# G SOME PROPERTIES OF MATRIX MULTIPLICATION

In the following exercise we should discover the properties of $2 \times 2$ matrix multiplication which are like those of ordinary number multiplication, and those which are not.

## EXERCISE 14G

1. For ordinary arithmetic $2 \times 3 = 3 \times 2$ and in algebra $ab = ba$.
   For matrices, is $\mathbf{AB} = \mathbf{BA}$ always?
   **Hint:** Try $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix}$ say.

2. If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ find $\mathbf{AO}$ and $\mathbf{OA}$.

3. For all real numbers $a$, $b$ and $c$ it is true that $a(b+c) = ab + ac$ and this is known as the **distributive law**.
   a. 'Make up' three $2 \times 2$ matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ and verify that $\mathbf{A(B + C) = AB + AC}$.
   b. Now let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$
      and prove that in general $\mathbf{A(B + C) = AB + AC}$.
   c. Use the matrices you 'made up' in a to verify that $\mathbf{(AB)C = A(BC)}$
   d. As in b prove that $\mathbf{(AB)C = A(BC)}$

4. a. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ i.e., $\mathbf{AX} = \mathbf{A}$, deduce that $w = z = 1$ and $x = y = 0$.
   b. For any real number $a$, it is true that $a \times 1 = 1 \times a = a$.
      Is there a matrix $\mathbf{I}$, say, such that $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all $2 \times 2$ matrices $\mathbf{A}$?
      [**Hint:** Use the results of a above.]

5. Suppose $\mathbf{A}^2 = \mathbf{AA}$, i.e., $\mathbf{A}$ multiplied by itself, and that $\mathbf{A}^3 = \mathbf{AAA}$.
   a. Find $\mathbf{A}^2$ if $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$
   b. Find $\mathbf{A}^3$ if $\mathbf{A} = \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}$.

6. a. If $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ try to find $\mathbf{A}^2$.
   b. When can $\mathbf{A}^2$ be found, i.e., under what conditions can we square a matrix?

7. Show that if $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $\mathbf{I}^2 = \mathbf{I}$ and $\mathbf{I}^3 = \mathbf{I}$.

$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the **identity matrix**.

You should have discovered from the above exercise that:

| Ordinary algebra | Matrix algebra |
|---|---|
| • If $a$ and $b$ are real numbers then so is $ab$. | • If $\mathbf{A}$ and $\mathbf{B}$ are matrices that can be multiplied then $\mathbf{AB}$ is also a matrix. {closure} |
| • $ab = ba$ for all $a$, $b$ | • In general $\mathbf{AB} \neq \mathbf{BA}$. {non-commutative} |
| • $a0 = 0a = 0$ for all $a$ | • If $\mathbf{O}$ is a zero matrix then $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$ for all $\mathbf{A}$. |
| • $a(b+c) = ab + ac$ | • $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ {distributive law} |
| • $a \times 1 = 1 \times a = a$ | • If $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all $2 \times 2$ matrices $\mathbf{A}$. {identity law} |
| • $a^n$ exists for all $a \geqslant 0$ | • $\mathbf{A}^n$ for $n \geqslant 2$ can be determined provided that $\mathbf{A}$ is a square and $n$ is an integer. |

**Example 7**

Expand and simplify where possible: **a** $(\mathbf{A}+2\mathbf{I})^2$ **b** $(\mathbf{A}-\mathbf{B})^2$

**a** $(\mathbf{A}+2\mathbf{I})^2$

$= (\mathbf{A}+2\mathbf{I})(\mathbf{A}+2\mathbf{I})$ {$\mathbf{X}^2 = \mathbf{XX}$ by definition}

$= (\mathbf{A}+2\mathbf{I})\mathbf{A} + (\mathbf{A}+2\mathbf{I})2\mathbf{I}$ {$\mathbf{B}(\mathbf{C}+\mathbf{D}) = \mathbf{BC}+\mathbf{BD}$}

$= \mathbf{A}^2 + 2\mathbf{IA} + 2\mathbf{AI} + 4\mathbf{I}^2$ {$\mathbf{B}(\mathbf{C}+\mathbf{D}) = \mathbf{BC}+\mathbf{BD}$ again, twice}

$= \mathbf{A}^2 + 2\mathbf{A} + 2\mathbf{A} + 4\mathbf{I}$ {$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ and $\mathbf{I}^2 = \mathbf{I}$}

$= \mathbf{A}^2 + 4\mathbf{A} + 4\mathbf{I}$

**b** $(\mathbf{A}-\mathbf{B})^2$

$= (\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B})$ {$\mathbf{X}^2 = \mathbf{XX}$ by definition}

$= (\mathbf{A}-\mathbf{B})\mathbf{A} - (\mathbf{A}-\mathbf{B})\mathbf{B}$ {$\mathbf{C}(\mathbf{D}-\mathbf{E}) = \mathbf{CD}-\mathbf{CE}$, three times}

$= \mathbf{A}^2 - \mathbf{BA} - \mathbf{AB} + \mathbf{B}^2$

**Note:** **b** cannot be simplified further as in general $\mathbf{AB} \neq \mathbf{BA}$.

**8** Given that all matrices are $2 \times 2$ and $\mathbf{I}$ is the identity matrix, explain and simplify:

**a** $\mathbf{A}(\mathbf{A}+\mathbf{I})$ **b** $(\mathbf{B}+2\mathbf{I})\mathbf{B}$ **c** $\mathbf{A}(\mathbf{A}^2 - 2\mathbf{A} + \mathbf{I})$

**d** $\mathbf{A}(\mathbf{A}^2 + \mathbf{A} - 2\mathbf{I})$ **e** $(\mathbf{A}+\mathbf{B})(\mathbf{C}+\mathbf{D})$ **f** $(\mathbf{A}+\mathbf{B})^2$

**g** $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})$ **h** $(\mathbf{A}+\mathbf{I})^2$ **i** $(3\mathbf{I}-\mathbf{B})^2$

**Example 8**

If $\mathbf{A}^2 = 2\mathbf{A} + 3\mathbf{I}$, find $\mathbf{A}^3$ and $\mathbf{A}^4$ in the form $k\mathbf{A} + l\mathbf{I}$, ($k$ and $l$ are scalars).

$\mathbf{A}^2 = 2\mathbf{A} + 3\mathbf{I}$, $\therefore$

$\mathbf{A}^3 = \mathbf{A} \times \mathbf{A}^2$
$= \mathbf{A}(2\mathbf{A}+3\mathbf{I})$
$= 2\mathbf{A}^2 + 3\mathbf{AI}$
$= 2(2\mathbf{A}+3\mathbf{I}) + 3\mathbf{AI}$
$= 7\mathbf{A} + 6\mathbf{I}$

and

$\mathbf{A}^4 = \mathbf{A} \times \mathbf{A}^3$
$= \mathbf{A}(7\mathbf{A}+6\mathbf{I})$
$= 7\mathbf{A}^2 + 6\mathbf{AI}$
$= 7(2\mathbf{A}+3\mathbf{I}) + 6\mathbf{A}$
$= 20\mathbf{A} + 21\mathbf{I}$

9 a If $\mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$, find $\mathbf{A}^3$ and $\mathbf{A}^4$ in linear form, $k\mathbf{A} + l\mathbf{I}$, ($k$ and $l$ are scalars).
b If $\mathbf{B}^2 = 2\mathbf{I} - \mathbf{B}$, find $\mathbf{B}^3$, $\mathbf{B}^4$ and $\mathbf{B}^5$ in linear form.
c If $\mathbf{C}^2 = 4\mathbf{C} - 3\mathbf{I}$, find $\mathbf{C}^3$ and $\mathbf{C}^5$ in linear form.

10 a If $\mathbf{A}^2 = \mathbf{I}$, simplify:
i $\mathbf{A}(\mathbf{A} + 2\mathbf{I})$ ii $(\mathbf{A} - \mathbf{I})^2$ iii $\mathbf{A}(\mathbf{A} + 3\mathbf{I})^2$
b If $\mathbf{A}^3 = \mathbf{I}$, simplify $\mathbf{A}^2(\mathbf{A} + \mathbf{I})^2$
c If $\mathbf{A}^2 = \mathbf{O}$, simplify:
i $\mathbf{A}(2\mathbf{A} - 3\mathbf{I})$ ii $\mathbf{A}(\mathbf{A} + 2\mathbf{I})(\mathbf{A} - \mathbf{I})$ iii $\mathbf{A}(\mathbf{A} + \mathbf{I})^3$

11 The result "if $ab = 0$ then $a = 0$ or $b = 0$" for real numbers does not have an equivalent result for matrices.
a If $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ find $\mathbf{AB}$.
This example provides us with evidence that
"If $\mathbf{AB} = \mathbf{O}$ then $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$" is a false statement.
b If $\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ determine $\mathbf{A}^2$.
c Comment on the following argument for a $2 \times 2$ matrix $\mathbf{A}$:
It is known that $\mathbf{A}^2 = \mathbf{A}$, $\therefore \mathbf{A}^2 - \mathbf{A} = \mathbf{O}$

$$\therefore \mathbf{A}(\mathbf{A} - \mathbf{I}) = \mathbf{O}$$
$$\therefore \mathbf{A} = \mathbf{O} \text{ or } \mathbf{A} - \mathbf{I} = \mathbf{O}$$
$$\therefore \mathbf{A} = \mathbf{O} \text{ or } \mathbf{I}$$

d Find **all** $2 \times 2$ matrices $\mathbf{A}$ for which $\mathbf{A}^2 = \mathbf{A}$. (**Hint:** Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.)

12 Give **one** example which shows that "if $\mathbf{A}^2 = \mathbf{O}$ then $\mathbf{A} = \mathbf{O}$" is a *false* statement.

**Example 9**

Find constants $a$ and $b$ such that $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$ for $\mathbf{A}$ equal to $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Since $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$, $\therefore \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = a\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{bmatrix} = \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$$

$$\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} = \begin{bmatrix} a+b & 2a \\ 3a & 4a+b \end{bmatrix}$$

Thus $a + b = 7$ and $2a = 10$
$\therefore a = 5$ and $b = 2$

Checking for consistency $3a = 3(5) = 15$ ✓ $4a + b = 4(5) + (2) = 22$ ✓

**13** Find constants $a$ and $b$ such that $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$ for $\mathbf{A}$ equal to:

**a** $\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}$

**14** If $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$, find constants $p$ and $q$ such that $\mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$.

**a** Hence, write $\mathbf{A}^3$ in linear form, $r\mathbf{A} + s\mathbf{I}$ where $r$ and $s$ are scalars.

**b** Also write $\mathbf{A}^4$ in linear form.

# H THE INVERSE OF A 2×2 MATRIX

We can solve $\begin{cases} 2x + 3y = 4 \\ 5x + 4y = 17 \end{cases}$ algebraically to get $x = 5$, $y = -2$.

Notice that this system can be written as a matrix equation $\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 17 \end{bmatrix}$.

The solution $x = 5$, $y = -2$ is easily checked as

$$\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 2(5) + 3(-2) \\ 5(5) + 4(-2) \end{bmatrix} = \begin{bmatrix} 4 \\ 17 \end{bmatrix} \quad \checkmark$$

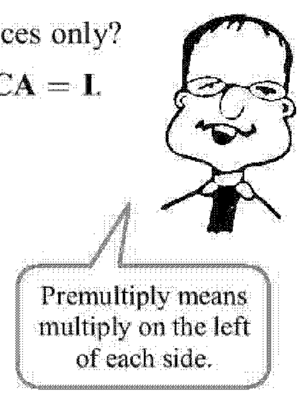
Notice that these matrix equations have form $\mathbf{AX} = \mathbf{B}$ where $\mathbf{A}$ is the matrix of coefficients, $\mathbf{X}$ is the unknown column matrix and $\mathbf{B}$ is the column matrix of constants.

The question arises: If $\mathbf{AX} = \mathbf{B}$, how can we find $\mathbf{X}$ using matrices only?

To answer this question, suppose there exists a matrix $\mathbf{C}$ such that $\mathbf{CA} = \mathbf{I}$.

If we *pre-multiply* each side of $\mathbf{AX} = \mathbf{B}$ by $\mathbf{C}$ we get

$$\mathbf{C}(\mathbf{AX}) = \mathbf{CB}$$
$$\therefore\ (\mathbf{CA})\mathbf{X} = \mathbf{CB}$$
$$\therefore\ \mathbf{IX} = \mathbf{CB}$$
$$\text{and so}\ \ \mathbf{X} = \mathbf{CB}$$




If it exists, we will call $\mathbf{C}$ such that $\mathbf{CA} = \mathbf{I}$, the **multiplication inverse** of $\mathbf{A}$ and we will use the notation $\mathbf{C} = \mathbf{A}^{-1}$.

In general, the **multiplication inverse of A**, if it exists, satisfies $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

Notice that $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$

Notice also that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = -2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2\mathbf{I}$

and that $\begin{bmatrix} 5 & 11 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -11 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 37 & 0 \\ 0 & 37 \end{bmatrix} = 37\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 37\mathbf{I}$

Notice that all answers are scalar multiples of $\mathbf{I}$.

## EXERCISE 14D

**1** **a** $3\mathbf{A}$ **b** $\mathbf{O}$ **c** $-\mathbf{C}$ **d** $\mathbf{O}$ **e** $2\mathbf{A}+2\mathbf{B}$
**f** $-\mathbf{A}-\mathbf{B}$ **g** $-2\mathbf{A}+\mathbf{C}$ **h** $4\mathbf{A}-\mathbf{B}$ **i** $3\mathbf{B}$

**2** **a** $\mathbf{X}=\mathbf{A}-\mathbf{B}$ **b** $\mathbf{X}=\mathbf{C}-\mathbf{B}$ **c** $\mathbf{X}=2\mathbf{C}-4\mathbf{B}$
**d** $\mathbf{X}=\frac{1}{2}\mathbf{A}$ **e** $\mathbf{X}=\frac{1}{3}\mathbf{B}$ **f** $\mathbf{X}=\mathbf{A}-\mathbf{B}$
**g** $\mathbf{X}=2\mathbf{C}$ **h** $\mathbf{X}=\frac{1}{2}\mathbf{B}-\mathbf{A}$ **i** $\mathbf{X}=\frac{1}{4}(\mathbf{A}-\mathbf{C})$

**3** **a** $\mathbf{X}=\begin{bmatrix}3 & 6\\ 9 & 18\end{bmatrix}$ **b** $\mathbf{X}=\begin{bmatrix}\frac{1}{2} & -\frac{1}{4}\\ \frac{3}{4} & \frac{5}{4}\end{bmatrix}$ **c** $\mathbf{X}=\begin{bmatrix}-1 & -6\\ 1 & -\frac{1}{2}\end{bmatrix}$

## EXERCISE 14E.1

**1** **a** $[11]$ **b** $[22]$ **c** $[16]$ **2** $\begin{bmatrix}w & x & y & z\end{bmatrix}\begin{bmatrix}\frac{1}{4}\\ \frac{1}{4}\\ \frac{1}{4}\\ \frac{1}{4}\end{bmatrix}$

**3** **a** $\mathbf{P}=\begin{bmatrix}27 & 35 & 39\end{bmatrix}$ $\mathbf{Q}=\begin{bmatrix}4\\ 3\\ 2\end{bmatrix}$

**b** total cost $=\begin{bmatrix}27 & 35 & 39\end{bmatrix}\begin{bmatrix}4\\ 3\\ 2\end{bmatrix}=\$291$

**4** **a** $\mathbf{P}=\begin{bmatrix}10 & 6 & 3 & 1\end{bmatrix}$ $\mathbf{N}=\begin{bmatrix}3\\ 2\\ 4\\ 2\end{bmatrix}$

**b** total points $=\begin{bmatrix}10 & 6 & 3 & 1\end{bmatrix}\begin{bmatrix}3\\ 2\\ 4\\ 2\end{bmatrix}=56$ points

## EXERCISE 14E.2

**1** Number of cols. in A does not equal no. of rows in B.

**2** **a** $m=n$ **b** $2\times 3$ **c** B has 3 columns, A has 2 rows

**3** **a** $\begin{bmatrix}28 & 29\end{bmatrix}$ **b** **i** $\begin{bmatrix}8\end{bmatrix}$ **ii** $\begin{bmatrix}2 & 0 & 3\\ 8 & 0 & 12\\ 4 & 0 & 6\end{bmatrix}$

**4** **a** $\begin{bmatrix}3 & 5 & 3\end{bmatrix}$ **b** $\begin{bmatrix}-2\\ 1\\ 1\end{bmatrix}$

**5** **a** $\mathbf{C}=\begin{bmatrix}12.5\\ 9.5\end{bmatrix}$ $\mathbf{N}=\begin{bmatrix}2375 & 5156\\ 2502 & 3612\end{bmatrix}$

**b** $\begin{bmatrix}78\,669.5\\ 65\,589\end{bmatrix}$ income from adult rides and children's rides **c** \$144 258.50

**6** **a** $\mathbf{R}=\begin{bmatrix}1 & 1\\ 1 & 2\\ 2 & 3\end{bmatrix}$ **b** $\mathbf{P}=\begin{bmatrix}7 & 3 & 19\\ 6 & 2 & 22\end{bmatrix}$ **c** $\begin{bmatrix}48 & 70\\ 52 & 76\end{bmatrix}$

**d** My costs at store A are \$48, my friend's costs at store B are \$76. **e** store A

## EXERCISE 14F

**1** **a** $\begin{bmatrix}16 & 18 & 15\\ 13 & 21 & 16\\ 10 & 22 & 24\end{bmatrix}$ **b** $\begin{bmatrix}10 & 6 & -7\\ 9 & 3 & 0\\ 4 & -4 & -10\end{bmatrix}$

**c** $\begin{bmatrix}22 & 0 & 132 & 176 & 198\\ 44 & 154 & 88 & 110 & 0\\ 176 & 44 & 88 & 88 & 132\end{bmatrix}$ **d** $\begin{bmatrix}115\\ 136\\ 46\\ 106\end{bmatrix}$

**2** **a** $\begin{bmatrix}3 & 3 & 2\end{bmatrix}$ **b** $\begin{bmatrix}125 & 150 & 140\\ 44 & 40 & 40\\ 75 & 80 & 65\end{bmatrix}$ **c** $\begin{bmatrix}657 & 730 & 670\end{bmatrix}$

**d** $\begin{bmatrix}369 & 420 & 385\end{bmatrix}$ **e** $\begin{bmatrix}657 & 730 & 670\\ 369 & 420 & 385\end{bmatrix}$

**3** \$224 660

**4** **a**
$$\begin{bmatrix}125 & 195 & 225\end{bmatrix}\times\begin{bmatrix}15 & 12 & 13 & 11 & 14 & 16 & 8\\ 4 & 3 & 6 & 2 & 0 & 4 & 7\\ 3 & 1 & 4 & 4 & 3 & 2 & 0\end{bmatrix}$$
$$-\begin{bmatrix}85 & 120 & 130\end{bmatrix}\times\begin{bmatrix}15 & 12 & 13 & 11 & 14 & 16 & 8\\ 4 & 3 & 6 & 2 & 0 & 4 & 7\\ 3 & 1 & 4 & 4 & 3 & 2 & 0\end{bmatrix}$$
$=\$7125$

**b**
$$\begin{bmatrix}125 & 195 & 225\end{bmatrix}\times\begin{bmatrix}15 & 12 & 13 & 11 & 14 & 16 & 8\\ 4 & 3 & 6 & 2 & 0 & 4 & 7\\ 3 & 1 & 4 & 4 & 3 & 2 & 0\end{bmatrix}$$
$$-\begin{bmatrix}85 & 120 & 130\end{bmatrix}\times\begin{bmatrix}20 & 20 & 20 & 20 & 20 & 20 & 20\\ 15 & 15 & 15 & 15 & 15 & 15 & 15\\ 5 & 5 & 5 & 5 & 5 & 5 & 5\end{bmatrix}$$
$=-\$9030$, i.e., a loss of \$9030

**c**
$$\left(\begin{bmatrix}125 & 195 & 225\end{bmatrix}-\begin{bmatrix}85 & 120 & 130\end{bmatrix}\right)\times\begin{bmatrix}15 & 12 & 13 & 11 & 14 & 16 & 8\\ 4 & 3 & 6 & 2 & 0 & 4 & 7\\ 3 & 1 & 4 & 4 & 3 & 2 & 0\end{bmatrix}$$

## EXERCISE 14G

**1** $\mathbf{AB}=\begin{bmatrix}-1 & 1\\ -1 & 7\end{bmatrix}$ $\mathbf{BA}=\begin{bmatrix}0 & 2\\ 3 & 6\end{bmatrix}$ $\mathbf{AB}\neq\mathbf{BA}$

**2** $\mathbf{AO}=\mathbf{OA}=\mathbf{O}$ **4** **b** $\mathbf{I}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

**5** **a** $\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}$ **b** $\begin{bmatrix}97 & -59\\ 118 & 38\end{bmatrix}$

**6** **a** $\mathbf{A}^2$ does not exist **b** when $\mathbf{A}$ is a square matrix

**8** **a** $\mathbf{A}^2+\mathbf{A}$ **b** $\mathbf{B}^2+2\mathbf{B}$ **c** $\mathbf{A}^3-2\mathbf{A}^2+\mathbf{A}$
**d** $\mathbf{A}^3+\mathbf{A}^2-2\mathbf{A}$ **e** $\mathbf{AC}+\mathbf{AD}+\mathbf{BC}+\mathbf{BD}$
**f** $\mathbf{A}^2+\mathbf{AB}+\mathbf{BA}+\mathbf{B}^2$ **g** $\mathbf{A}^2-\mathbf{AB}+\mathbf{BA}-\mathbf{B}^2$
**h** $\mathbf{A}^2+2\mathbf{A}+\mathbf{I}$ **i** $9\mathbf{I}-6\mathbf{B}+\mathbf{B}^2$

**9** **a** $\mathbf{A}^3=3\mathbf{A}-2\mathbf{I}$ $\mathbf{A}^4=4\mathbf{A}-3\mathbf{I}$
**b** $\mathbf{B}^3=3\mathbf{B}-2\mathbf{I}$ $\mathbf{B}^4=6\mathbf{I}-5\mathbf{B}$ $\mathbf{B}^5=11\mathbf{B}-10\mathbf{I}$
**c** $\mathbf{C}^3=13\mathbf{C}-12\mathbf{I}$ $\mathbf{C}^5=121\mathbf{C}-120\mathbf{I}$

**10** **a** **i** $\mathbf{I}+2\mathbf{A}$ **ii** $2\mathbf{I}-2\mathbf{A}$ **iii** $10\mathbf{A}+6\mathbf{I}$
**b** $\mathbf{A}^2+\mathbf{A}+2\mathbf{I}$ **c** **i** $-3\mathbf{A}$ **ii** $-2\mathbf{A}$ **iii** $\mathbf{A}$

**11** **a** $\mathbf{AB}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ **b** $\mathbf{A}^2=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$

**c** false as $\mathbf{A}(\mathbf{A}-\mathbf{I})=\mathbf{O}$ does not imply that $\mathbf{A}=\mathbf{O}$ or $\mathbf{A}-\mathbf{I}=\mathbf{O}$

**d** $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $\begin{bmatrix}a & b\\ \frac{a-a^2}{b} & 1-a\end{bmatrix}$, $b\neq 0$

**12** For example, $\mathbf{A}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, gives $\mathbf{A}^2=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

**13** **a** $a=3$, $b=-4$ **b** $a=1$, $b=8$

**14** $p=-2$, $q=1$ **a** $\mathbf{A}^3=5\mathbf{A}-2\mathbf{I}$ **b** $\mathbf{A}^4=-12\mathbf{A}+5\mathbf{I}$

## EXERCISE 14H

**1** **a** $\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}=3\mathbf{I}$, $\begin{bmatrix}1 & -2\\ -\frac{2}{3} & \frac{5}{3}\end{bmatrix}$

**b** $\begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix}=10\mathbf{I}$, $\begin{bmatrix}0.2 & 0.4\\ -0.1 & 0.3\end{bmatrix}$

**c** $\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}=2\mathbf{I}$, $\begin{bmatrix}-\frac{11}{2} & \frac{9}{2} & \frac{15}{2}\\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ 4 & -3 & -5\end{bmatrix}$