently, a vector parallel to the required line is

$$\langle 3 - 0, 0 - 4, 4 - 2 \rangle = \langle 3, -4, 2 \rangle$$

Using $(3, 0, 4)$ for (x_0, y_0, z_0), we get

$$\frac{x - 3}{3} = \frac{y - 0}{-4} = \frac{z - 4}{2}$$

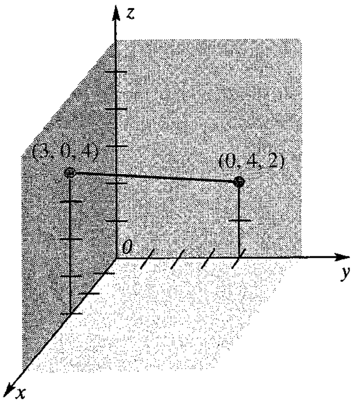


Figure 5

An alternative solution is based on the fact that the line of intersection of two planes is perpendicular to both of their normals. The vector $\mathbf{u} = \langle 2, -1, -5 \rangle$ is normal to the first plane; $\mathbf{v} = \langle 4, 5, 4 \rangle$ is normal to the second. Since

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -5 \\ 4 & 5 & 4 \end{vmatrix} = 21\mathbf{i} = 28\mathbf{j} + 14\mathbf{k}$$

the vector $\mathbf{w} = \langle 21, -28, 14 \rangle$ is parallel to the required line. This implies that $\frac{1}{7}\mathbf{w} = \langle 3, -4, 2 \rangle$ also has this property. Next, find any point on the line of intersection, for example, $(3, 0, 4)$, and proceed as in the earlier solution. ■

EXAMPLE 4 Find parametric equations of the line through $(1, -2, 3)$ that is perpendicular to both the x-axis and the line

$$\frac{x-4}{2} = \frac{y-3}{-1} = \frac{z}{5}$$

Solution The x-axis and the given line have directions $\mathbf{u} = \langle 1, 0, 0\rangle$ and $\mathbf{v} = \langle 2, -1, 5\rangle$, respectively. A vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 2 & -1 & 5 \end{vmatrix} = 0\mathbf{i} - 5\mathbf{j} - \mathbf{k}$$

The required line is parallel to $\langle 0, -5, -1\rangle$ and so also to $\langle 0, 5, 1\rangle$. Since the first direction number is zero, the line does not have symmetric equations. Its parametric equations are

$$x = 1, \qquad y = -2 + 5t, \qquad z = 3 + t \qquad \blacksquare$$

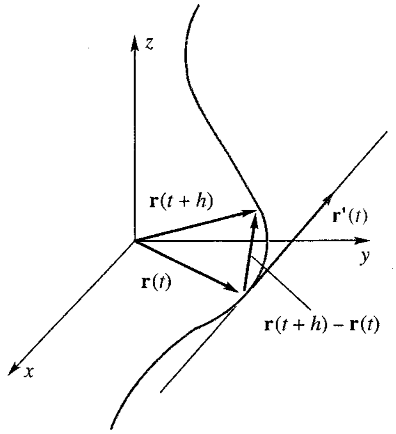


Figure 6

Tangent Line to a Curve Let

$$\mathbf{r} = \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

be the position vector determining a curve in three-space (Figure 6). In complete analogy with what we did in the plane (Section 13.4), we define $\mathbf{r}'(t)$ by

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

It follows that $\mathbf{r}'(t)$, if it exists, has the direction of the tangent line to the curve at the point $P(t)$ corresponding to t. Moreover, $\mathbf{r}'(t)$ exists if and only if $f'(t)$, $g'(t)$, and $h'(t)$ exist and, in this case,

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

Thus, $f'(t)$, $g'(t)$, and $h'(t)$ are direction numbers for the tangent line at P.

EXAMPLE 5 Find the symmetric equations for the tangent line to the curve determined by

$$\mathbf{r}(t) = t\mathbf{i} + \tfrac{1}{2}t^2\mathbf{j} + \tfrac{1}{3}t^3\mathbf{k}$$

at $P(2) = \left(2, 2, \frac{8}{3}\right)$.

Solution

$$\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$$

and

$$\mathbf{r}'(2) = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

so the tangent line has direction $\langle 1, 2, 4\rangle$. Its symmetric equations are

$$\frac{x-2}{1} = \frac{y-2}{2} = \frac{z-\frac{8}{3}}{4} \qquad \blacksquare$$

Concepts Review

1. The parametric equations for a line through $(1,-3,2)$ parallel to the vector $\langle 4,-2,-1\rangle$ are $x =$ ________, $y =$ ________, $z =$ ________.

2. The symmetric equations for the line of Question 1 are ________.

3. If $\mathbf{r}(t) = t^2\mathbf{i} - 3t\mathbf{j} + t^3\mathbf{k}$, then $\mathbf{r}'(t) =$ ________.

4. A vector parallel to the tangent line at $t = 1$ of the curve determined by the position vector $\mathbf{r}(t)$ of Question 3 is ________. This tangent line has symmetric equations ________.

Problem Set 14.4

In Problems 1–4, find the parametric equations of the line through the given pair of points.

1. $(1,-2,3), (4,5,6)$

2. $(2,-1,-5), (7,-2,3)$

3. $(4,2,3), (6,2,-1)$

4. $(5,-3,-3), (5,4,2)$

In Problems 5–8, write both the parametric equations and the symmetric equations for the line through the given point parallel to the given vector.

5. $(4,5,6), \langle 3,2,1\rangle$

6. $(-1,3,-6), \langle -2,0,5\rangle$

7. $(1,1,1), \langle -10,-100,-1000\rangle$

8. $(-2,2,-2), \langle 7,-6,3\rangle$

In Problems 9–12, find the symmetric equations of the line of intersection of the given pair of planes.

9. $4x + 3y - 7z = 1, 10x + 6y - 5z = 10$

10. $x + y - z = 2, 3x - 2y + z = 3$

11. $x + 4y - 2z = 13, 2x - y - 2z = 5$

12. $x - 3y + z = -1, 6x - 5y + 4z = 9$

13. Find the symmetric equations of the line through $(4,0,6)$ and perpendicular to the plane $x - 5y + 2z = 10$.

14. Find the symmetric equations of the line through $(-5,7,-2)$ and perpendicular to both $\langle 2,1,-3\rangle$ and $\langle 5,4,-1\rangle$.

15. Find the parametric equations of the line through $(5,-3,4)$ that intersects the z-axis at right angles.

16. Find the symmetric equations of the line through $(2,-4,5)$ that is parallel to the plane $3x + y - 2z = 5$ and perpendicular to the line

$$\frac{x+8}{2} = \frac{y-5}{3} = \frac{z-1}{-1}$$

17. Find the equation of the plane that contains the parallel lines

$$\begin{cases} x = -2 + 2t \\ y = 1 + 4t \\ z = 2 - t \end{cases} \quad \text{and} \quad \begin{cases} x = 2 - 2t \\ y = 3 - 4t \\ z = 1 + t \end{cases}$$

18. Show that the lines

$$\frac{x-1}{-4} = \frac{y-2}{3} = \frac{z-4}{-2}$$

and

$$\frac{x-2}{-1} = \frac{y-1}{1} = \frac{z+2}{6}$$

intersect, and find the equation of the plane that they determine.

19. Find the equation of the plane containing the line $x = 1 + 2t$, $y = -1 + 3t$, $z = 4 + t$ and the point $(1,-1,5)$.

20. Find the equation of the plane containing the line $x = 3t$, $y = 1 + t$, $z = 2t$ and parallel to the intersection of the planes $2x - y + z = 0$ and $y + z + 1 = 0$.

21. Find the distance between the skew (nonintersecting and nonparallel) lines $x = 2 - t$, $y = 3 + 4t$, $z = 2t$ and $x = -1 + t$, $y = 2$, $z = -1 + 2t$ by using the following steps.

(a) Note by putting $t = 0$ that $(2,3,0)$ is on the first line.

(b) Find the equation of the plane π through $(2,3,0)$ parallel to both given lines (i.e., with normal perpendicular to both).

(c) Find a point Q on the second line.

(d) Find the distance from Q to the plane π. (See Example 6 of Section 14.2.)

See Problem 30 for another way to do this problem.

22. Find the distance between the skew lines $x = 1 + 2t$, $y = -3 + 4t$, $z = -1 - t$ and $x = 4 - 2t$, $y = 1 + 3t$, $z = 2t$ (see Problem 21).

23. Find the symmetric equations of the tangent line to the curve with equation

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 6\sin t\,\mathbf{j} + t\mathbf{k}$$

at $t = \pi/3$.

24. Find the parametric equations of the tangent line to the curve $x = 2t^2$, $y = 4t$, $z = t^3$ at $t = 1$.

25. Find the equation of the plane perpendicular to the curve $x = 3t$, $y = 2t^2$, $z = t^5$ at $t = -1$.

26. Find the equation of the plane perpendicular to the curve

$$\mathbf{r}(t) = t\sin t\,\mathbf{i} + 3t\mathbf{j} + 2t\cos t\,\mathbf{k}$$

at $t = \pi/2$.

27. Consider the curve $\mathbf{r}(t) = \sin t\cos t\,\mathbf{i} + \sin^2 t\,\mathbf{j} + \cos t\,\mathbf{k}$, $0 \le t \le 2\pi$.

(a) Show that the curve lies on a sphere centered at the origin.

(b) Where does the tangent line at $t = \pi/6$ intersect the xy-plane?

28. Point to Plane Let P be a point on a plane with normal $\mathbf{n}$ and Q be a point off the plane (Figure 7). Show that the distance d from Q to the plane is given by

$$d = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{|\mathbf{n}|}$$

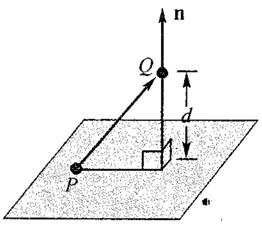
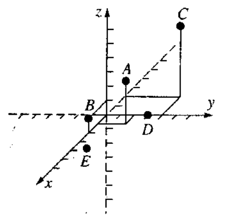


Figure 7

Problem Set 14.1

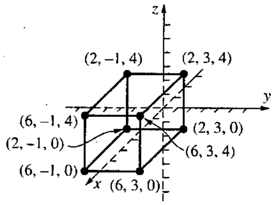
1. $A(1,2,3), B(2,0,1), C(-2,4,5), D(0,3,0), E(-1,-2,-3)$



3. $x = 0; x = 0, y = 0$

5. **(a)** $\sqrt{43}$; **(b)** 5; **(c)** $\sqrt{(e + \pi)^2 + (\pi + 4)^2 + 3}$

9.



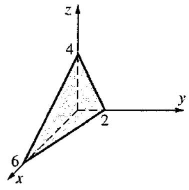
11. **(a)** $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 25$;

(b) $(x + 2)^2 + (y + 3)^2 + (z + 6)^2 = 5$;

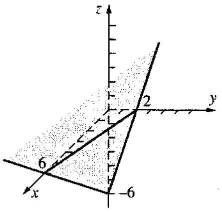
(c) $(x - \pi)^2 + (y - e)^2 + (z - \sqrt{2})^2 = \pi$

13. $(6, -7, 4); 10$ **15.** $(\frac{1}{2}, -1, -2); \sqrt{\frac{17}{2}}$

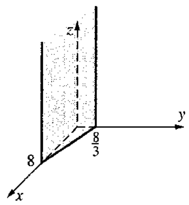
17.



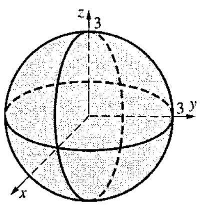
19.



21.



23.



25. $(x - 1)^2 + (y - 1)^2 + (z - \frac{11}{2})^2 = \frac{53}{4}$

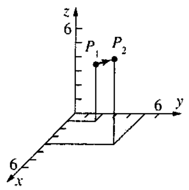
27. $(x - 6)^2 + (y - 6)^2 + (z - 6)^2 = 36$

29. **(a)** Plane parallel to and 2 units above the xy-plane;
(b) Plane perpendicular to the xy-plane, whose trace in the xy-plane is the line $x = y$;
(c) Union of the yz-plane ($x = 0$) and the xz-plane ($y = 0$);
(d) Union of the three coordinate planes;
(e) Cylinder of radius 2, parallel to the z-axis;
(f) Top half of the sphere with center $(0, 0, 0)$ and radius 3

33. $\frac{11\pi}{12}$

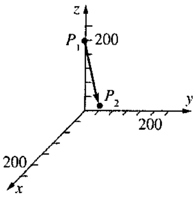
Problem Set 14.2

1. **(a)**



$3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

(b)



$-13\mathbf{i} + 55\mathbf{j} - 178\mathbf{k}$

3. **(a)** length $= \sqrt{21}$; $\cos\alpha = \frac{4}{\sqrt{21}}, \cos\beta = \frac{1}{\sqrt{21}}$, $\cos\gamma = \frac{2}{\sqrt{21}}$

(b) length $= \sqrt{62}$; $\cos\alpha = -\frac{2}{\sqrt{62}}, \cos\beta = -\frac{3}{\sqrt{62}}$, $\cos\gamma = \frac{7}{\sqrt{62}}$

5. $\left\langle \frac{3}{5\sqrt{2}}, -\frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle; \left\langle -\frac{3}{\sqrt{2}}, \frac{4}{\sqrt{2}}, -\frac{5}{\sqrt{2}} \right\rangle$

7. $\cos^{-1}\left(-\frac{2}{\sqrt{247}}\right)$

9. $\frac{10}{\sqrt{593}}\mathbf{i} - \frac{40}{\sqrt{593}}\mathbf{j} + \frac{240}{\sqrt{593}}\mathbf{k}$;
$-\frac{10}{\sqrt{593}}\mathbf{i} + \frac{40}{\sqrt{593}}\mathbf{j} - \frac{240}{\sqrt{593}}\mathbf{k}$

11. $\cos^{-1}\frac{11}{\sqrt{129}}$ **13.** $\sqrt{3}$

15. $\mathbf{m} = -\frac{48}{43}\mathbf{i} + \frac{80}{43}\mathbf{j} - \frac{48}{43}\mathbf{k}; \mathbf{n} = -\frac{81}{43}\mathbf{i} + \frac{6}{43}\mathbf{j} + \frac{91}{43}\mathbf{k}$

17. **(a)** $\alpha \approx 143.30°, \beta \approx 57.69°, \gamma \approx 74.50°$
(b) $\alpha \approx 63.75°, \beta \approx 27.79°, \gamma \approx 98.48°$

19. $2\sqrt{3}$ **23.** (a), (b), (f)

25. **(a)** $2x - 4y + 3z = -15$; **(b)** $3x - 2y - z = -4$

27. 56.91° **29.** **(a)** $z = 2$; **(b)** $2x - 3y - 4z = -13$

31. 0 **33.** $\frac{12}{\sqrt{38}}$ **37.** 35.26° **39.** 37.86°

41. 32 joules **43.** 15 joules **45.** $(3, 2, 1)$ **47.** 0

49. $\left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2}, \frac{a_3 + b_3}{2}\right); \frac{1}{2}|\mathbf{a} - \mathbf{b}|$ **51.** $(4, 2, 3)$

Problem Set 14.3

1. **(a)** $-4\mathbf{i} - 10\mathbf{j} - 4\mathbf{k}$; **(b)** $-6\mathbf{i} - 36\mathbf{j} - 27\mathbf{k}$; **(c)** 8;
(d) $-98\mathbf{i} - 59\mathbf{j} + 88\mathbf{k}$

3. $c(-14\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$, c in $\mathbb{R}$ **5.** $\pm\left\langle \frac{7}{\sqrt{86}}, -\frac{1}{\sqrt{86}}, \frac{6}{\sqrt{86}} \right\rangle$

7. $2\sqrt{74}$ **9.** $4\sqrt{6}$ **11.** $2x - y - z = -3$

13. $7x + 5y + 4z = -5$ **15.** $-x + 10y + 17z = -3$

17. 69 **19.** **(a)** 9; **(b)** $\sqrt{35}$; **(c)** 40.01° **21.** (c), (d)

29. $\frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}$

Problem Set 14.4

1. $x = 1 + 3t, y = -2 + 7t, z = 3 + 3t$

3. $x = 4 + t, y = 2, z = 3 - 2t$

5. $x = 4 + 3t, y = 5 + 2t, z = 6 + t; \dfrac{x-4}{3} = \dfrac{y-5}{2} = \dfrac{z-6}{1}$

7. $x = 1 + t, y = 1 + 10t, z = 1 + 100t;$

$$\frac{x-1}{1} = \frac{y-1}{10} = \frac{z-1}{100}$$

9. $\dfrac{x-4}{27} = \dfrac{y+5}{-50} = \dfrac{z}{-6}$ **11.** $\dfrac{x+8}{10} = \dfrac{y}{2} = \dfrac{z+\frac{21}{2}}{9}$

13. $\dfrac{x-4}{1} = \dfrac{y}{-5} = \dfrac{z-6}{2}$ **15.** $x = 5t, y = -3t, z = 4$

17. $x + y + 6z = 11$ **19.** $3x - 2y = 5$

21. **(b)** $2x + y - z = 7$; **(c)** $(-1, 2, -1)$; **(d)** $\sqrt{6}$

23. $\dfrac{x-1}{-\sqrt{3}} = \dfrac{y-3\sqrt{3}}{3} = \dfrac{z-\frac{\pi}{3}}{1}$ **25.** $3x - 4y + 5z = -22$

27. **(b)** $\left(\dfrac{3\sqrt{3}}{4}, \dfrac{7}{4}, 0\right)$ **29.** **(a)** $\dfrac{8\sqrt{2}}{3}$; **(b)** $\dfrac{3\sqrt{26}}{7}$

Section 14.5

1. $\mathbf{v}(1) = 4\mathbf{i} + 10\mathbf{j} + 2\mathbf{k}; \mathbf{a}(1) = 10\mathbf{j}; s(1) = 2\sqrt{30}$

3. $\mathbf{v}(2) = -\frac{1}{4}\mathbf{i} - \frac{4}{9}\mathbf{j} + 80\mathbf{k}; \mathbf{a}(2) = \frac{1}{4}\mathbf{i} + \frac{26}{27}\mathbf{j} + 160\mathbf{k};$

$s(2) = \dfrac{\sqrt{8{,}294{,}737}}{36}$

5. $\mathbf{v}(2) = 4\mathbf{j} + \dfrac{2^{2/3}}{3}\mathbf{k}; \mathbf{a}(2) = 4\mathbf{j} - \dfrac{1}{9\sqrt[3]{2}}\mathbf{k};$

$s(2) = \sqrt{16 + \dfrac{2^{4/3}}{9}}$

7. $\mathbf{v}(\pi) = -\mathbf{j} + \mathbf{k}; \mathbf{a}(\pi) = \mathbf{i}, s(\pi) = \sqrt{2}$

9. $\mathbf{v}(\frac{\pi}{4}) = 2\mathbf{i} + 3e^{\pi/4}\mathbf{j}; \mathbf{a}(\frac{\pi}{4}) = 4\mathbf{i} + 3e^{\pi/4}\mathbf{j} + 16\mathbf{k};$

$s(\frac{\pi}{4}) = \sqrt{4 + 9e^{\pi/2}}$

11. $\mathbf{v}(2) = 2\pi\mathbf{i} + \mathbf{j} - e^{-2}\mathbf{k}; \mathbf{a}(2) = 2\pi\mathbf{i} - 2\pi^2\mathbf{j} + e^{-2}\mathbf{k};$

$s(2) = \sqrt{4\pi^2 + 1 + e^{-4}}$

15. $2\sqrt{2}$ **17.** 144 **19.** $\sqrt{41}$ **21.** $\frac{3}{2}$

23. $\sqrt{2}\sinh\pi$

25. $\kappa = \dfrac{\sqrt{6}}{10\sqrt{5}}; \mathbf{T} = \dfrac{2}{\sqrt{5}}\mathbf{i} + \dfrac{1}{\sqrt{5}}\mathbf{j};$

$\mathbf{N} = \dfrac{1}{\sqrt{30}}\mathbf{i} - \dfrac{2}{\sqrt{30}}\mathbf{j} + \dfrac{5}{\sqrt{30}}\mathbf{k};$

$\mathbf{B} = \dfrac{1}{\sqrt{6}}\mathbf{i} - \dfrac{2}{\sqrt{6}}\mathbf{j} - \dfrac{1}{\sqrt{6}}\mathbf{k}$

27. $\dfrac{\sqrt{11}}{21\sqrt{7}}; \mathbf{T} = \dfrac{2}{\sqrt{21}}\mathbf{i} + \dfrac{1}{\sqrt{21}}\mathbf{j} + \dfrac{4}{\sqrt{21}}\mathbf{k};$

$\mathbf{N} = -\dfrac{5}{\sqrt{77}}\mathbf{i} - \dfrac{6}{\sqrt{77}}\mathbf{j} + \dfrac{4}{\sqrt{77}}\mathbf{k};$

$\mathbf{B} = \dfrac{4}{\sqrt{33}}\mathbf{i} - \dfrac{4}{\sqrt{33}}\mathbf{j} - \dfrac{1}{\sqrt{33}}\mathbf{k}$

29. $\kappa = \frac{9}{91}; \mathbf{T} = \left\langle -\dfrac{3}{\sqrt{13}}, 0, \dfrac{2}{\sqrt{13}} \right\rangle; \mathbf{N} = \langle 0, 1, 0 \rangle;$

$\mathbf{B} = \left\langle -\dfrac{2}{\sqrt{13}}, 0, -\dfrac{3}{\sqrt{13}} \right\rangle$

31. $\kappa = \frac{1}{3}\operatorname{sech}^2\frac{1}{3}; \mathbf{T} = \tanh\frac{1}{3}\mathbf{i} + \operatorname{sech}\frac{1}{3}\mathbf{j};$

$\mathbf{N} = \operatorname{sech}\frac{1}{3}\mathbf{i} - \tanh\frac{1}{3}\mathbf{j}; \mathbf{B} = -\mathbf{k}$

33. $\kappa = \dfrac{1}{2\sqrt{2}}; \mathbf{T} = -\dfrac{1}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j} + \dfrac{1}{\sqrt{2}}\mathbf{k};$

$\mathbf{N} = \dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}; \mathbf{B} = -\dfrac{1}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j} - \dfrac{1}{\sqrt{2}}\mathbf{k}$

35. $a_T(t) = \dfrac{4t}{\sqrt{10 + 4t^2}}; a_N(t) = 2\sqrt{\dfrac{5}{5 + 2t^2}}$

37. $a_T(t) = \dfrac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + 4 + e^{-2t}}}; a_N(t) = 2\sqrt{\dfrac{e^{2t} + 1 + e^{-2t}}{e^{2t} + 4 + e^{-2t}}}$

39. $a_T(t) = \dfrac{4t^3}{\sqrt{2t^4 + 3}}; a_N(t) = \dfrac{2\sqrt{6}t}{\sqrt{2t^4 + 3}}$

41. $a_T(t) = \dfrac{\tan t \sec^2 t - \cot t \csc^2 t}{\sqrt{1 + \cot^2 t + \tan^2 t}};$

$a_N(t) = \dfrac{\sqrt{\csc^4 t + 4\csc^2 t \sec^2 t + \sec^4 t}}{\sqrt{1 + \cot^2 t + \tan^2 t}}$

43. $\mathbf{T} = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle; \mathbf{N} = \langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \rangle; \mathbf{B} = \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle$

45. $\mathbf{T} = \dfrac{1}{\sqrt{\cosh^2\frac{\pi}{6c} + 1}}\left(\cosh\dfrac{\pi}{6c}\mathbf{i} + \mathbf{k}\right);$

$\mathbf{N} = \dfrac{1}{\sqrt{\cosh^2\frac{\pi}{6c} + 1}}\left(\mathbf{i} - \cosh\dfrac{\pi}{6c}\mathbf{k}\right);$

$\mathbf{B} = \mathbf{j}$

47. $\mathbf{T} = \dfrac{1}{\sqrt{1 + 5\pi^2}}(-\mathbf{i} - \pi\mathbf{j} + 2\pi\mathbf{k});$

$\mathbf{N} = \dfrac{[(5\pi^3 + 6\pi)\mathbf{i} + (-2 - 5\pi^2)\mathbf{j} + 2\mathbf{k}]}{\sqrt{(8 + 16\pi^2 + 5\pi^4)(1 + 5\pi^2)}};$

$\mathbf{B} = \dfrac{1}{\sqrt{8 + 16\pi^2 + 5\pi^4}}[2\pi\mathbf{i} + (2 + 2\pi^2)\mathbf{j} + (2 + \pi^2)\mathbf{k}]$

49. $\mathbf{T} = \dfrac{1}{\sqrt{3}}\mathbf{i} + \dfrac{1}{2}\left(\dfrac{1}{\sqrt{3}} - 1\right)\mathbf{j} + \dfrac{1}{2}\left(\dfrac{1}{\sqrt{3}} + 1\right)\mathbf{k};$

$\mathbf{N} = -\dfrac{1 + \sqrt{3}}{2\sqrt{2}}\mathbf{j} + \dfrac{1 - \sqrt{3}}{2\sqrt{2}}\mathbf{k};$

$\mathbf{B} = \sqrt{\frac{2}{3}}\mathbf{i} + \frac{1}{12}(3\sqrt{2} - \sqrt{6})\mathbf{j} - \frac{1}{12}(3\sqrt{2} + \sqrt{6})\mathbf{k}$

51. $\mathbf{T} = \dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\operatorname{sech}\dfrac{\pi}{3}\mathbf{j} + \dfrac{1}{\sqrt{2}}\tanh\dfrac{\pi}{3}\mathbf{k};$

$\mathbf{N} = -\tanh\frac{\pi}{3}\mathbf{j} + \operatorname{sech}\frac{\pi}{3}\mathbf{k};$

$\mathbf{B} = \dfrac{1}{\sqrt{2}}\mathbf{i} - \dfrac{1}{\sqrt{2}}\operatorname{sech}\dfrac{\pi}{3}\mathbf{j} - \dfrac{1}{\sqrt{2}}\tanh\dfrac{\pi}{3}\mathbf{k}$

53. **(b)** $R_p = 10\,R_m; t = \frac{\pi}{9}$

55. **(a)** Winding upward around the right circular cylinder $x = \sin t, y = \cos t$, as t increases.
(b) Same as part (a), but winding much faster by a factor of $3t^2$.