[08-09-11-Y11]

23. Let the weights at A, B, and C of Problem 21 be $3w$, $4w$, and $5w$, respectively. Determine the three angles at P at equilibrium. What geometric quantity (as in Problem 22) is now minimized?

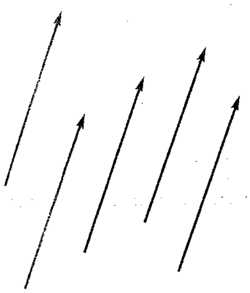
24. A company will build a plant to manufacture refrigerators to be sold in cities A, B, and C in quantities a, b, and c, respectively, each year. Where is the best location for the plant, that is, the location that will minimize delivery costs (see Problem 23)?

Answers to Concepts Review: **1.** magnitude; direction **2.** they have the same magnitude and direction **3.** the tail of **u**; the head of **v** **4.** force; velocity

---

13.3 Vectors in the Plane: Algebraic Approach

From the geometric perspective of the previous section, a vector can be described as a family of arrows all having the same length and direction (Figure 1). Our aim now is to place vectors in an algebraic context. Here is how we do it.

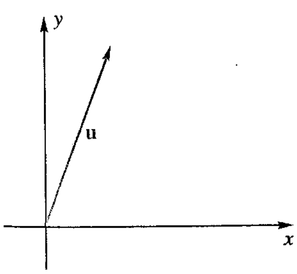


The vector **u**

Figure 1

We begin by imposing a Cartesian coordinate system on the plane. Then, for a given vector **u**, we pick as its representative the arrow that has its tail at the origin (Figure 2). This arrow is uniquely determined by the coordinates u_1 and u_2 of its head; that is, the vector **u** is completely described by the ordered pair $\langle u_1, u_2 \rangle$ (Figure 3). This being so, we consider $\langle u_1, u_2 \rangle$ to be the vector; it is the vector **u** in its algebraic clothes. Incidently, we use $\langle u_1, u_2 \rangle$ rather than (u_1, u_2) because the latter symbol already has two meanings—an open interval and a point in the plane.

But why do we offer this new interpretation? There are two good answers. First, for vectors, as we have tried to emphasize for other topics throughout this book, the interplay between their geometric and algebraic aspects greatly enriches and clarifies the subject. Second, it is the algebraic viewpoint that most easily generalizes to higher dimensions. This is because it is almost as easy to talk about an ordered n-tuple $\langle u_1, u_2, \ldots, u_n \rangle$ as it is an ordered pair $\langle u_1, u_2 \rangle$.



Pick a special representative

Figure 2

Operations on Vectors

The numbers u_1 and u_2 are called **components** of $\mathbf{u} = \langle u_1, u_2 \rangle$. Two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$. To add **u** and **v**, we add corresponding components; that is,

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$$

To multiply **u** by a scalar c, we multiply each component by c. Thus,

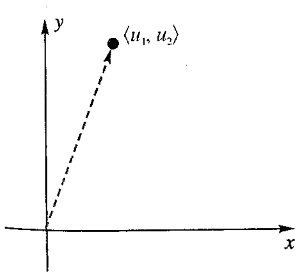
$$\mathbf{u}c = c\mathbf{u} = \langle cu_1, cu_2 \rangle$$

In particular,

$$-\mathbf{u} = \langle -u_1, -u_2 \rangle$$

and

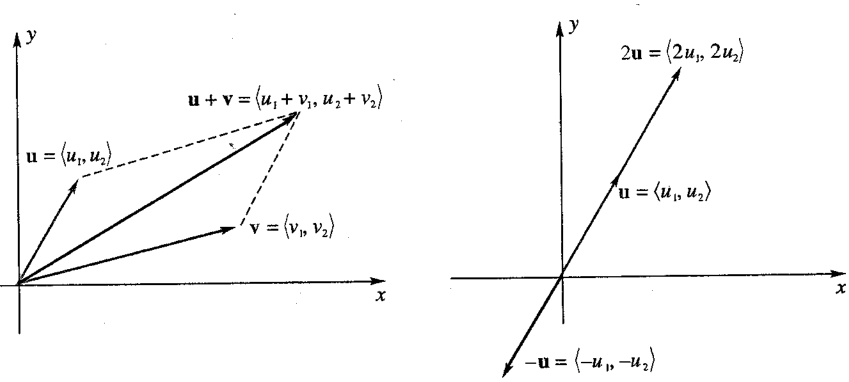
$$\mathbf{0} = 0\mathbf{u} = \langle 0, 0 \rangle$$



Identify **u** with the ordered pair $\langle u_1, u_2 \rangle$

Figure 3

Figure 4 shows that these definitions are equivalent to the earlier geometric ones.



Vector Addition

Scalar Multiplication

Figure 4

Using the algebraic interpretation of vectors, the following rules for operating with vectors are easily established.

Theorem A

For any vectors **u**, **v**, and **w** and any scalars a and b, the following relationships hold.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $a(b\mathbf{u}) = (ab)\mathbf{u} = \mathbf{u}(ab)$
6. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
7. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

Proof We illustrate the proof by demonstrating Rule 6. Be sure that you understand why each of the following steps is valid.

$$\begin{aligned} a(\mathbf{u} + \mathbf{v}) &= a(\langle u_1, u_2\rangle + \langle v_1, v_2\rangle) \\ &= a\langle u_1 + v_1, u_2 + v_2\rangle \\ &= \langle a(u_1 + v_1), a(u_2 + v_2)\rangle \\ &= \langle au_1 + av_1, au_2 + av_2\rangle \\ &= \langle au_1, au_2\rangle + \langle av_1, av_2\rangle \\ &= a\langle u_1, u_2\rangle + a\langle v_1, v_2\rangle \\ &= a\mathbf{u} + a\mathbf{v} \quad \blacklozenge \end{aligned}$$

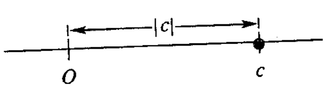


Figure 5

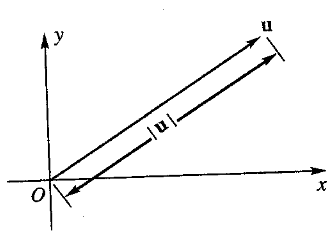


Figure 6

Length and the Dot Product The **length** (or **magnitude**) $|\mathbf{u}|$ of the vector $\mathbf{u} = \langle u_1, u_2\rangle$ is given by

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2}$$

For example, if $\mathbf{u} = \langle 4, -2\rangle$, then $|\mathbf{u}| = \sqrt{4^2 + (-2)^2} = 2\sqrt{5}$. If we multiply **u** by the scalar c, we multiply its length by $|c|$; that is,

$$|c\mathbf{u}| = |c|\,|\mathbf{u}|$$

Do not be confused by the apparent double usage of the symbol $|\ |$. The symbol $|c|$, called the *absolute value of* c, is the distance from the origin to c on the real line (Figure 5). The symbol $|\mathbf{u}|$, called the *length* of **u**, is the distance from the origin to the head of **u** in the plane (Figure 6).

EXAMPLE 1 Let $\mathbf{u} = \langle 4, -3\rangle$. Find $|\mathbf{u}|$ and $|-2\mathbf{u}|$. Also find a vector **v** with the same direction as **u**, but with length 1.

Solution $|\mathbf{u}| = \sqrt{4^2 + (-3)^2} = 5$ and $|-2\mathbf{u}| = |-2|\,|\mathbf{u}| = 2 \cdot 5 = 10$. To find **v**, simply divide **u** by its length $|\mathbf{u}|$; that is,

$$\mathbf{v} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\langle 4, -3\rangle}{5} = \frac{1}{5}\langle 4, -3\rangle = \left\langle \frac{4}{5}, \frac{-3}{5} \right\rangle$$

The length of **v** is then

$$|\mathbf{v}| = \left|\frac{\mathbf{u}}{|\mathbf{u}|}\right| = \frac{1}{|\mathbf{u}|}|\mathbf{u}| = 1 \quad \blacksquare$$

We have discussed scalar multiplication, that is, the multiplication of a vector **u** by a scalar c. The result $c\mathbf{u}$ is always a vector. Now we introduce a multiplication for two vectors **u** and **v**. It is called the **dot product** and is symbolized by $\mathbf{u} \cdot \mathbf{v}$. We define it by the formula

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$$

For example, $\langle 4, -3\rangle \cdot \langle 3, 2\rangle = 12 + (-6) = 6$. Note that the dot product of two vectors is a scalar.

The properties of the dot product are easy to establish; we state them without proof.

Theorem B

If **u**, **v**, and **w** are vectors and c is a scalar, then these properties hold.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
3. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$
4. $\mathbf{0} \cdot \mathbf{u} = 0$
5. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$

To understand the significance of the dot product, we offer an alternative formula for it. If **u** and **v** are nonzero vectors, then

$$\boxed{\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta}$$

where θ is the angle between **u** and **v**. By the **angle between u and v**, we mean the smallest nonnegative angle between **u** and **v**, so $0 \le \theta \le \pi$.

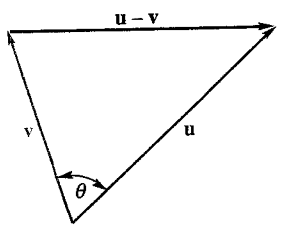


Figure 7

To derive this formula, apply the Law of Cosines to the triangle in Figure 7.

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}|\cos\theta$$

On the other hand, from the properties of the dot product stated in Theorem B,

$$\begin{aligned}|\mathbf{u} - \mathbf{v}|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot (\mathbf{u} - \mathbf{v}) - \mathbf{v} \cdot (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\mathbf{u} \cdot \mathbf{v}\end{aligned}$$

Equating these two expressions for $|\mathbf{u} - \mathbf{v}|^2$ gives the desired result.

An extremely important consequence of the formula just obtained is the following theorem.

Theorem C Perpendicularity Criterion

Two vectors **u** and **v** are perpendicular if and only if their dot product, $\mathbf{u} \cdot \mathbf{v}$, is 0.

Proof Two nonzero vectors are perpendicular if and only if the angle θ between them is $\pi/2$; that is, if and only if $\cos\theta = 0$. But $\cos\theta = 0$ if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. The result is valid for zero vectors, provided that we agree that a zero vector is perpendicular to every vector. ♦

Vectors that are perpendicular are said to be **orthogonal**.

EXAMPLE 2 Find b so that $\mathbf{u} = \langle 8, 6\rangle$ and $\mathbf{v} = \langle 3, b\rangle$ are orthogonal.

Solution

$$\mathbf{u} \cdot \mathbf{v} = (8)(3) + (6)(b) = 24 + 6b = 0$$

Thus, $b = -4$. ■

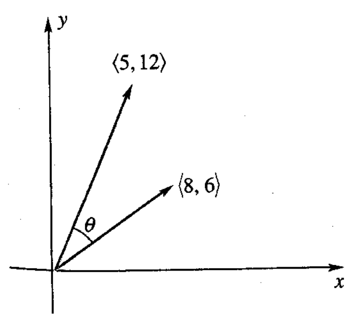


Figure 8

EXAMPLE 3 Find the angle between $\mathbf{u} = \langle 8, 6\rangle$ and $\mathbf{v} = \langle 5, 12\rangle$ (see Figure 8).

Solution

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{(8)(5) + (6)(12)}{(10)(13)} = \frac{112}{130} \approx 0.862$$

Then

$$\theta \approx \cos^{-1}(0.862) \approx 0.532 \text{ (or } 30.5^\circ)$$ ■

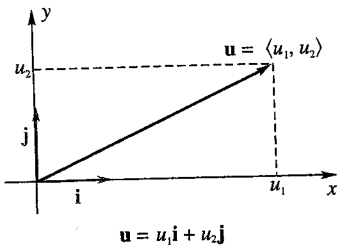


Figure 9

Basis Vectors Let $\mathbf{i} = \langle 1,0 \rangle$ and $\mathbf{j} = \langle 0,1 \rangle$, and note that these two vectors are perpendicular and of unit length. They are called **basis vectors**, because any vector $\mathbf{u} = \langle u_1, u_2 \rangle$ can be represented in a unique way in terms of $\mathbf{i}$ and $\mathbf{j}$. In fact,

$$\mathbf{u} = \langle u_1, u_2 \rangle = u_1\langle 1,0 \rangle + u_2\langle 0,1 \rangle = u_1\mathbf{i} + u_2\mathbf{j}$$

The geometric interpretation is shown in Figure 9.

EXAMPLE 4 If $\mathbf{u}$ is the arrow from the point $P(2,-1)$ to $Q(-3,7)$, write $\mathbf{u}$ in the form $u_1\mathbf{i} + u_2\mathbf{j}$.

Solution We first translate this arrow so that it emanates from the origin (Figure 10). This can always be accomplished by subtracting the components of the initial point from those of the terminal point. Thus, the corresponding algebraic vector is $\langle -3-2, 7-(-1) \rangle = \langle -5, 8 \rangle$. We conclude that

$$\mathbf{u} = -5\mathbf{i} + 8\mathbf{j}$$

■

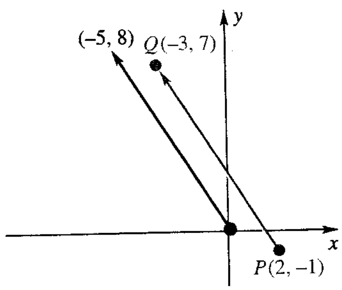


Figure 10

EXAMPLE 5 Find the measure of the angle ABC, where $A = (4,3)$, $B(1,-1)$, and $C(6,-4)$, as in Figure 11.

Solution

$$\mathbf{u} = \overrightarrow{BA} = (4-1)\mathbf{i} + (3+1)\mathbf{j} = 3\mathbf{i} + 4\mathbf{j} = \langle 3, 4 \rangle$$
$$\mathbf{v} = \overrightarrow{BC} = (6-1)\mathbf{i} + (-4+1)\mathbf{j} = 5\mathbf{i} - 3\mathbf{j} = \langle 5, -3 \rangle$$
$$|\mathbf{u}| = \sqrt{3^2 + 4^2} = 5$$
$$|\mathbf{v}| = \sqrt{5^2 + (-3)^2} = \sqrt{34}$$
$$\mathbf{u} \cdot \mathbf{v} = (3)(5) + (4)(-3) = 3$$
$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{3}{5\sqrt{34}} \approx 0.1029$$
$$\theta \approx 1.468 \text{ (about } 84.09°)$$

■

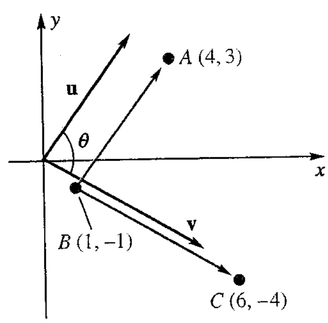


Figure 11

Projections Let $\mathbf{u}$ and $\mathbf{v}$ be vectors, and let θ be the angle between them. For now, we assume that $0 \le \theta \le \pi/2$. Let $\mathbf{w}$ be the vector in the direction of $\mathbf{v}$ that has magnitude $|\mathbf{u}|\cos\theta$ (see Figure 12). Since $\mathbf{w}$ has the same direction as $\mathbf{v}$, we know that $\mathbf{w} = c\mathbf{v}$ for some positive scalar c. On the other hand, the *magnitude* of $\mathbf{w}$ must be $|\mathbf{u}|\cos\theta$. Thus,

$$|\mathbf{u}|\cos\theta = |\mathbf{w}| = |c\mathbf{v}| = c|\mathbf{v}|$$

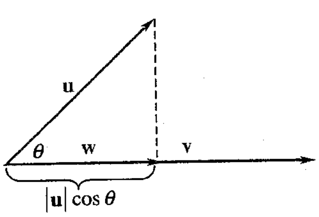


Figure 12

The constant c is therefore

$$c = \frac{|\mathbf{u}|}{|\mathbf{v}|}\cos\theta = \frac{|\mathbf{u}|}{|\mathbf{v}|}\,\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}$$

Thus,

$$\mathbf{w} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

For $\pi/2 < \theta \le \pi$, we define $\mathbf{w}$ to be the vector in the line determined by $\mathbf{v}$, but pointing in the direction opposite $\mathbf{v}$ (see Figure 13). The magnitude of this vector is $|\mathbf{w}| = -|\mathbf{u}|\cos\theta = c|\mathbf{v}|$ for some positive scalar c. Thus, $c = (-|\mathbf{u}|\cos\theta)/(|\mathbf{v}|) = -\mathbf{u} \cdot \mathbf{v}/|\mathbf{v}|^2$. Since $\mathbf{w}$ points in the direction *opposite* $\mathbf{v}$, we have $\mathbf{w} = -c\mathbf{v} = (\mathbf{u} \cdot \mathbf{v}/|\mathbf{v}|^2)\mathbf{v}$. Thus, in both cases we have $\mathbf{w} = (\mathbf{u} \cdot \mathbf{v}/|\mathbf{v}|^2)\mathbf{v}$. The vector $\mathbf{w}$ is called the **vector projection of u on v**, or sometimes just the **projection of u on v**, and is denoted $\text{pr}_\mathbf{v}\,\mathbf{u}$:

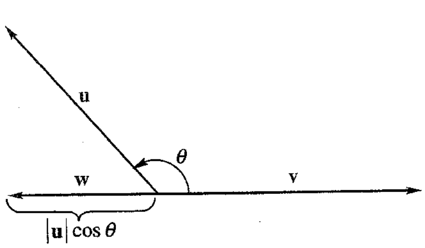


Figure 13

$$\text{pr}_\mathbf{v}\,\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

The **scalar projection of u on v** is defined to be $|\mathbf{u}|\cos\theta$. It is positive, zero, or negative, depending on whether θ is acute, right, or obtuse. When $0 \le \theta \le \pi/2$, the scalar projection is equal to the magnitude of $\text{pr}_\mathbf{v}\,\mathbf{u}$, and when $\pi/2 < \theta \le \pi$, the scalar projection is equal to the opposite of the magnitude of $\text{pr}_\mathbf{v}\,\mathbf{u}$.

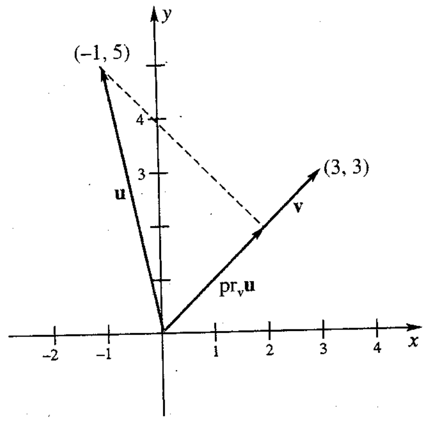


Figure 14

EXAMPLE 6 Let $\mathbf{u} = \langle -1, 5\rangle$ and $\mathbf{v} = \langle 3, 3\rangle$. Find the vector projection of **u** on **v** and the scalar projection of **u** on **v**.

Solution Figure 14 shows the two vectors. The vector projection is

$$\text{pr}_{\langle 3,3\rangle}\langle -1, 5\rangle = \left(\frac{\langle -1,5\rangle\cdot\langle 3,3\rangle}{|\langle 3,3\rangle|^2}\right)\langle 3,3\rangle = \frac{-3+15}{3^2+3^2}\langle 3,3\rangle = 2\mathbf{i} + 2\mathbf{j}$$

and the scalar projection is

$$|\mathbf{u}|\cos\theta = |\langle -1,5\rangle|\frac{\langle -1,5\rangle\cdot\langle 3,3\rangle}{|\langle -1,5\rangle|\,|\langle 3,3\rangle|} = \frac{-3+15}{\sqrt{3^2+3^2}} = 2\sqrt{2}$$ ■

The work done by a constant force **F** in moving an object along the line from P to Q is the magnitude of the force in the direction of the motion times the distance moved. Thus, if **D** is the vector from P to Q, the work done is

$$(\text{Scalar projection of } \mathbf{F} \text{ on } \mathbf{D})|\mathbf{D}| = |\mathbf{F}|\cos\theta|\mathbf{D}|$$

That is,

$$\boxed{\text{Work} = \mathbf{F}\cdot\mathbf{D}}$$

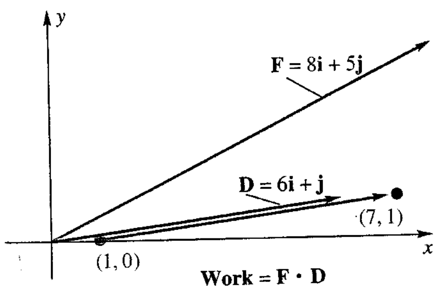


Figure 15

EXAMPLE 7 A force $\mathbf{F} = 8\mathbf{i} + 5\mathbf{j}$ in newtons moves an object from $(1, 0)$ to $(7, 1)$, where distance is measured in meters (Figure 15). How much work is done?

Solution Let **D** be the vector from $(1, 0)$ to $(7, 1)$; that is, let $\mathbf{D} = 6\mathbf{i} + \mathbf{j}$. Then,

$$\text{Work} = \mathbf{F}\cdot\mathbf{D} = (8)(6) + (5)(1) = 53 \text{ joules}$$ ■

Concepts Review

1. If $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ are vectors, then $\mathbf{u} + \mathbf{v} =$ ________, $c\mathbf{u} =$ ________, and $|\mathbf{u}| =$ ________.

2. The dot product of $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ is defined by $\mathbf{u}\cdot\mathbf{v} =$ ________. The corresponding geometric formula for $\mathbf{u}\cdot\mathbf{v}$ is ________.

3. The vectors $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$ are called ________ because any vector can be uniquely represented in the form $a\mathbf{i} + b\mathbf{j}$.

4. The work done by a force **F** in moving an object along the vector **D** is given by ________.

Problem Set 13.3

1. Let $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j}$, and $\mathbf{c} = -5\mathbf{j}$. Find each of the following:
(a) $2\mathbf{a} - 4\mathbf{b}$
(b) $\mathbf{a}\cdot\mathbf{b}$
(c) $\mathbf{a}\cdot(\mathbf{b} + \mathbf{c})$
(d) $(-2\mathbf{a} + 3\mathbf{b})\cdot 5\mathbf{c}$
(e) $|\mathbf{a}|\mathbf{c}\cdot\mathbf{a}$
(f) $\mathbf{b}\cdot\mathbf{b} - |\mathbf{b}|$

2. Let $\mathbf{a} = \langle 3, -1\rangle$, $\mathbf{b} = \langle 1, -1\rangle$, and $\mathbf{c} = \langle 0, 5\rangle$. Find each of the following:
(a) $-4\mathbf{a} + 3\mathbf{b}$
(b) $\mathbf{b}\cdot\mathbf{c}$
(c) $(\mathbf{a} + \mathbf{b})\cdot\mathbf{c}$
(d) $2\mathbf{c}\cdot(3\mathbf{a} + 4\mathbf{b})$
(e) $|\mathbf{b}|\mathbf{b}\cdot\mathbf{a}$
(f) $|\mathbf{c}|^2 - \mathbf{c}\cdot\mathbf{c}$

3. Find the cosine of the angle between **a** and **b** and make a sketch.
(a) $\mathbf{a} = \langle 1, -3\rangle$, $\mathbf{b} = \langle -1, 2\rangle$
(b) $\mathbf{a} = \langle -1, -2\rangle$, $\mathbf{b} = \langle 6, 0\rangle$
(c) $\mathbf{a} = \langle 2, -1\rangle$, $\mathbf{b} = \langle -2, -4\rangle$
(d) $\mathbf{a} = \langle 4, -7\rangle$, $\mathbf{b} = \langle -8, 10\rangle$

4. Find the angle between **a** and **b** and make a sketch.
(a) $\mathbf{a} = 12\mathbf{i}$, $\mathbf{b} = -5\mathbf{i}$
(b) $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = -8\mathbf{i} - 6\mathbf{j}$
(c) $\mathbf{a} = -\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} - 6\mathbf{j}$
(d) $\mathbf{a} = \sqrt{3}\mathbf{i} + \mathbf{j}$, $\mathbf{b} = 3\mathbf{i} + \sqrt{3}\mathbf{j}$

5. Write the vector represented by $\overrightarrow{AB}$ in the form $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$.

(a) $A(2,2), B(-3,4)$ (b) $A(0,4), B(-6,0)$

(c) $A(\sqrt{2},-\pi), B(0,0)$ (d) $A(-7,e), B(-4,\frac{1}{3})$

6. Find a unit vector $\mathbf{u}$ in the direction of $\mathbf{a}$ and express it in the form $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$.

(a) $\mathbf{a} = \langle -3,-4\rangle$ (b) $\mathbf{a} = \langle 1,-7\rangle$

(c) $\mathbf{a} = \langle 0,-4\rangle$ (d) $\mathbf{a} = \langle -5,-12\rangle$

7. If $\mathbf{u} + \mathbf{v}$ is perpendicular to $\mathbf{u} - \mathbf{v}$, what can you say about the relative magnitudes of $\mathbf{u}$ and $\mathbf{v}$?

8. Show that $(\mathbf{u} + \mathbf{v}) \cdot (3\mathbf{u} - \mathbf{v}) = 3|\mathbf{u}|^2 - |\mathbf{v}|^2 + 2\mathbf{u}\cdot\mathbf{v}$.

In Problems 9–12, give a proof of the indicated property. Use $\mathbf{u} = \langle u_1, u_2\rangle$, $\mathbf{v} = \langle v_1, v_2\rangle$, *and* $\mathbf{w} = \langle w_1, w_2\rangle$.

9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

10. $\mathbf{u}\cdot(\mathbf{v} + \mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$

11. $c(\mathbf{u}\cdot\mathbf{v}) = (c\mathbf{u})\cdot\mathbf{v}$

12. If $\mathbf{u} + \mathbf{v} = \mathbf{u}$, then $\mathbf{v} = \mathbf{0}$.

13. Find a vector that has the same direction as $6\mathbf{i} - 8\mathbf{j}$ and three times its length.

14. Find a vector that has the opposite direction to $-5\mathbf{i} + 12\mathbf{j}$ and unit length.

15. Show that the vectors $\langle 6,3\rangle$ and $\langle -1,2\rangle$ are orthogonal (perpendicular).

16. Show that the vectors $\langle -5,\sqrt{3}\rangle$ and $\langle \sqrt{27},15\rangle$ are orthogonal.

17. For what numbers c are $\langle c,6\rangle$ and $\langle c,-4\rangle$ orthogonal?

18. For what numbers c are $2c\mathbf{i} - 8\mathbf{j}$ and $3\mathbf{i} + c\mathbf{j}$ orthogonal?

19. Given $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = -3\mathbf{i} + 4\mathbf{j}$, two noncollinear vectors (i.e., vectors such that the angle θ between them satisfies $0 < \theta < \pi$), and another vector $\mathbf{r} = 7\mathbf{i} - 8\mathbf{j}$, find scalars k and m such that $\mathbf{r} = k\mathbf{a} + m\mathbf{b}$.

20. Given $\mathbf{a} = -4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$ (two noncollinear vectors) and another vector $\mathbf{r} = 6\mathbf{i} - 7\mathbf{j}$, find scalars k and m such that $\mathbf{r} = k\mathbf{a} + m\mathbf{b}$.

21. Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ be noncollinear vectors. If $\mathbf{r} = r_1\mathbf{i} + r_2\mathbf{j}$ is an arbitrarily chosen vector in the plane of $\mathbf{a}$ and $\mathbf{b}$, find scalars k and m such that $\mathbf{r} = k\mathbf{a} + m\mathbf{b}$.

22. Show that the vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$ is perpendicular to the line with equation $ax + by = c$. *Hint:* Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points on the line. Show that $\mathbf{n}\cdot\overrightarrow{P_1P_2} = 0$.

23. Find the work done by the force $\mathbf{F} = 3\mathbf{i} + 10\mathbf{j}$ newtons in moving an object 10 meters north (i.e., in the $\mathbf{j}$ direction).

24. Find the work done by a force of 100 newtons acting in the direction S 70° E in moving an object 30 meters east.

25. Find the work done by the force $\mathbf{F} = 6\mathbf{i} + 8\mathbf{j}$ pounds in moving an object from $(1,0)$ to $(6,8)$, where distance is in feet.

26. Find the work done by a force $\mathbf{F} = -5\mathbf{i} + 8\mathbf{j}$ newtons in moving an object 12 meters north.

27. Prove the **Cauchy–Schwarz Inequality**:

$$|\mathbf{u}\cdot\mathbf{v}| \le |\mathbf{u}|\,|\mathbf{v}|$$

28. Prove the **Triangle Inequality** (see Figure 16):

$$|\mathbf{u} + \mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$$

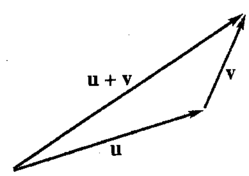


Figure 16

Hint:

$$|\mathbf{u} + \mathbf{v}|^2 = (\mathbf{u} + \mathbf{v})\cdot(\mathbf{u} + \mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}$$
$$= |\mathbf{u}|^2 + 2\mathbf{u}\cdot\mathbf{v} + |\mathbf{v}|^2$$

Now apply Problem 27.

29. What can you say about the nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in each case?

(a) $|\mathbf{u}|^2 + |\mathbf{v}|^2 = 2\mathbf{u}\cdot\mathbf{v}$ (b) $|\mathbf{u}|^2 = |\mathbf{v}|^2 = 2\mathbf{u}\cdot\mathbf{v}$

30. Show that the diagonals of a rhombus (parallelogram with equal sides) are perpendicular. *Hint:* Let $\mathbf{u}$ and $\mathbf{v}$ denote adjacent edges. Calculate $(\mathbf{u} + \mathbf{v})\cdot(\mathbf{u} - \mathbf{v})$.

31. Let A and B be the ends of the diameter of a circle and C be any other point on the circle. Use vector methods to show that AC is perpendicular to BC.

32. Prove that $\mathbf{w} = |\mathbf{v}|\mathbf{u} + |\mathbf{u}|\mathbf{v}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

33. Find $\text{pr}_{\mathbf{v}}\,\mathbf{u}$ for each of the following:

(a) $\mathbf{u} = \langle 0,5\rangle, \mathbf{v} = \langle 3,4\rangle$ (b) $\mathbf{u} = \langle -3,2\rangle, \mathbf{v} = \langle 3,4\rangle$

34. Find a simple expression for each of the following:

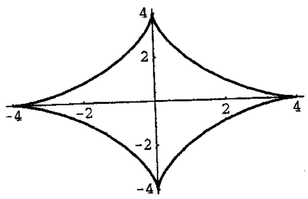
(a) $\text{pr}_{\mathbf{u}}\,\mathbf{u}$ (b) $\text{pr}_{-\mathbf{u}}\,\mathbf{u}$

(c) $\text{pr}_{\mathbf{u}}(-\mathbf{u})$ (d) $\text{pr}_{-\mathbf{u}}(-\mathbf{u})$

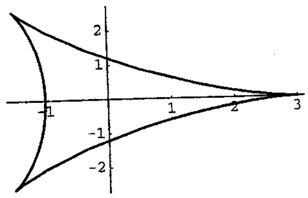
35. Derive the formula $d = |ax_0 + by_0 + c|/\sqrt{a^2 + b^2}$ for the distance from $P(x_0, y_0)$ to the line $ax + by + c = 0$. *Hint:* See Problem 22. Also let $Q(x_1, y_1)$ be a point on the line. Find the magnitude of the scalar projection of $\overrightarrow{QP}$ on $\mathbf{n}$.

Answers to Concepts Review: **1.** $\langle u_1 + v_1, u_2 + v_2\rangle$; $\langle cu_1, cu_2\rangle$; $\sqrt{u_1^2 + u_2^2}$ **2.** $u_1v_1 + u_2v_2$; $|\mathbf{u}|\,|\mathbf{v}|\cos\theta$ **3.** basis vectors **4.** $\mathbf{F}\cdot\mathbf{D}$

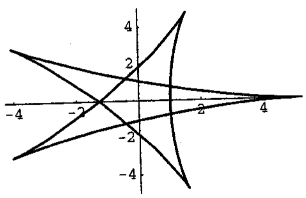
71. **(a)**



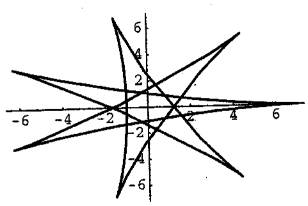
(b)



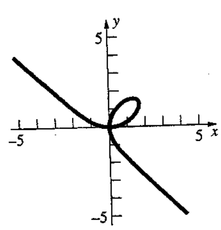
(c)



(d)

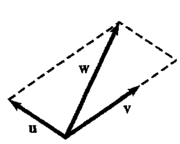


73. Quadrant I for $t > 0$, quadrant II for $-1 < t < 0$, quadrant III for no t, quadrant IV for $t < -1$.

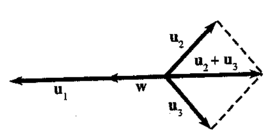


Problem Set 13.2

1.



3.



5. $\frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v}$ **7.** 1 **9.** $|\mathbf{w}| \approx 79.34$; S 7.5° W

11. 150 N **13.** N 2.08° E; 467 mi/h **15.** 80 mi/h

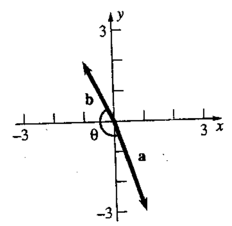
23. $\alpha + \beta = 143.13°, \beta + \gamma = 126.87°, \alpha + \gamma = 90°$

Problem Set 13.3

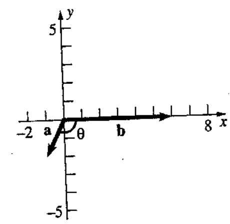
1. **(a)** $-12\mathbf{i} + 18\mathbf{j}$; **(b)** -13; **(c)** -28; **(d)** 375;

(e) $-15\sqrt{13}$ **(f)** $13 - \sqrt{13}$

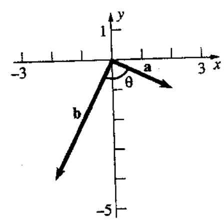
3. **(a)** $-\dfrac{7}{5\sqrt{2}}$



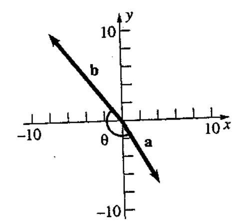
(b) $-\dfrac{1}{\sqrt{5}}$



(c) 0



(d) $-\dfrac{51}{\sqrt{2665}}$



5. **(a)** $-5\mathbf{i} + 2\mathbf{j}$; **(b)** $-6\mathbf{i} - 4\mathbf{j}$; **(c)** $-\sqrt{2}\mathbf{i} + \pi\mathbf{j}$;
(d) $3\mathbf{i} + (\frac{1}{3} - e)\mathbf{j}$

7. $|\mathbf{u}| = |\mathbf{v}|$ **13.** $18\mathbf{i} - 24\mathbf{j}$ **15.** $\langle 6,3\rangle \cdot \langle -1,2\rangle = 0$

17. $\pm 2\sqrt{6}$ **19.** $k = \frac{2}{3}, m = -\frac{5}{3}$

21. $k = \dfrac{r_1 b_2 - r_2 b_1}{a_1 b_2 - a_2 b_1}, m = \dfrac{a_1 r_2 - a_2 r_1}{a_1 b_2 - a_2 b_1}$ **23.** 100 Joules

25. 94 ft-lb **29.** **(a)** $\mathbf{u} = \mathbf{v}$; **(b)** $\theta = 60°$

33. **(a)** $\langle \frac{12}{5}, \frac{16}{5} \rangle$; **(b)** $\langle -\frac{3}{25}, -\frac{4}{25} \rangle$

Problem Set 13.4

1. $2\mathbf{i} - \mathbf{j}$ **3.** $\frac{1}{2}\mathbf{i} - 4\mathbf{j}$ **5.** $\mathbf{i}$ **7.** Does not exist

9. **(a)** $\{t \in \mathbb{R}: t \le 3\}$; **(b)** $\{t \in \mathbb{R}: t \le 20\}$

11. **(a)** $\{t \in \mathbb{R}: t < 3\}$; **(b)** $\{t \in \mathbb{R}: t < 20, t^2 \text{ not an integer}\}$

13. **(a)** $9(3t + 4)^2\mathbf{i} + 2te^{t^2}\mathbf{j}$; $54(3t + 4)\mathbf{i} + 2(2t^2 + 1)e^{t^2}\mathbf{j}$;

(b) $\sin 2t\,\mathbf{i} - 3\sin 3t\,\mathbf{j}$; $2\cos 2t\,\mathbf{i} - 9\cos 3t\,\mathbf{j}$

15. $2e^{-2t} + \dfrac{4}{t^2} - \dfrac{2}{t^2}\ln(t^2)$

17. $-\dfrac{e^{-3t}}{2}\left(\dfrac{6t - 7}{\sqrt{t - 1}}\right)\mathbf{i} + e^{-3t}\ln(2t^2)$

19. $-6t\sin(3t^2 - 4)\mathbf{i} + 18te^{9t^2-12}\mathbf{j}$

21. $(e - 1)\mathbf{i} + (1 - e^{-1})\mathbf{j}$

23. $\mathbf{v} = -e^{-1}\mathbf{i} + e\mathbf{j}$; $\mathbf{a} = e^{-1}\mathbf{i} + e\mathbf{j}$; $|\mathbf{v}| = \sqrt{e^{-2} + e^2}$