2.1 EXERCISES

In Exercises 1 and 2, find the indicated matrix if it is defined; if it is undefined, explain why. Let

$$A = \begin{bmatrix} 7 & 0 & -1 \\ -1 & 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 4 & 1 \\ 5 & -3 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 4 \\ -4 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

1. $-2A$, $B - 2A$, AC, CD

2. $A + B$, $3C - E$, CB, EB

3. Let $A = \begin{bmatrix} 4 & -1 \\ 3 & -2 \end{bmatrix}$. Compute $3I_2 - A$ and $(3I_2)A$.

4. Compute $A - I_3$ and $A - 2I_3$, when

$$A = \begin{bmatrix} 4 & -5 & 3 \\ 5 & 7 & -2 \\ -3 & 2 & -1 \end{bmatrix}$$

In Exercises 5 and 6, compute the product AB in two ways: (1) by the definition, where $A\mathbf{b}_1$ and $A\mathbf{b}_2$ are computed separately; and (2) by the row–column rule for computing a product.

5. $A = \begin{bmatrix} 1 & 3 \\ 0 & 4 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$

6. $A = \begin{bmatrix} 3 & 4 \\ 5 & 0 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$

7. If a matrix A is 3×5 and the product AB is 3×7, what is the size of B?

8. How many rows does B have if BA is a 2×6 matrix?

9. With C, D, and E as in Exercises 1 and 2, compute $(CD)E$ and $C(DE)$. Which product requires fewer multiplications?

10. Let $A = \begin{bmatrix} 3 & -4 \\ -5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 4 \\ 5 & k \end{bmatrix}$. What value(s) of k, if any, will make $AB = BA$?

11. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -8 \\ 2 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & 2 \\ 1 & -2 \end{bmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.

12. Let $A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}$. Find a 2×2 matrix B such that $AB = 0$. Use two different nonzero columns for B.

13. Let $\mathbf{r}_1, \dots, \mathbf{r}_p$ be vectors in $\mathbb{R}^n$, and let Q be an $m \times n$ matrix. Write the matrix $[\, Q\mathbf{r}_1 \ \cdots \ Q\mathbf{r}_p \,]$ as a *product* of two matrices (neither of which is an identity matrix).

14. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Compute AD and DA. Explain how the columns or rows of A change when A is multiplied by D on the right or on the left. Find a 3×3 diagonal matrix B (with $B \neq I$) such that $AB = BA$.

Exercises 15 and 16 concern arbitrary matrices A, B, and C for which the indicated sums and products are defined. Mark each statement True or False. Justify each answer.

15. a. If A and B are 2×2 with columns $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{b}_1, \mathbf{b}_2$, respectively, then $AB = [\, \mathbf{a}_1\mathbf{b}_1 \ \ \mathbf{a}_2\mathbf{b}_2 \,]$.
    b. Each column of AB is a linear combination of the columns of B using weights from the corresponding column of A.
    c. $AB + AC = A(B + C)$
    d. $A^T + B^T = (A + B)^T$
    e. The transpose of a product of matrices equals the product of their transposes.

16. a. If A and B are 3×3 and $B = [\, \mathbf{b}_1 \ \ \mathbf{b}_2 \ \ \mathbf{b}_3 \,]$, then $AB = [A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3]$.
    b. The second row of AB is the second row of A multiplied on the right by B.
    c. $(AB)C = (AC)B$
    d. $(AB)^T = A^T B^T$
    e. The transpose of a sum of matrices equals the sum of their transposes.

17. If $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ and $AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$, determine the first and second columns of B.

18. Suppose that the first two columns of B, $\mathbf{b}_1$ and $\mathbf{b}_2$, are equal. What can you say about the columns of AB (if AB is defined)? Why?

In Exercises 19–24, assume that the matrix product AB is defined.

19. Suppose that the third column of B is the sum of the first two columns. What can you say about the third column of AB? Why?

20. Suppose that the second column of B is all zeros. What can you say about the second column of AB?

21. Suppose that the last column of AB is entirely zero but B itself has no column of zeros. What can you say about the columns of A?

22. Show that if the columns of B are linearly dependent, then so are the columns of AB.

23. Suppose that $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. (This exercise and Exercise 24 will be cited in Section 2.3.)

24. Suppose that $AD = I_n$. Show that for any $\mathbf{y}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{y}$ has a solution. [*Hint:* Start with the fact that the equation $I_n\mathbf{u} = \mathbf{y}$ has a solution.]

In Exercises 25 and 26, view vectors in $\mathbb{R}^n$ as $n \times 1$ matrices. For $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^n$, the matrix product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix, called the **scalar product**, or **inner product**, of $\mathbf{u}$ and $\mathbf{v}$. It is usually written as a single real number without brackets. The matrix product $\mathbf{u}\mathbf{v}^T$ is an $n \times n$ matrix, called the **outer product** of $\mathbf{u}$ and $\mathbf{v}$. The products $\mathbf{u}^T\mathbf{v}$ and $\mathbf{u}\mathbf{v}^T$ will appear later in the text.

25. Let $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Compute $\mathbf{u}^T\mathbf{v}$, $\mathbf{v}^T\mathbf{u}$, $\mathbf{u}\mathbf{v}^T$, and $\mathbf{v}\mathbf{u}^T$.

26. If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^n$, how are $\mathbf{u}^T\mathbf{v}$ and $\mathbf{v}^T\mathbf{u}$ related? How are $\mathbf{u}\mathbf{v}^T$ and $\mathbf{v}\mathbf{u}^T$ related?

27. Prove Theorem 2(b) and 2(c). Use the row–column rule. The (i, j)-entry in $A(B + C)$ can be written as

$$a_{i1}(b_{1j} + c_{1j}) + \cdots + a_{in}(b_{nj} + c_{nj}) \quad \text{or} \quad \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})$$

CHAPTER 2

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1. $\begin{bmatrix} -14 & 0 & 2 \\ 2 & -10 & -4 \end{bmatrix}$, $\begin{bmatrix} -15 & 4 & 3 \\ 7 & -13 & -4 \end{bmatrix}$, not defined, $\begin{bmatrix} -7 & 4 \\ -4 & 0 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 1 \\ -3 & 5 \end{bmatrix}$, $\begin{bmatrix} 12 & -3 \\ 9 & -6 \end{bmatrix}$

5. $\begin{bmatrix} 5 & 0 \\ 4 & -4 \\ -3 & -7 \end{bmatrix}$

7. 5×7

9. $(CD)E$ takes 12 multiplications; $C(DE)$ takes 8. Both products equal $\begin{bmatrix} -61 \\ -28 \end{bmatrix}$.

11. $AB = AC = \begin{bmatrix} 7 & -2 \\ 21 & -6 \end{bmatrix}$

13. *Hint:* One of the two matrices is Q.

15. Answer the questions before checking the *Study Guide.*

17. $\mathbf{b}_1 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -8 \\ -5 \end{bmatrix}$

19. The third column of AB is the sum of the first two columns of AB. Here's why. Denote the first three columns of B by $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$. If $\mathbf{b}_3 = \mathbf{b}_1 + \mathbf{b}_2$, then the third column of AB is $A\mathbf{b}_3 = A\mathbf{b}_1 + A\mathbf{b}_2$, by a property of matrix-vector multiplication.

21. The columns of A are linearly dependent. Why?

23. *Hint:* Suppose that $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{0}$, and show that $\mathbf{x}$ must be zero.

25. $\mathbf{u}^T\mathbf{v} = -2a + 3b - 4c$,

$$\mathbf{v}^T\mathbf{u} = [a \quad b \quad c]\begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix} = -2a + 3b - 4c,$$

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}[a \quad b \quad c] = \begin{bmatrix} -2a & -2b & -2c \\ 3a & 3b & 3c \\ -4a & -4b & -4c \end{bmatrix}$$