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Elementary Linear Algebra

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1.2 Gaussian Elimination and Gauss-Jordan Elimination

In Section 1.1, Gaussian elimination was introduced as a procedure for solving a system of linear equations. In this section you will study this procedure more thoroughly, beginning with some definitions. The first is the definition of a **matrix.**

Definition of a Matrix

If m and n are positive integers, then an $m \times n$ **matrix** is a rectangular array

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \left.\vphantom{\begin{bmatrix} a \\ a \\ a \\ \vdots \\ a \end{bmatrix}}\right\} m \text{ rows}$$

in which each **entry,** a_{ij}, of the matrix is a number. An $m \times n$ matrix (read "m by n") has m **rows** (horizontal lines) and n **columns** (vertical lines).

REMARK: The plural of matrix is *matrices*. If each entry of a matrix is a *real* number, then the matrix is called a **real matrix.** Unless stated otherwise, all matrices in this text are assumed to be real matrices.

The entry a_{ij} is located in the ith row and the jth column. The index i is called the **row subscript** because it gives the position in the horizontal lines, and the index j is called the **column subscript** because it gives the position in the vertical lines.

A matrix with m rows and n columns (an $m \times n$ matrix) is said to be of **size** $m \times n$. If $m = n$, the matrix is called **square** of **order** n. For a square matrix, the entries a_{11}, a_{22}, $a_{33}, \ldots$ are called the **main diagonal** entries.

EXAMPLE 1 *Examples of Matrices*

The following matrices have the indicated sizes.

(a) Size: 1×1

$$[2]$$

(b) Size: 2×2

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(c) Size: 1×4

$$\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$$

(d) Size: 3×2

$$\begin{bmatrix} e & \pi \\ 2 & \sqrt{2} \\ -7 & 4 \end{bmatrix}$$

One very common use of matrices is to represent a system of linear equations. The matrix derived from the coefficients and constant terms of a system of linear equations is called the **augmented matrix** of the system. The matrix containing only the coefficients of the system is called the **coefficient matrix** of the system. Here is an example.

System

$$\begin{aligned} x - 4y + 3z &= 5 \\ -x + 3y - z &= -3 \\ 2x - 4z &= 6 \end{aligned}$$

Augmented Matrix

$$\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{bmatrix}$$

Coefficient Matrix

$$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$$

REMARK: Note the use of 0 to indicate that the coefficient of y in the third equation is zero. Also note the fourth column of constant terms in the augmented matrix.

When forming either the coefficient matrix or the augmented matrix of a system, you should begin by aligning the variables in the equations vertically.

Given System

$$\begin{aligned} x_1 + 3x_2 &= 9 \\ -x_2 + 4x_3 &= -2 \\ x_1 - 5x_3 &= 0 \end{aligned}$$

Line Up Variables

$$\begin{aligned} x_1 + 3x_2 &= 9 \\ -x_2 + 4x_3 &= -2 \\ x_1 - 5x_3 &= 0 \end{aligned}$$

Form Augmented Matrix

$$\begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & -1 & 4 & -2 \\ 1 & 0 & -5 & 0 \end{bmatrix}$$

Elementary Row Operations

In the previous section you studied three operations that can be used on a system of linear equations to produce equivalent systems.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

In matrix terminology these three operations correspond to **elementary row operations.** An elementary row operation on an augmented matrix corresponding to a given system of linear equations produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are said to be **row-equivalent** if one can be obtained from the other by a finite sequence of elementary row operations.

Elementary Row Operations

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

Although elementary row operations are simple to perform, they involve a lot of arithmetic. Because it is easy to make a mistake, you should get in the habit of noting the elementary row operation performed in each step so that you can go back to check your work.

Because solving some systems involves several steps, it is helpful to use a shorthand method of notation to keep track of each elementary row operation you perform. This notation is introduced in the next example.

EXAMPLE 2 *Elementary Row Operations*

(a) Interchange the first and second rows.

$$\begin{array}{ccc} \textit{Original Matrix} & \textit{New Row Equivalent Matrix} & \textit{Notation} \\ \begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix} & \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix} & \boldsymbol{R_1 \leftrightarrow R_2} \end{array}$$

(b) Multiply the first row by $\frac{1}{2}$ to produce a new first row.

$$\begin{array}{ccc} \textit{Original Matrix} & \textit{New Row Equivalent Matrix} & \textit{Notation} \\ \begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix} & \boldsymbol{(\tfrac{1}{2})R_1 \rightarrow R_1} \end{array}$$

(c) Add -2 times the first row to the third row to produce a new third row.

$$\begin{array}{ccc} \textit{Original Matrix} & \textit{New Row Equivalent Matrix} & \textit{Notation} \\ \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix} & \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix} & \boldsymbol{R_3 + (-2)R_1 \rightarrow R_3} \end{array}$$

syntax reversed for TI 86

REMARK: Notice in Example 2(c) that adding -2 times row 1 to row 3 does not change row 1.

Technology Note

Many graphing calculators and computer software programs can perform elementary row operations on matrices. If you are using a *TI-86*, your screens for Example 2(c) will look like the following:

```
A
      [[1 2 -4 3 ]
       [0 3 -2 -1]
       [2 1 5  -2]]
```

```
mRAdd(-2,A,1,3)
      [[1 2  -4 3 ]
       [0 3  -2 -1]
       [0 -3 13 -8]]
```

In Example 7 in Section 1.1, you used Gaussian elimination with back-substitution to solve a system of linear equations. You will now learn the matrix version of Gaussian elimination. The two methods used in the following example are essentially the same. The basic difference is that with the matrix method you do not need to keep writing the variables.

EXAMPLE 3 *Using Elementary Row Operations to Solve a System*

Linear System	*Associated Augmented Matrix*	
$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \quad &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$	
Add the first equation to the second equation.	Add the first row to the second row to produce a new second row.	
$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix}$	$R_2 + R_1 \rightarrow R_2$
Add -2 times the first equation to the third equation.	Add -2 times the first row to the third row to produce a new third row.	
$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix}$	$R_3 + (-2)R_1 \rightarrow R_3$
Add the second equation to the third equation.	Add the second row to the third row to produce a new third row.	
$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$	$R_3 + R_2 \rightarrow R_3$

Multiply the third equation by $\frac{1}{2}$.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

Multiply the third row by $\frac{1}{2}$ to produce a new third row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad \left(\tfrac{1}{2}\right)R_3 \rightarrow R_3$$

Here you can use back-substitution to find the solution, as in Example 6 in Section 1.1.

The last matrix in Example 3 is said to be in **row-echelon form.** The term *echelon* refers to the stair-step pattern formed by the nonzero elements of the matrix. To be in this form, a matrix must have the following properties.

Definition of Row-Echelon Form of a Matrix

A matrix in **row-echelon form** has the following properties.

1. All rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

REMARK: A matrix in row-echelon form is in **reduced row-echelon form** if every column that has a leading 1 has zeros in every position above and below its leading 1.

EXAMPLE 4 *Row-Echelon Form*

The following matrices are in row-echelon form.

(a) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The matrices shown in parts (b) and (d) are in *reduced* row-echelon form. The following matrices are not in row-echelon form.

(e) $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

It can be shown that every matrix is row-equivalent to a matrix in row-echelon form. For instance, in Example 4 you could change the matrix in part (e) to row-echelon form by multiplying the second row in the matrix by $\frac{1}{2}$.

The method of using Gaussian elimination with back-substitution to solve a system is as follows.

Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the augmented matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

Gaussian elimination with back-substitution works well as an algorithmic method for solving systems of linear equations with a computer. For this algorithm, the order in which the elementary row operations are performed is important. Move from *left to right by columns*, changing all entries directly below the leading 1's to zeros.

look carefully

EXAMPLE 5 *Gaussian Elimination with Back-Substitution*

Solve the following system.

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 \phantom{{}- 2x_4} &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

Solution The augmented matrix for this system is

$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}.$$

need to [illegible] calculator now

Obtain a leading 1 in the upper left corner and zeros elsewhere in the first column.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

← **The first two rows are interchanged.** $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

← **Adding -2 times the first row to the third row produces a new third row.** $R_3 + (-2)R_1 \rightarrow R_3$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{bmatrix}$$ ← **Adding −1 times the first row to the fourth row produces a new fourth row.** $R_4 + (-1)R_1 \rightarrow R_4$

Now that the first column is in the desired form, you should change the second column as follows.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -13 & -39 \end{bmatrix}$$ ← **Adding 6 times the second row to the fourth row produces a new fourth row.** $R_4 + (6)R_2 \rightarrow R_4$

To write the third column in proper form, multiply the third row by $\frac{1}{3}$.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -13 & -39 \end{bmatrix}$$ ← **Multiplying the third row by $\frac{1}{3}$ produces a new third row.** $(\frac{1}{3})R_3 \rightarrow R_3$

Similarly, to write the fourth column in proper form, you should multiply the fourth row by $-\frac{1}{13}$.

why does calc give a different ref? lots of decimals

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$ ← **Multiplying the fourth row by $-\frac{1}{13}$ produces a new fourth row.** $(-\frac{1}{13})R_4 \rightarrow R_4$

The matrix is now in row-echelon form, and the corresponding system of linear equations is as follows.

$$\begin{aligned} x_1 + 2x_2 - x_3 \qquad &= 2 \\ x_2 + x_3 - 2x_4 &= -3 \\ x_3 - x_4 &= -2 \\ x_4 &= 3 \end{aligned}$$

Using back-substitution, you can determine that the solution is

$$x_1 = -1, \quad x_2 = 2, \quad x_3 = 1, \quad x_4 = 3.$$

Technology Note

Some graphing calculators and computer software programs (for example, the *TI-86*, *HP-38*, Maple, and MATLAB) will produce the reduced row-echelon form of a matrix for you. If your tool has this capability, verify that the reduced row-echelon form of the matrix in part (f) of Example 4 is

$$\begin{bmatrix} 1 & 0 & -5 & 10 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

When solving a system of linear equations, remember that it is possible for the system to have no solution. If, in the elimination process, you obtain a row with zeros except for the last entry, it is unnecessary to continue the elimination process. You can simply conclude that the system is inconsistent and therefore has no solution.

EXAMPLE 6 *A System with No Solution*

Solve the following system.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_1 + x_3 &= 6 \\ 2x_1 - 3x_2 + 5x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

Solution The augmented matrix for this system is

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix}.$$

Apply Gaussian elimination to the augmented matrix as follows.

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \qquad R_2 + (-1)R_1 \rightarrow R_2$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \qquad R_3 + (-2)R_1 \rightarrow R_3$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -4 \\ 0 & 5 & -7 & -11 \end{bmatrix} \qquad R_4 + (-3)R_1 \rightarrow R_4$$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 5 & -7 & -11 \end{bmatrix} \qquad R_3 + R_2 \rightarrow R_3$$

Note that the third row of this matrix consists of zeros except for the last entry. This means that the original system of linear equations is *inconsistent*. You can see why this is true by converting back to a system of linear equations.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_2 - x_3 &= 2 \\ 0 &= -2 \\ 5x_2 - 7x_3 &= -11 \end{aligned}$$

Because the third "equation" is a false statement, it follows that the system has no solution.

Gauss-Jordan Elimination

With Gaussian elimination, you apply elementary row operations to a matrix to obtain a (row-equivalent) row-echelon form. A second method of elimination, called **Gauss-Jordan elimination** after Carl Gauss and Wilhelm Jordan (1842–1899), continues the reduction process until a *reduced* row-echelon form is obtained. This procedure is demonstrated in the following example.

EXAMPLE 7 *Gauss-Jordan Elimination*

Use Gauss-Jordan elimination to solve the system

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17. \end{aligned}$$

Solution In Example 3, Gaussian elimination was used to obtain the following row-echelon form.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Now, rather than using back-substitution, apply elementary row operations until you obtain a matrix in reduced row-echelon form. To do this, you must produce zeros above each of the leading 1's, as follows.

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_1 + (2)R_2 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_2 + (-3)R_3 \to R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad R_1 + (-9)R_3 \to R_1$$

Now, converting back to a system of linear equations, you have

$$\begin{aligned} x \qquad\qquad &= 1 \\ y \qquad &= -1 \\ z &= 2. \end{aligned}$$

The Gaussian and Gauss-Jordan elimination procedures employ an algorithmic approach that is easily adapted to computer use. However, these elimination procedures make no effort to avoid fractional coefficients. For instance, if the system in Example 7 had been listed as

$$\begin{aligned} 2x - 5y + 5z &= 17 \\ x - 2y + 3z &= 9 \\ -x + 3y \qquad &= -4 \end{aligned}$$

both procedures would have required multiplying the first row by $\frac{1}{2}$, which would have introduced fractions in the first row. For hand computations, fractions can sometimes be avoided by judiciously choosing the order in which elementary row operations are applied. Moreover, it can be shown that no matter which order you use, the reduced row-echelon form will be the same.

The next example demonstrates how Gauss-Jordan elimination can be used to solve a system with an infinite number of solutions.

EXAMPLE 8 *A System with an Infinite Number of Solutions*

Solve the following system of linear equations.

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 \qquad &= 1 \end{aligned}$$

Solution The augmented matrix of the system of linear equations is $\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix}$.

Using a graphing calculator, a computer software program, or Gauss-Jordan elimination, you can verify that the reduced row-echelon form of the matrix is

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}.$$

The corresponding system of equations is

$$\begin{aligned} x_1 \qquad + 5x_3 &= 2 \\ x_2 - 3x_3 &= -1. \end{aligned}$$

Now, using the parameter t to represent the *nonleading* variable x_3, you have

$$x_1 = 2 - 5t, \qquad x_2 = -1 + 3t, \qquad x_3 = t$$

where t is any real number.

REMARK: Note that in Example 8 an arbitrary parameter was assigned to the nonleading variable x_3. Subsequently you solved for the leading variables x_1 and x_2 as functions of t.

You have looked now at two elimination methods for solving a system of linear equations. Which is better? To some degree the answer depends on personal preference. In real-life applications of linear algebra, systems of linear equations are usually solved by computer. Most computer programs use a form of Gaussian elimination, with special emphasis on ways to reduce rounding errors and minimize storage of data. Because the examples and exercises in this text are generally much simpler and focus on the underlying concepts, you will need to know both methods.

Homogeneous Systems of Linear Equations

As the final topic in this section, you will look at systems of linear equations in which each of the constant terms is zero. We call such systems **homogeneous.** For example, a homogeneous system of m equations in n variables has the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= 0\\
&\ \vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0.
\end{aligned}$$

It is easy to see that a homogeneous system must have at least one solution. Specifically, if all variables in a homogeneous system have the value zero, then each of the equations must be satisfied. Such a solution is called **trivial** (or **obvious**). For instance, a homogeneous system of three equations in the three variables x_1, x_2, and x_3 must have $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ as a trivial solution.

EXAMPLE 9 *Solving a Homogeneous System of Linear Equations*

Solve the following system of linear equations.

$$\begin{aligned}
x_1 - x_2 + 3x_3 &= 0\\
2x_1 + x_2 + 3x_3 &= 0
\end{aligned}$$

Solution Applying Gauss-Jordan elimination to the augmented matrix

$$\begin{bmatrix} 1 & -1 & 3 & 0\\ 2 & 1 & 3 & 0 \end{bmatrix}$$

on calc, ref of this gives
1 .5 1.5 0
0 1 -1 0

yields the following.

$$\begin{bmatrix} 1 & -1 & 3 & 0\\ 0 & 3 & -3 & 0 \end{bmatrix} \qquad R_2 + (-2)R_1 \rightarrow R_2$$

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \quad (\tfrac{1}{3})R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \quad R_1 + R_2 \rightarrow R_1$$

The system of equations corresponding to this matrix is

$$\begin{aligned} x_1 \quad\quad + 2x_3 &= 0 \\ x_2 - \quad x_3 &= 0. \end{aligned}$$

Using the parameter $t = x_3$, the solution set is given by

$$x_1 = -2t, \quad x_2 = t, \quad x_3 = t, \quad t \text{ is any real number.}$$

Therefore this system of equations has an infinite number of solutions, one of which is the trivial solution (given by $t = 0$).

Example 9 illustrates an important point about homogeneous systems of linear equations. You began with two equations in three variables and discovered that the system has an infinite number of solutions. In general, a homogeneous system with fewer equations than variables has an infinite number of solutions.

Theorem 1.1 *The Number of Solutions of a Homogeneous System*

Every homogeneous system of linear equations is consistent. Moreover, if the system has fewer equations than variables, then it must have an infinite number of solutions.

Section 1.2 Exercises

In Exercises 1–6, determine the size of the given matrix.

1. $\begin{bmatrix} 1 & 2 & -4 \\ 3 & -4 & 6 \\ 0 & 1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 2 & -1 & -1 & 1 \\ -6 & 2 & 0 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 3 & 4 & -10 \end{bmatrix}$

4. $\begin{bmatrix} -1 \end{bmatrix}$

5. $\begin{bmatrix} 8 & 6 & 4 & 1 & 3 \\ 2 & 1 & -7 & 4 & 1 \\ 1 & 1 & -1 & 2 & 1 \\ 1 & -1 & 2 & 0 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$

In Exercises 7–12, determine whether the given matrix is in row-echelon form. If it is, determine whether it is also in reduced row-echelon form.

7. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 0 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 13–18, find the solution set of the system of linear equations represented by the given augmented matrix.

13. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

15. $\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

In Exercises 19–32, solve the given system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination.

19. $\begin{aligned} x + 2y &= 7 \\ 2x + y &= 8 \end{aligned}$

20. $\begin{aligned} 2x + 6y &= 16 \\ -2x - 6y &= -16 \end{aligned}$

21. $\begin{aligned} -x + 2y &= 1.5 \\ 2x - 4y &= 3 \end{aligned}$

22. $\begin{aligned} 2x - y &= -0.1 \\ 3x + 2y &= 1.6 \end{aligned}$

23. $\begin{aligned} -3x + 5y &= -22 \\ 3x + 4y &= 4 \\ 4x - 8y &= 32 \end{aligned}$

24. $\begin{aligned} x + 2y &= 0 \\ x + y &= 6 \\ 3x - 2y &= 8 \end{aligned}$

25. $\begin{aligned} x_1 \qquad\quad - 3x_3 &= -2 \\ 3x_1 + x_2 - 2x_3 &= 5 \\ 2x_1 + 2x_2 + x_3 &= 4 \end{aligned}$

26. $\begin{aligned} 2x_1 - x_2 + 3x_3 &= 24 \\ 2x_2 - x_3 &= 14 \\ 7x_1 - 5x_2 \qquad &= 6 \end{aligned}$

27. $\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 \qquad - 2x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$

28. $\begin{aligned} 2x_1 \qquad + 3x_3 &= 3 \\ 4x_1 - 3x_2 + 7x_3 &= 5 \\ 8x_1 - 9x_2 + 15x_3 &= 10 \end{aligned}$

29. $\begin{aligned} 4x + 12y - 7z - 20w &= 22 \\ 3x + 9y - 5z - 28w &= 30 \end{aligned}$

30. $\begin{aligned} x + 2y + z &= 8 \\ -3x - 6y - 3z &= -21 \end{aligned}$

31. $\begin{aligned} 3x + 3y + 12z &= 6 \\ x + y + 4z &= 2 \\ 2x + 5y + 20z &= 10 \\ -x + 2y + 8z &= 4 \end{aligned}$

32. $\begin{aligned} 2x + y - z + 2w &= -6 \\ 3x + 4y \qquad + w &= 1 \\ x + 5y + 2z + 6w &= -3 \\ 5x + 2y - z - w &= 3 \end{aligned}$

C In Exercises 33–36, use a computer or graphing calculator to solve the given system of linear equations.

33. $\begin{aligned} x_1 - 2x_2 + 5x_3 - 3x_4 &= 23.6 \\ x_1 + 4x_2 - 7x_3 - 2x_4 &= 45.7 \\ 3x_1 - 5x_2 + 7x_3 + 4x_4 &= 29.9 \end{aligned}$

34. $\begin{aligned} 23.4x - 45.8y + 43.7z &= 87.2 \\ 86.4x + 12.3y - 56.9z &= 14.5 \\ 93.6x - 50.7y + 12.6z &= 44.4 \end{aligned}$

35. $\begin{aligned} x_1 - x_2 + 2x_3 + 2x_4 + 6x_5 &= 6 \\ 3x_1 - 2x_2 + 4x_3 + 4x_4 + 12x_5 &= 14 \\ x_2 - x_3 - x_4 - 3x_5 &= -3 \\ 2x_1 - 2x_2 + 4x_3 + 5x_4 + 15x_5 &= 10 \\ 2x_1 - 2x_2 + 4x_3 + 4x_4 + 13x_5 &= 13 \end{aligned}$

36. $\begin{aligned} x_1 + x_2 - 2x_3 + 3x_4 + 2x_5 &= 9 \\ 3x_1 + 3x_2 - x_3 + x_4 + x_5 &= 5 \\ 2x_1 + 2x_2 - x_3 + x_4 - 2x_5 &= 1 \\ 4x_1 + 4x_2 + x_3 \qquad - 3x_5 &= 4 \\ 8x_1 + 5x_2 - 2x_3 - x_4 + 2x_5 &= 3 \end{aligned}$

In Exercises 37–40, solve the homogeneous linear system corresponding to the given coefficient matrix.

37. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

38. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

39. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

40. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

41. Consider the matrix $A = \begin{bmatrix} 1 & k & 2 \\ -3 & 4 & 1 \end{bmatrix}$.

(a) If A is the *augmented* matrix of a system of linear equations, determine the number of equations and unknowns.

(b) If A is the *augmented* matrix of a system of linear equations, find the value(s) of k such that the system is consistent.

(c) If A is the *coefficient* matrix of a *homogeneous* system of linear equations, determine the number of equations and unknowns.

(d) If A is the *coefficient* matrix of a *homogeneous* system of linear equations, find the value(s) of k such that the system is consistent.

42. Consider the matrix $A = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 2 & k \\ 4 & -2 & 6 \end{bmatrix}$.

(a) If A is the *augmented* matrix of a system of linear equations, determine the number of equations and unknowns.

(b) If A is the *augmented* matrix of a system of linear equations, find the value(s) of k such that the system is consistent.

(c) If A is the *coefficient* matrix of a *homogeneous* system of linear equations, determine the number of equations and unknowns.

(d) If A is the *coefficient* matrix of a *homogeneous* system of linear equations, find the value(s) of k such that the system is consistent.

In Exercises 43 and 44, find values of a, b, and c (if possible) such that the given system of linear equations has (a) a unique solution, (b) no solution, and (c) an infinite number of solutions.

43.
$$\begin{aligned} x + y \phantom{{}+ z} &= 2 \\ y + z &= 2 \\ x \phantom{{}+y} + z &= 2 \\ ax + by + cz &= 0 \end{aligned}$$

44.
$$\begin{aligned} x + y \phantom{{}+ z} &= 0 \\ y + z &= 0 \\ x \phantom{{}+y} + z &= 0 \\ ax + by + cz &= 0 \end{aligned}$$

45. The following system has one solution: $x = 1$, $y = -1$, and $z = 2$.

$$\begin{aligned} 4x - 2y + 5z &= 16 && \textbf{Equation 1} \\ x + y \phantom{{}+5z} &= 0 && \textbf{Equation 2} \\ -x - 3y + 2z &= 6 && \textbf{Equation 3} \end{aligned}$$

Solve the systems given by (a) Equations 1 and 2, (b) Equations 1 and 3, and (c) Equations 2 and 3. (d) How many solutions does each of these systems have?

46. Assume that the following system has a unique solution.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 && \textbf{Equation 1} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 && \textbf{Equation 2} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 && \textbf{Equation 3} \end{aligned}$$

Does the system composed of Equations 1 and 2 have (a) a unique solution, (b) no solution, or (c) an infinite number of solutions?

In Exercises 47 and 48, find the unique reduced row-echelon matrix that is row-equivalent to the given matrix.

47. $\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$

48. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

W **49.** Describe all possible 2×2 reduced row-echelon matrices. Support your answer with examples.

W **50.** Describe all possible 3×3 reduced row-echelon matrices. Support your answer with examples.

True or False? In Exercises 51 and 52, determine whether the statement is true or false. If it is true, give a reason or cite an appropriate statement in the text. If it is false, provide an example that shows that the statement is not true in all cases or cite an appropriate statement in the text.

51. (a) A 6×3 matrix has six rows. T

(b) Every matrix is row-equivalent to a matrix in row-echelon form. T

(c) If the row-echelon form of the augmented matrix of a system of linear equations contains the row $[1\ 0\ 0\ 0\ 0]$, then the original system is inconsistent. F

(d) A homogeneous system of four linear equations in six unknowns has an infinite number of solutions. T

52. (a) A 4×7 matrix has four columns.

(b) Every matrix has a unique reduced row-echelon form.

(c) A homogeneous system of four linear equations in four unknowns is always consistent.

(d) Multiplying a row of a matrix by a constant is one of the elementary row operations.

In Exercises 53 and 54, determine conditions on a, b, c, and d such that the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ will be row-equivalent to the given matrix.

53. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

54. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

In Exercises 55 and 56, find all values of λ (the Greek letter lambda) such that the given homogeneous system of linear equations will have nontrivial solutions.

55.
$$\begin{aligned} (\lambda - 2)x + y &= 0 \\ x + (\lambda - 2)y &= 0 \end{aligned}$$

56.
$$\begin{aligned} (\lambda - 1)x + 2y &= 0 \\ x + \lambda y &= 0 \end{aligned}$$

W **57.** Is it possible that a system of linear equations with fewer equations than variables may have no solution? If so, give an example.

W **58.** Does a matrix have a unique row-echelon form? Illustrate your answer with examples. Is the reduced row-echelon form unique?

59. Solve the following system for α and β, $0 \le \alpha, \beta \le 2\pi$.

$$\begin{aligned} 2\cos\alpha - \sin\beta &= 0 \\ 4\cos\alpha + 2\sin\beta &= 4 \end{aligned}$$

60. Solve the following system of equations for x and y.

$$\begin{aligned} x^2 + 2y^3 &= 2 \\ 3x^2 - y^3 &= 13 \end{aligned}$$

W **61.** Consider the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Perform the following sequence of row operations.

(a) Add (-1) times the second row to the first row.

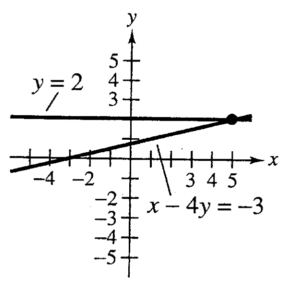
(b) Add 1 times the first row to the second row.

(c) Add (-1) times the second row to the first row.

(d) Multiply the first row by (-1).

What has happened to the original matrix? Describe in general how to interchange two rows of a matrix using only the second and third elementary row operations.

$$\begin{aligned} x - 4y &= -3 \\ y &= 2 \end{aligned}$$



$$\begin{aligned} x &= 5 \\ y &= 2 \end{aligned}$$