Chapter 2

Limits of Functions

Our purpose in this chapter is to study the limit concept as applied to functions. The notion of 'limit as x tends to a of a function f' is fundamental to all further ideas in real analysis. We shall develop it here and then use it to define 'continuous function' and to obtain the elementary properties of continuous functions.

In the last chapter, we began the study of the limit concept as applied to sequences that are special types of functions having for a domain the set of all natural numbers, **N**. In this chapter, we apply the concept to functions having as domains, arbitrary subsets of the set **R** of real numbers.

2.1 Functions, Limits at Infinity

The concept of function is central to the rest of this book, and so we discuss it at some length. As can be seen from the treatment in the appendix, functions are collections of ordered pairs, where these collections satisfy certain constraints. Implicit in the definition is the notion of an **ordered pair**. These are defined in the appendix, but the property that is important, in fact essential, is that *ordered pairs are equal if and only if they are equal coordinatewise*, namely,

$$(a, b) = (c, d) \text{ .if and only if. } a = c \text{ and } b = d.$$

Here, a is called the **first coordinate** and b the **second coordinate**. We use this property to make the following definition:

Definition. A **function** is a set f of ordered pairs such that if x is the first coordinate of an ordered pair in f, then there is exactly one y such that $(x, y) \in f$.

Discussion. The trouble with this definition is it is too pat. The subtleties are hidden, and the motivation for the definition has long since disappeared. Analysis grew out of graphs and rules for computations and geometry, not out of rigorous discussions of ordered pairs. And so we should look to these for intuition (graphs, rules for computation, geometry). Let us examine these in turn.

Functions, as treated in the last century, were essentially rules for computation. By this we mean for any given number that was a proper input, one could obtain a unique output by applying the rule. When thinking about functions in this way, we are really identifying the function with a computing machine. For a concrete example of this idea, the reader should think of a calculator that will compute the square root of nonnegative numbers. Obviously, not every number is suitable for computation purposes by all machines. This gives rise to the idea of **domain.** (The domain of the 'root x' machine is the nonnegative reals.) Also, we notice that not all numbers will appear as outputs, which gives rise to the idea of **range.** (The range of 'root x' is the nonnegative reals.) The main features of this discussion are that the rule is **single-valued** (that is, it produces a unique output for each accepted input), has a domain, and has a range. The trouble with the machine concept is that it is not suitable for mathematical analysis. The important fact is that with each element of the domain we associate a unique element of the range. This can be captured by considering sets of ordered pairs, and the machine in the middle can be dispensed with. The notion of 'rule' has not disappeared, however. We often specify functions by giving the rule for computation, together with the domain; for example,

$$f(x) = 2x + 5, \; x \in \mathbf{R}.$$

In set notation, this type of specification for functions takes the form

$$\{(x, 2x + 5) : x \in \mathbf{R}\}.$$

Either type of specification gives us immediate access to an algebraic rule for computation. It is through judicious manipulation of this rule that we will be able to establish the interesting properties of a particular function. For concrete examples of this, the reader should review our computations with particular sequences. Manipulation of the rule was the method by which limits were established. The student should also note that regardless of the notation, a function f is a set of ordered pairs and that '$f(x)$' does not denote a function, but a particular output of the function f, called the **value of the function at** x.

We shall consider only ordered pairs of real numbers, so our functions are real valued functions of a real variable. We shall frequently use the arrow notation, namely, $f : D \to \mathbf{R}$, where the domain is the subset D of $\mathbf{R}$ and the range is contained in $\mathbf{R}$. We shall use Dmn f and Rng f throughout the book to denote the domain and the range of a function f. □

Graphs provide us with pictures of functions. Pictures, in turn, lead naturally to questions about geometry. As such, pictures can be important aids to our intuition, and we should use them freely. To see how a graph can lead us to natural geometric questions, look at the graph of $f(x) = 2x$, $x \in \mathbf{R}$. The graph of this function (Figure 2.1.1) can be drawn by a single motion of the pencil across the page. The result is a continuous line, and mathematicians began to wonder what analytic properties of functions would lead to graphs that can be produced with a single stroke of a pen.

Graphs can also be misleading. For example, the graph of

$$f(x) = \begin{cases} x & x \in \mathbf{Q} \\ 1 - x & x \notin \mathbf{Q} \end{cases}$$

pictured in Figure 2.1.2 does not appear to represent a function because the rule appears to associate more than one y with a single value of x. However, examination of the rule shows that in fact there is a single unique y associated with each x, and so the rule is indeed single-valued, even though the picture cannot make it seem so.

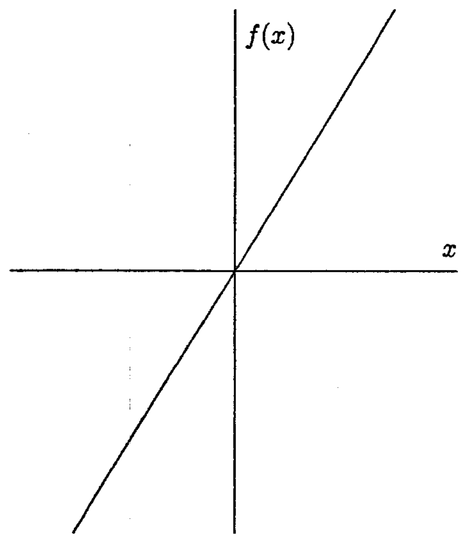

Figure 2.1.1 A graph of $f(x) = 2x$, $x \in \mathbf{R}$, showing that it can be drawn without lifting the pencil from paper.

In summary, we will consistently emphasize functions from the point of view of a domain, D, and a rule $f(x) = \cdots$, $x \in D$. We will draw graphs whenever these will aid our intuition, but we will always test our intuition by using the algebraic and analytic tools we have developed. Once again, we state that pictures are essential to the generation of intuition, but they cannot form a part of an analytic proof!

Unless explicitly stated otherwise, *all functions in this book will have both their domain and range contained in* $\mathbf{R}$. With respect to domains, the reader should keep in mind the standard domains of the elementary calculus, namely, open intervals (a, b), closed intervals $[a, b]$, and infinite intervals (a, ∞) or $(-\infty, a]$ to illustrate the various possibilities. It is essential that the reader first understand the content of our theorems and definitions for functions having intervals for their domains, before attempting to deal with the subtleties that arise for functions having domains that are arbitrary subsets of the real numbers. For this reason, functions having interval domains will be emphasized and many of the subtleties left to the exercises.

Definition. Let f be a function with domain D. We say that f **has a limit as x tends to ∞** provided there exists an $A \in \mathbf{R}$ such that for every positive ϵ,

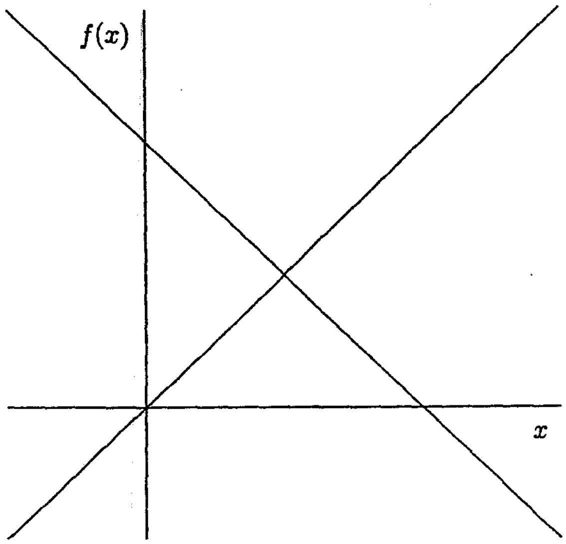

Figure 2.1.2 An intuitive graph of
$f(x) = \begin{cases} x, & x \in \mathbf{Q} \\ 1-x, & x \in \mathbf{R} \sim \mathbf{Q}. \end{cases}$ Note: Both lines are part of the graph and both lines have infinitely many 'holes'.

there is an $M \in \mathbf{R}$ such that

$$x \in D \text{ and } x > M \text{ .implies. } |f(x) - A| < \epsilon.$$

In the case that a number A satisfying the definition exists, we say that A is the **limit** of $f(x)$ as $x \to \infty$, and we write

$$\lim_{x\to\infty} f(x) = A.$$

Notation. We will often use the symbol $+\infty$, instead of ∞, to emphasize we are looking at positive infinity, as opposed to negative infinity, which is denoted by $-\infty$.

Discussion. Let us compare this definition with that of the limit definition for sequences. Both require us to find a fixed real number A as a first step. Both use small values of ϵ as a test for closeness. Both require us to find an M, which depends on ϵ, such that the functional values (sequence values) are within a distance ϵ of A provided that x (or n, in the case of sequences) is at least as large as M. So what is the difference? Only that functions have a domain D that is an arbitrary subset of the real numbers, while for sequences the domain is the set of all natural numbers. In dealing with a sequence, we know that there are natural numbers—elements of the domain—that exceed any choice of M. For a function having an arbitrary domain, this is not necessarily the case; indeed, $(M, \infty) \cap D$ may very well be empty for some choices of M. Thus, we may expect that the differences between this definition of limit at $+\infty$ and the corresponding definition for sequences will arise for functions whose domain is bounded above. Such a difference is given in Exercise 3.