NEW ADDITIONAL MATHEMATICS

PURE

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23.1 BASIC CONCEPTS

Scalar and Vector Quantities

In the earlier chapters, we calculate distances between points, areas of regions enclosed by curves and volumes of solids of revolution. These quantities involving only magnitude are called **scalar** quantities. However, when studying the motion of a particle moving along the x-axis, we use positive and negative signs to indicate direction of displacement, velocity and acceleration.

For example, in Fig. 23.1, when a particle P is moving with a velocity $v = -3 \text{ m s}^{-1}$ at a certain instant, P is actually moving with a speed of 3 m s^{-1} in the direction opposite to the positive direction of the x-axis.

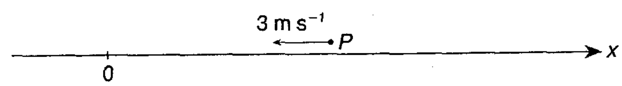


Fig. 23.1

Displacement, velocity and acceleration, which involve not only magnitude but also direction, are known as **vector** quantities.

In this chapter, we shall explore the basic concepts on vectors and study the general properties of vectors and its applications in simple geometry. These same properties for all vector quantities will be seen in their applications in other areas such as mechanics and geometry.

Representation of Vectors by Directed Line Segments

A simple way to study vectors is to represent them by directed line segments and define operations on them using the intuitive properties of directed line segments.

For example, a displacement is represented by the directed line segment $\overrightarrow{AB}$ and a velocity of 25 km h^{-1} in the direction N 30° E is represented by the directed line segment $\overrightarrow{PQ}$ as shown in Fig. 23.2.

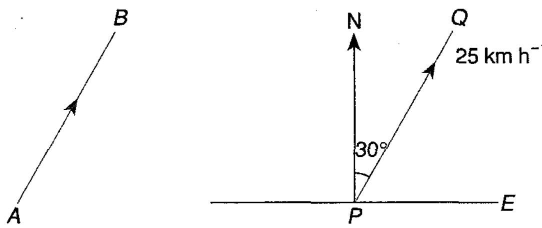


Fig. 23.2

For the displacement, its magnitude is represented by the length AB and its direction is indicated by the arrow from A to B. Similarly, the magnitude of the velocity is represented by the length PQ and its direction is from P to Q.

The above displacement represented by $\overrightarrow{AB}$ can be written as **AB** or denoted by a single letter $\underset{\sim}{d}$ or **d**. The magnitude of the directed line segment $\overrightarrow{AB}$ is denoted by $|\overrightarrow{AB}|$, $|\mathbf{AB}|$, $|\underset{\sim}{d}|$, $|\mathbf{d}|$ or simply d.

Equal Vectors

Fig. 23.3 shows a parallelogram $PQRS$, the directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{SR}$ have the same direction and are of equal length. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{SR}$ are said to be equal and we write $\overrightarrow{PQ} = \overrightarrow{SR}$. Similarly, $\overrightarrow{PS} = \overrightarrow{QR}$.

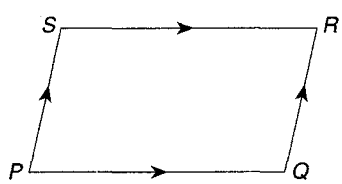


Fig. 23.3

In general,

> **a** and **b** are equal $\Leftrightarrow$ **a** and **b** have the same direction and $|\mathbf{a}| = |\mathbf{b}|$.

Negative Vectors

Refer to the parallelogram $PQRS$ in Fig. 23.4. The directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are of equal length but in opposite directions. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are negative vectors of each other and we write:

$$\overrightarrow{PQ} = -\overrightarrow{RS} \quad \text{or} \quad \overrightarrow{RS} = -\overrightarrow{PQ}$$

Similarly,

$$\overrightarrow{PS} = -\overrightarrow{RQ} \quad \text{or} \quad \overrightarrow{RQ} = -\overrightarrow{PS}$$

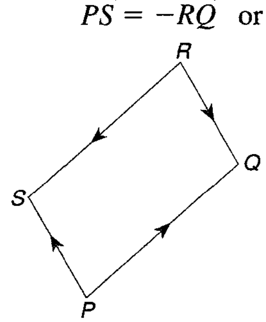


Fig. 23.4

In general, $\mathbf{a} = -\mathbf{b} \iff \mathbf{a}$ and $\mathbf{b}$ are in opposite directions and $|\mathbf{a}| = |\mathbf{b}|$.

Zero Vector

The directed line segment $\overrightarrow{AA}$ has zero magnitude and it represents a zero vector denoted by $\underset{\sim}{0}$ or $\mathbf{0}$.

In general, $\mathbf{a}$ is a zero vector $\iff |\mathbf{a}| = 0$.

Scalar Multiplication of a Vector

In Fig. 23.5, the points M and N are midpoints of the sides PQ and PR of the triangle PQR.

Since $\overrightarrow{QR}$ and $\overrightarrow{MN}$ have the same direction,

$$QR = 2MN \Rightarrow \overrightarrow{QR} = 2\overrightarrow{MN},$$

and

$$MN = \frac{1}{2}QR \Rightarrow \overrightarrow{MN} = \frac{1}{2}\overrightarrow{QR}.$$

Further,

$$\overrightarrow{QR} = 2\overrightarrow{MN} \Rightarrow \overrightarrow{RQ} = -2\overrightarrow{MN},$$
$$\overrightarrow{MN} = \frac{1}{2}\overrightarrow{QR} \Rightarrow \overrightarrow{MN} = -\frac{1}{2}\overrightarrow{RQ}.$$

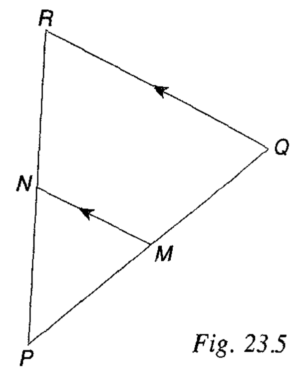


Fig. 23.5

Note that the negative sign indicates that the related vectors are opposite in direction.

In general,

$\mathbf{b} = k\mathbf{a}$ and $k > 0 \iff \mathbf{a}$ and $\mathbf{b}$ are in the same direction and $|\mathbf{b}| = |k||\mathbf{a}|$,
$\mathbf{b} = k\mathbf{a}$ and $k < 0 \iff \mathbf{a}$ and $\mathbf{b}$ are opposite in direction and $|\mathbf{b}| = |k||\mathbf{a}|$.

Unit Vectors

If $|\mathbf{u}| = 1$, $\mathbf{u}$ is known as a unit vector.

The unit vector in the direction of a given vector $\mathbf{a}$ is $\frac{\mathbf{a}}{|\mathbf{a}|}$ and we write:

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Example 1 Fig. 23.6 shows a grid with parallel lines. $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{MN} = \mathbf{b}$.

(a) Express **a** and **b** in terms of **u**.

(b) If **u** is a unit vector, find the magnitude of **a** and of **b**.

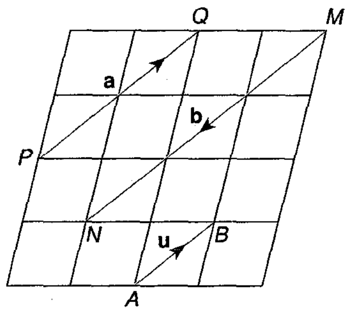


Fig. 23.6

Solution:

(a) $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ are in the same direction and $PQ = 2AB$.

$$\overrightarrow{PQ} = 2\overrightarrow{AB}$$
$$\mathbf{a} = \underline{\underline{2\mathbf{u}}}$$

$\overrightarrow{MN}$ and $\overrightarrow{AB}$ are in opposite directions and $MN = 3AB$.

$$\overrightarrow{MN} = -3\overrightarrow{AB}$$
$$\mathbf{b} = \underline{\underline{-3\mathbf{u}}}$$

(b) $|\mathbf{a}| = 2|\mathbf{u}| = \underline{\underline{2 \text{ units}}}$ $\quad (\because \ |\mathbf{u}| = 1)$

$|\mathbf{b}| = |-3|\,|\mathbf{u}| = \underline{\underline{3 \text{ units}}}$

Addition of Vectors

In the triangle PQR (Fig. 23.7), the vectors $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{PR}$ are related as follows:

Let $\overrightarrow{PQ}$ be the displacement from P to Q,
$\overrightarrow{QR}$ be the displacement from Q to R, and
$\overrightarrow{PR}$ be the displacement from P to R.

In physical situations, the displacement $\overrightarrow{PQ}$ followed by the displacement $\overrightarrow{QR}$ is the displacement $\overrightarrow{PR}$ and we write:

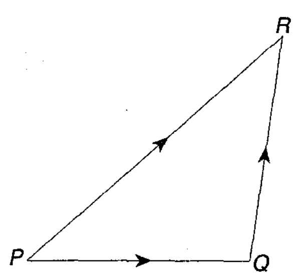


Fig. 23.7

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

This process of adding the two vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ using the triangle PQR is known as the **triangle law of addition.**

Refer to the parallelogram $PQRS$ in Fig. 23.8.

$$\overrightarrow{PS} = \overrightarrow{QR}$$
$$\overrightarrow{PQ} + \overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QR}$$
$$= \overrightarrow{PR}$$

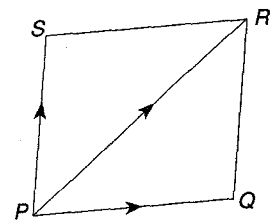


Fig. 23.8

Hence,

$$\overrightarrow{PQ} + \overrightarrow{PS} = \overrightarrow{PR}.$$

This process of adding $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ using the parallelogram $PQRS$ is known as the **parallelogram law**.

In Fig. 23.9, $OACB$ is a parallelogram. We have $\overrightarrow{OA} = \overrightarrow{BC} = \mathbf{a}$, $\overrightarrow{OB} = \overrightarrow{AC} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

By the triangle law,

$$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$$

and $\overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC}$

i.e. $\mathbf{a} + \mathbf{b} = \mathbf{c}$

and $\mathbf{b} + \mathbf{a} = \mathbf{c}$

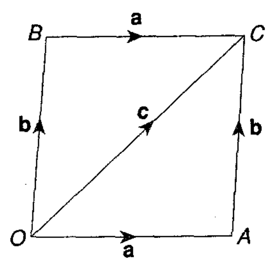


Fig. 23.9

Hence,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

and so vector addition is said to be **commutative**.

By the triangle law, $\overrightarrow{OA} + \overrightarrow{AA} = \overrightarrow{OA}$

which gives

$$\mathbf{a} + \mathbf{0} = \mathbf{a},$$
$$\mathbf{0} + \mathbf{a} = \mathbf{a}.$$

Subtraction of Vectors

Refer to the triangle PQR in Fig. 23.10.

$$\overrightarrow{QR} = \overrightarrow{QP} + \overrightarrow{PR}$$
$$= \overrightarrow{PR} + \overrightarrow{QP}$$
$$= \overrightarrow{PR} + (-\overrightarrow{PQ})$$
$$= \overrightarrow{PR} - \overrightarrow{PQ}$$

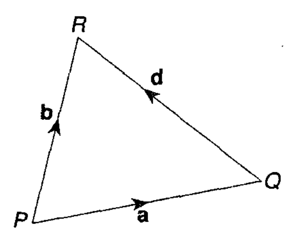


Fig. 23.10

Hence,

$$\overrightarrow{QR} = \overrightarrow{PR} - \overrightarrow{PQ}.$$

Suppose $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{PR} = \mathbf{b}$ and $\overrightarrow{QR} = \mathbf{d}$,
we have

$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} \Rightarrow \overrightarrow{QR} = \overrightarrow{PR} - \overrightarrow{PQ},$$

i.e.

$$\mathbf{b} = \mathbf{a} + \mathbf{d} \Rightarrow \mathbf{d} = \mathbf{b} - \mathbf{a}.$$

If $\mathbf{a} = \mathbf{b}$, i.e. $R = Q$, $\mathbf{d} = \overrightarrow{QR} = \overrightarrow{QQ} = \mathbf{0}$,

i.e.

$$\mathbf{a} = \mathbf{b} \Leftrightarrow \mathbf{b} - \mathbf{a} = \mathbf{0}.$$

Example 2 Fig. 23.11 shows a regular hexagon with vertices $ABCDEF$ and centre O. Express each of the following sums as a single vector.

(a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$ **(b)** $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{AF}$

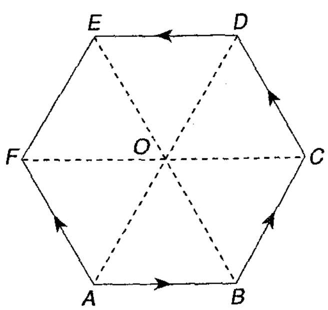


Fig. 23.11

Solution: **(a)**

$$\begin{aligned} \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} &= (\overrightarrow{AB} + \overrightarrow{BC}) + (\overrightarrow{CD} + \overrightarrow{DE}) \\ &= \overrightarrow{AC} + \overrightarrow{CE} && \textit{(triangle law)} \\ &= \underline{\underline{\overrightarrow{AE}}} && \textit{(triangle law)} \end{aligned}$$

(b)

$$\begin{aligned} \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{AF} &= (\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{AF} \\ &= \overrightarrow{AC} + \overrightarrow{AF} && \textit{(triangle law)} \\ &= \underline{\underline{\overrightarrow{AD}}} && \textit{(parallelogram law)} \end{aligned}$$

Example 3 *ABCDEF* is a regular hexagon. Express the sum of the following vectors in terms of a single vector.

(a) $\overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AE}$

(b) $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AE}, \overrightarrow{AF}$

(c) $\overrightarrow{AB}, \overrightarrow{FE}, \overrightarrow{CD}$

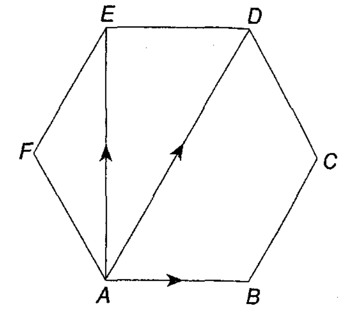


Fig. 23.12

Solution:

(a) $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{AE}$

$= (\overrightarrow{AB} + \overrightarrow{AE}) + \overrightarrow{AD}$

$= \overrightarrow{AD} + \overrightarrow{AD}$ *(parallelogram law)*

$= \underline{\underline{2\overrightarrow{AD}}}$

(b) $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AE} + \overrightarrow{AF} = (\overrightarrow{AB} + \overrightarrow{AE}) + (\overrightarrow{AC} + \overrightarrow{AF})$

$= \overrightarrow{AD} + \overrightarrow{AD}$ *(parallelogram law)*

$= \underline{\underline{2\overrightarrow{AD}}}$

(c) $\overrightarrow{AB} + \overrightarrow{FE} + \overrightarrow{CD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$ $(\overrightarrow{FE} = \overrightarrow{BC})$

$= (\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CD}$

$= \overrightarrow{AC} + \overrightarrow{CD}$ *(triangle law)*

$= \underline{\underline{\overrightarrow{AD}}}$ *(triangle law)*

Example 4 In Fig. 23.13, *ABCDEF* is a regular hexagon and $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{BC} = \mathbf{q}$.

(a) Express $\overrightarrow{FD}$, $\overrightarrow{CD}$ and $\overrightarrow{BE}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

(b) Given that $\mathbf{p}$ is a unit vector, evaluate $|\overrightarrow{AC}|$.

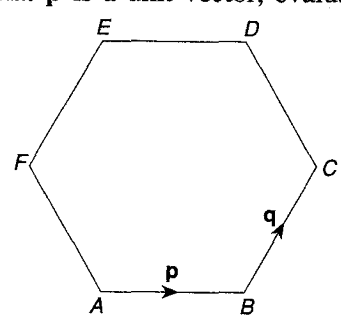


Fig. 23.13

Solution:

(a) Consider the parallelogram *ACDF*.

$\overrightarrow{FD} = \overrightarrow{AC}$

$= \overrightarrow{AB} + \overrightarrow{BC} = \underline{\underline{\mathbf{p} + \mathbf{q}}}$

Consider the $\triangle ACD$.

$\overrightarrow{CD} = \overrightarrow{AD} - \overrightarrow{AC}$

$= 2\mathbf{q} - (\mathbf{p} + \mathbf{q}) = \underline{\underline{\mathbf{q} - \mathbf{p}}}$

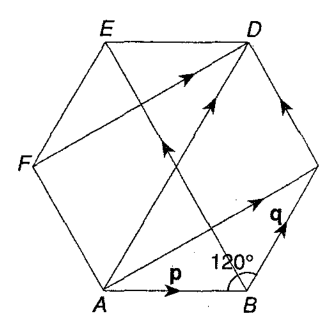


Fig. 23.14

Consider the trapezium $BCDE$.

$\overrightarrow{BE} = 2\overrightarrow{CD}$

$= \underline{\underline{2(\mathbf{q} - \mathbf{p})}}$

(b) $|\mathbf{p}| = 1 \Rightarrow |\mathbf{q}| = 1$

i.e. $AB = 1$ and $BC = 1$

By the cosine rule,

$$AC^2 = AB^2 + BC^2 - 2AB \, . \, BC \cos 120°$$
$$= 1 + 1 - 2 \times \left(-\frac{1}{2}\right)$$
$$= 3$$
$$AC = \sqrt{3} \Rightarrow |\overrightarrow{AC}| = \underline{\underline{\sqrt{3} \text{ units}}}$$

Example 5 $PQRS$ is a quadrilateral. The points A, B, C and D are midpoints of PQ, QR, RS and SP respectively. Show that $ABCD$ is a parallelogram.

Solution:

$$\overrightarrow{AB} = \overrightarrow{QB} - \overrightarrow{QA}$$
$$= \frac{1}{2}\overrightarrow{QR} - \frac{1}{2}\overrightarrow{QP}$$
$$= \frac{1}{2}(\overrightarrow{QR} - \overrightarrow{QP})$$
$$= \frac{1}{2}\overrightarrow{PR}$$

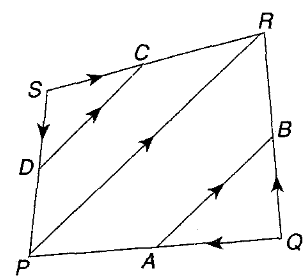


Fig. 23.15

Similarly,

$$\overrightarrow{DC} = \frac{1}{2}\overrightarrow{PR}.$$

Hence $\overrightarrow{AB} = \overrightarrow{DC} \Rightarrow ABCD$ is a parallelogram.

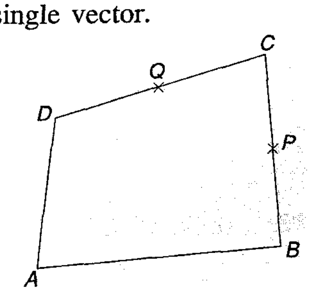
Exercise 23.1 *(Answers on p. 590)*

1. In the figure, $ABCD$ is a quadrilateral. P and Q are the midpoints of BC and CD respectively. Express each of the following sums as a single vector.

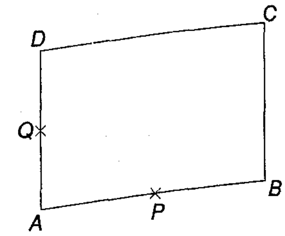
   (a) $\overrightarrow{AB} + \overrightarrow{BQ} + \overrightarrow{QP}$

   (b) $\overrightarrow{AC} + \overrightarrow{CD} + \overrightarrow{DB}$

   (c) $\overrightarrow{AQ} + \overrightarrow{QB} + \overrightarrow{BD}$



2. $ABCD$ is a parallelogram. P and Q are the midpoints of AB and AD respectively. Show that



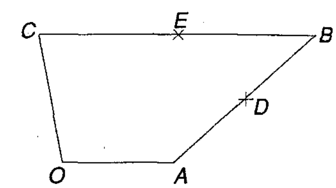
(a) $\overrightarrow{AP} + \overrightarrow{AQ} = \frac{1}{2}\overrightarrow{AC}$,

(b) $\overrightarrow{PC} + \overrightarrow{QC} = \frac{3}{2}\overrightarrow{AC}$,

(c) $\overrightarrow{PD} + \overrightarrow{QB} = \frac{1}{2}\overrightarrow{AC}$.

3. OAB is a triangle and M is the midpoint of AB. Express $\overrightarrow{AM}$ and $\overrightarrow{OM}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

4. $OABC$ is a trapezium with $\overrightarrow{CB} = 3\overrightarrow{OA}$. The points D and E are midpoints of AB and BC.



(a) Express $\overrightarrow{OD}$, $\overrightarrow{OE}$ and $\overrightarrow{DE}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

(b) Express $\overrightarrow{OD}$, $\overrightarrow{OE}$ and $\overrightarrow{DE}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OC}$.

5. $OABC$ is a parallelogram. Express the sum of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ in terms of a single vector.

6. $ABCD$ is a rectangle. Given that $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{BC} = \mathbf{q}$, express in terms of $\mathbf{p}$ and $\mathbf{q}$,

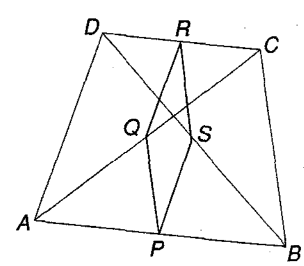
(a) $\overrightarrow{AC}$,

(b) $\overrightarrow{DB}$,

(c) $\overrightarrow{BX}$, where X is the midpoint of CD.

Given that $|\mathbf{p}| = 2|\mathbf{q}|$ and $\mathbf{q}$ is a unit vector, evaluate $|\overrightarrow{BX}|$.

7. In the figure, P, Q, R and S are the mid-points of the sides AB, AC, DC, DB of the quadrilateral $ABCD$. Show that $PQRS$ is a parallelogram.



*8. $ABCDEFGH$ is a regular octagon and $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{BC} = \mathbf{q}$. Express $\overrightarrow{AH}$ in terms of $\mathbf{p}$ and $\mathbf{q}$ and show that $\overrightarrow{AE} + \overrightarrow{BH} + \overrightarrow{CG} + \overrightarrow{DF} = 2(2 + \sqrt{2})(\mathbf{q} - \sqrt{2}\mathbf{p})$.

9. *ABCDEF* is a regular hexagon. Express in terms of a single vector the sum of the vectors

(a) $\overrightarrow{AB}$, $\overrightarrow{AE}$, (b) $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AE}$, $\overrightarrow{AF}$,

(c) $\overrightarrow{AB}$, $\overrightarrow{AF}$, (d) $4\overrightarrow{AB}$, $2\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$, $5\overrightarrow{AF}$. *(C)*

10. *ABCDEF* is a regular hexagon. Given that $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{BC} = \mathbf{q}$, express the following in terms of one, or both, **p** and **q**.

(a) $\overrightarrow{AD} + \overrightarrow{BE}$ (b) $\overrightarrow{BD} + \overrightarrow{CE}$

23.2 VECTORS EXPRESSED IN TERMS OF TWO NON-PARALLEL VECTORS

In Example 4, **p** and **q** are non-parallel vectors and

$\overrightarrow{FD} = \mathbf{p} + \mathbf{q}$,

$\overrightarrow{CD} = \mathbf{q} - \mathbf{p} = -\mathbf{p} + \mathbf{q}$,

$\overrightarrow{BE} = 2(\mathbf{q} - \mathbf{p}) = -2\mathbf{p} + 2\mathbf{q}$.

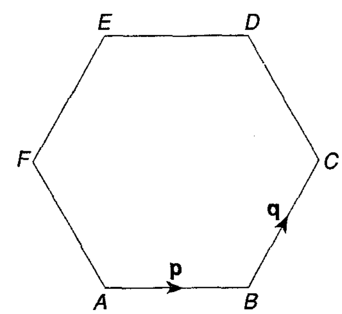


Fig. 23.16

Each of these vectors is expressed in terms of the non-parallel vectors **p** and **q** and is of the form $m\mathbf{p} + n\mathbf{q}$, where m and n are scalars.

In general:

> If **a** and **b** are two non-zero and non-parallel vectors, any vector $\overrightarrow{OP}$ in the plane containing **a** and **b** can be expressed in terms of **a** and **b**. That is,
> $\overrightarrow{OP} = m\mathbf{a} + n\mathbf{b}$, where m and n are constants.

This result may be explained as follows:
For any vector $\overrightarrow{OP}$, there is a parallelogram $OA'PB'$ such that

$$\overrightarrow{OA'} = m\mathbf{a}, \quad \overrightarrow{OB'} = n\mathbf{b}$$

and $\overrightarrow{OP} = \overrightarrow{OA'} + \overrightarrow{OB'}$ *(parallelogram rule)*

$= m\mathbf{a} + n\mathbf{b}$

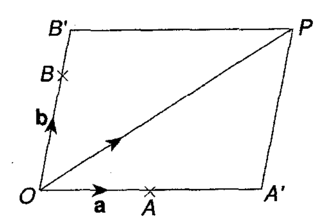


Fig. 23.17

An important nature of vectors is that two non-parallel vectors cannot be equal. This enables us to have the following results:

If non-zero vectors **a** and **b** are non-parallel, then

(a) $\lambda\mathbf{a} = \mu\mathbf{a} \Rightarrow \lambda\mathbf{a} = \mathbf{0}$ and $\mu\mathbf{a} = \mathbf{0}$
$\qquad\Rightarrow \lambda = 0$ and $\mu = 0$,

(b) $p\mathbf{a} + q\mathbf{b} = r\mathbf{a} + s\mathbf{b} \Rightarrow (p - r)\mathbf{a} = (s - q)\mathbf{b}$
$\qquad\Rightarrow (p - r) = 0$ and $(s - q) = 0$,

which gives

$$p\mathbf{a} + q\mathbf{b} = r\mathbf{a} + s\mathbf{b} \Rightarrow p = r \text{ and } q = s.$$

Example 6 Given that vectors **a** and **b** are non-parallel and non-zero, find the values of t and s if

$$\mathbf{a} + t(\mathbf{b} + 2\mathbf{a}) = 2\mathbf{a} + \mathbf{b} + s(\mathbf{a} - \mathbf{b}).$$

Solution:

$\mathbf{a} + t(\mathbf{b} + 2\mathbf{a}) = 2\mathbf{a} + \mathbf{b} + s(\mathbf{a} - \mathbf{b})$
$(1 + 2t)\mathbf{a} + t\mathbf{b} = (2 + s)\mathbf{a} + (1 - s)\mathbf{b}$
$\Rightarrow \quad 1 + 2t = 2 + s$
$\qquad 2t = 1 + s$ .... (1)
and $\qquad t = 1 - s$ .... (2)
Adding (1) and (2),

$$3t = 2 \Rightarrow t = \frac{2}{3} \text{ and so } s = \frac{1}{3}$$

Example 7 In Fig. 23.18, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OP} = 2\mathbf{a} + 3\mathbf{b}$. Given that $\overrightarrow{OM} = \lambda\overrightarrow{OP}$ and $\overrightarrow{AM} = \mu\overrightarrow{AB}$, express $\overrightarrow{OM}$

(a) in terms of λ, **a** and **b**,
(b) in terms of μ, **a** and **b**.

Hence find the values of λ and μ and express $\overrightarrow{OM}$ in terms of **a** and **b**.

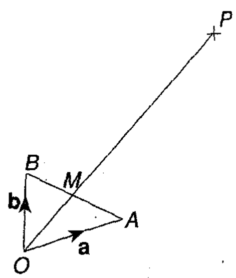


Fig. 23.18

Solution:

(a) $\overrightarrow{OM} = \lambda\overrightarrow{OP}$
$\qquad = \lambda(2\mathbf{a} + 3\mathbf{b})$ .... (1)

(b) $\overrightarrow{AM} = \mu\overrightarrow{AB}$
$\qquad = \mu(\overrightarrow{OB} - \overrightarrow{OA})$
$\qquad = \mu(\mathbf{b} - \mathbf{a})$

By the triangle law,

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$$
$$= \mathbf{a} + \mu(\mathbf{b} - \mathbf{a})$$
$$= \underline{\underline{(1 - \mu)\mathbf{a} + \mu\mathbf{b}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1) and (2),

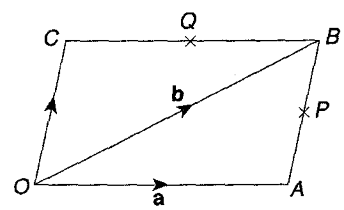
$$2\lambda\mathbf{a} + 3\lambda\mathbf{b} = (1 - \mu)\mathbf{a} + \mu\mathbf{b}$$
$$\Rightarrow \quad 2\lambda = 1 - \mu \text{ and } 3\lambda = \mu$$
$$\Rightarrow \quad \lambda = \underline{\underline{\tfrac{1}{5}}}, \quad \mu = \underline{\underline{\tfrac{3}{5}}}$$

and $\overrightarrow{OM} = \frac{1}{5}(2\mathbf{a} + 3\mathbf{b})$

$$= \underline{\underline{\tfrac{2}{5}\mathbf{a} + \tfrac{3}{5}\mathbf{b}}}$$

Exercise 23.2 (Answers on p. 590)

1. $OABC$ is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. P and Q are the midpoints of AB and BC respectively. Find $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{AQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.



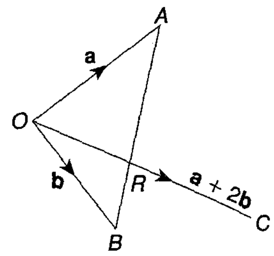
2. $ABCDEF$ is a regular hexagon with centre O. Suppose $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{AF} = \mathbf{q}$, express $\overrightarrow{AO}$, $\overrightarrow{AC}$, $\overrightarrow{AE}$ and $\overrightarrow{CE}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

3. Given that the non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are non-parallel and that
$$3\mathbf{a} + t(2\mathbf{a} - 3\mathbf{b}) = \mathbf{a} + \mathbf{b} + s(\mathbf{a} + 2\mathbf{b}),$$
find the value of t and of s.

4. Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are non-parallel. If $2\mathbf{a} + t(\mathbf{a} - \mathbf{b})$ and $2\mathbf{b} + \mathbf{a} + t\mathbf{b}$ are parallel, find the values of t.

5. $OABC$ is a square with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. M is the midpoint of BC and AM intersects OC produced at P. Given that $\overrightarrow{AP} = k\overrightarrow{AM}$ and $\overrightarrow{OP} = n\overrightarrow{OC}$, express $\overrightarrow{OP}$
   (a) in terms of k, $\mathbf{a}$ and $\mathbf{c}$,
   (b) in terms of n, $\mathbf{a}$ and $\mathbf{c}$.
   Hence, find the value of n and of k.

6. Given that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{a} + 2\mathbf{b}$ as shown in the figure. AB meets OC at R so that $\overrightarrow{AR} = k\overrightarrow{AB}$ and $\overrightarrow{OR} = n\overrightarrow{OC}$. Express $\overrightarrow{OR}$
   (a) in terms of k, $\mathbf{a}$ and $\mathbf{b}$,
   (b) in terms of n, $\mathbf{a}$ and $\mathbf{b}$.
   Hence, find the value of n and of k.



7. Given that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OP} = \frac{2}{3}\overrightarrow{OA}$ and $\overrightarrow{OQ} = 2\mathbf{b}$, express $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
   PQ meets AB at R so that $\overrightarrow{PR} = n\overrightarrow{PQ}$ and $\overrightarrow{AR} = k\overrightarrow{AB}$. Express $\overrightarrow{OR}$
   (a) in terms of n, $\mathbf{a}$ and $\mathbf{b}$,
   (b) in terms of k, $\mathbf{a}$ and $\mathbf{b}$.
   Hence find the value of n and of k.

23.3 POSITION VECTORS

In Fig. 23.19 the position of a point P with respect to an origin O is indicated by the directed line segment $\overrightarrow{OP}$. The vector $\overrightarrow{OP}$ is called the **position vector** of P.

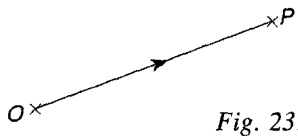


Fig. 23.19

Example 8 Relative to an origin O, the position vectors of the points A and B are $\mathbf{a}$ and $\mathbf{b}$. C is the point such that $OACB$ is a parallelogram and M is the point of intersection of the diagonals OC and AB. Find the position vectors of C and M.

Solution: By the parallelogram law,

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$$
$$= \mathbf{a} + \mathbf{b}$$

The position vector of C is $\underline{\underline{\mathbf{a} + \mathbf{b}}}$.

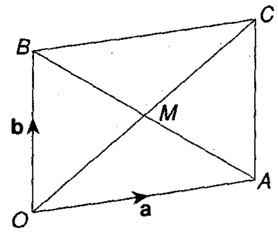


Fig. 23.20

Since the diagonals bisect each other,

$$\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OC}$$
$$= \frac{1}{2}(\mathbf{a} + \mathbf{b})$$

The position vector of M is $\underline{\underline{\frac{1}{2}(\mathbf{a} + \mathbf{b})}}$.

Example 9 Fig. 23.21 shows an equilateral triangle OAB and the position vectors of A and B relative to O are $\mathbf{a}$ and $\mathbf{b}$ respectively. Indicate on the diagram the points P and Q whose position vectors are $2\mathbf{a} + \mathbf{b}$ and $3\mathbf{a} - \mathbf{b}$ respectively.

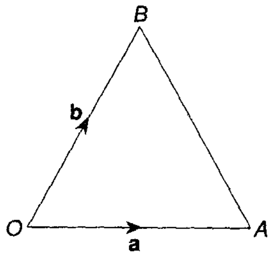


Fig. 23.21

Solution: The points with position vectors $2\mathbf{a}$ and $\mathbf{b}$ are A_1 and B. Then the point with position vector $2\mathbf{a} + \mathbf{b}$ is P where OA_1PB is a parallelogram.
The points with position vectors $3\mathbf{a}$ and $-\mathbf{b}$ are A_2 and B_1. The point with position vector $3\mathbf{a} - \mathbf{b}$ is Q where OB_1QA_2 is a parallelogram. P and Q are as shown in Fig. 23.22.

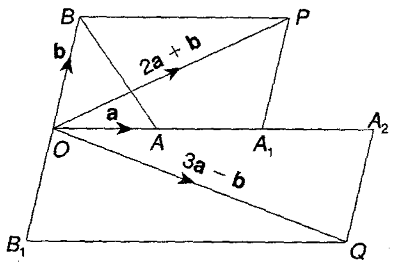


Fig. 23.22

Collinear Points

Fig. 23.23 shows three distinct points A, B and C. If A, B and C lie on a straight line, the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel. The converse is also true. Hence, we have the following result:

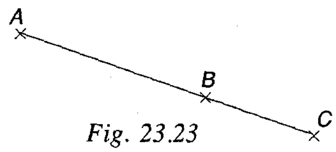


Fig. 23.23

Three distinct points A, B and C are collinear
$\Leftrightarrow \overrightarrow{AB} = k\overrightarrow{BC}$, where k is a scalar.

Example 10 Relative to an origin O, the position vectors of A, B and C are $2\mathbf{p} - 2\mathbf{q}$, $3\mathbf{p} + \lambda\mathbf{q}$ and $(2 + \lambda)\mathbf{p} + 6\mathbf{q}$ where $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors.

(a) Express $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in terms of λ, $\mathbf{p}$ and $\mathbf{q}$.

(b) Given that A, B and C are collinear, find the possible values of λ.

Solution: (a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (3\mathbf{p} + \lambda\mathbf{q}) - (2\mathbf{p} - 2\mathbf{q})$

$= \underline{\underline{\mathbf{p} + (\lambda + 2)\mathbf{q}}}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (2 + \lambda)\mathbf{p} + 6\mathbf{q} - (3\mathbf{p} + \lambda\mathbf{q})$

$= \underline{\underline{(\lambda - 1)\mathbf{p} + (6 - \lambda)\mathbf{q}}}$

(b) Since A, B and C are collinear,

$$\overrightarrow{BC} = k\overrightarrow{AB}$$

$$(\lambda - 1)\mathbf{p} + (6 - \lambda)\mathbf{q} = k(\mathbf{p} + (\lambda + 2)\mathbf{q})$$

$\Rightarrow \quad \lambda - 1 = k$ and $6 - \lambda = k(\lambda + 2)$

$\Rightarrow \quad 6 - \lambda = (\lambda - 1)(\lambda + 2)$

$\Rightarrow \quad \lambda^2 + 2\lambda - 8 = 0$

$\Rightarrow \quad (\lambda - 2)(\lambda + 4) = 0$

$\lambda = 2$ or -4

The possible values of λ are $\underline{\underline{2 \text{ and } -4}}$.

Example 11 Relative to an origin O, the position vectors of A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively. Given that $\overrightarrow{OC} = \frac{2}{3}\overrightarrow{OA}$, $\overrightarrow{OD} = 2\overrightarrow{OB}$ and $\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AB}$,

(a) express $\overrightarrow{CE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$,

(b) express $\overrightarrow{ED}$ in terms of $\mathbf{a}$ and $\mathbf{b}$,

(c) show that C, D and E are collinear and find the ratio $CE : ED$.

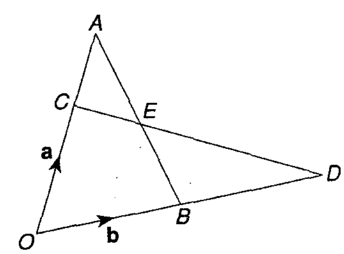


Fig. 23.24

Solution: (a) $\overrightarrow{CE} = \overrightarrow{CA} + \overrightarrow{AE}$

$= \frac{1}{3}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

$= \frac{1}{3}\mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$

$= \underline{\underline{\frac{1}{6}(-\mathbf{a} + 3\mathbf{b})}}$

(b) $\overrightarrow{ED} = \overrightarrow{EB} + \overrightarrow{BD}$

$= \frac{1}{2}\overrightarrow{AB} + \overrightarrow{OB}$

$= \frac{1}{2}(\mathbf{b} - \mathbf{a}) + \mathbf{b}$

$= \underline{\underline{\frac{1}{2}(-\mathbf{a} + 3\mathbf{b})}}$

(c) From (a) and (b),

$\overrightarrow{ED} = 3\overrightarrow{CE} \Rightarrow C$, D and E are collinear and

$CE : ED = \underline{\underline{1:3}}$.

Example 12 $OABC$ is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. The midpoint of AB is X and CX meets OB at Y.

Given that $\overrightarrow{OY} = \lambda\overrightarrow{OB}$ and $\overrightarrow{CY} = \mu\overrightarrow{CX}$,

(a) express $\overrightarrow{OY}$ in terms of λ, $\mathbf{a}$ and $\mathbf{c}$,

(b) express $\overrightarrow{OY}$ in terms of μ, $\mathbf{a}$ and $\mathbf{c}$,

(c) find the value of λ and of μ and the vector $\overrightarrow{OY}$ in terms of $\mathbf{a}$ and $\mathbf{c}$.

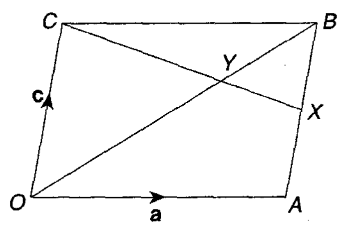


Fig. 23.25

Solution:

(a) $\overrightarrow{OY} = \lambda\overrightarrow{OB} = \underline{\underline{\lambda(\mathbf{a} + \mathbf{c})}}$ .... (1)

(b) $\overrightarrow{OY} = \overrightarrow{OC} + \overrightarrow{CY}$

$= \overrightarrow{OC} + \mu\overrightarrow{CX}$

$= \mathbf{c} + \mu\left(\mathbf{a} - \frac{1}{2}\mathbf{c}\right)$ $\quad (\overrightarrow{CX} = \overrightarrow{CB} + \overrightarrow{BX})$

$\overrightarrow{OY} = \underline{\underline{\mu\mathbf{a} + \left(1 - \frac{\mu}{2}\right)\mathbf{c}}}$ .... (2)

(c) Solving (1) and (2),

$$\lambda\mathbf{a} + \lambda\mathbf{c} = \mu\mathbf{a} + \left(1 - \frac{\mu}{2}\right)\mathbf{c}$$

$$\Rightarrow \lambda = \mu \text{ and } \lambda = 1 - \frac{\mu}{2}$$

$$\Rightarrow \lambda = \mu = \underline{\underline{\frac{2}{3}}}$$

From (1), $\overrightarrow{OY} = \lambda(\mathbf{a} + \mathbf{c})$

$= \underline{\underline{\frac{2}{3}(\mathbf{a} + \mathbf{c})}}$

Exercise 23.3 (Answers on p. 590)

1. The position vectors of the points A and B relative to an origin O are $\mathbf{a}$ and $\mathbf{b}$ respectively. Given that M and N lie between A and B, $AM = MN = NB$, express the position vectors of M and N in terms of $\mathbf{a}$ and $\mathbf{b}$.

2. $OABC$ is a trapezium and $\overrightarrow{BC} = m\overrightarrow{OA}$. Given that the position vectors of A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively, express the position vector of C in terms of m, $\mathbf{a}$ and $\mathbf{b}$.

3. The points P, Q and R have position vectors $\mathbf{a} + \mathbf{b}$, $3\mathbf{a} - \mathbf{b}$ and $6\mathbf{a} - 4\mathbf{b}$ respectively. Find $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and hence show that P, Q and R are collinear.

4. The points A, B and C have position vectors $\mathbf{p} + \mathbf{q}$, $3\mathbf{p} - 2\mathbf{q}$ and $6\mathbf{p} + k\mathbf{q}$ respectively relative to an origin O. Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. If $\overrightarrow{AB} = \lambda\overrightarrow{AC}$, find the value of k and of λ.

5. The points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $m\mathbf{a} + n\mathbf{b}$ respectively. If A, B and C are collinear, show that $m + n = 1$.

6. A, B and C are points with position vectors $\mathbf{p} - \mathbf{q}$, $\lambda(\mathbf{p} + \mathbf{q})$, and $\mathbf{p} + \lambda\mathbf{q}$ respectively, relative to an origin O. Obtain expressions for $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Given that A, B and C are collinear, find the value of λ.

7. $ABCD$ is a parallelogram whose diagonals meet at E. M is the midpoint of DC. Given that $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{AD} = \mathbf{q}$, express in terms of $\mathbf{p}$ and $\mathbf{q}$,
   (a) $\overrightarrow{AE}$, (b) $\overrightarrow{BD}$, (c) $\overrightarrow{MB}$.
   AD is produced to N where $AD = DN$. Prove, by a vector method, that N, M and B are collinear. *(C)*

8. Relative to an origin O, the position vectors of A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively. OA is produced to C so that $OC = kOA$, and C is joined to the midpoint, D, of OB. E is the point on CD such that $CE = nCD$.
   (a) Express the position vector of E in terms of $\mathbf{a}$, $\mathbf{b}$, k and n.
   (b) If A, E and B are collinear, find k in terms of n.

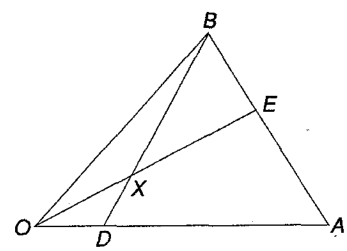
9. P, Q and R are points with position vectors $\mathbf{p}$, $\mathbf{q}$ and $3\mathbf{q} - 9\mathbf{p}$ respectively. S is the point on PQ produced such that $\overrightarrow{QS} = m\overrightarrow{PQ}$ and $\overrightarrow{RS} = n\overrightarrow{OQ}$ where m and n are constants.
   Find the position vector of S in terms of
   (a) $\mathbf{p}$, $\mathbf{q}$ and m, (b) $\mathbf{p}$, $\mathbf{q}$ and n.
   Hence, evaluate m and n and find the position vector of S.

10. Points A and B have position vectors **a** and **b** respectively relative to an origin O. P is the point on OA produced such that $\overrightarrow{OP} = 3\overrightarrow{OA}$ and Q is on OB produced such that $\overrightarrow{OQ} = 2\overrightarrow{OB}$. The lines AQ and BP meet at R. Express $\overrightarrow{AQ}$ and $\overrightarrow{BP}$ in terms of **a** and **b**. Given that $\overrightarrow{AR} = \lambda\overrightarrow{AQ}$ and $\overrightarrow{BR} = \mu\overrightarrow{BP}$, express $\overrightarrow{OR}$ in terms of

(a) λ, **a** and **b**, (b) μ, **a** and **b**.

Hence, determine the values of λ and μ and find the position vector of R.

11. Points A and B have position vectors **a** and **b** respectively relative to an origin O. The point D is such that $\overrightarrow{OD} = k\overrightarrow{OA}$ and the point E is such that $\overrightarrow{AE} = l\overrightarrow{AB}$. The line segments BD and OE intersect at X. If $\overrightarrow{OX} = \frac{2}{5}\overrightarrow{OE}$ and $\overrightarrow{XB} = \frac{4}{5}\overrightarrow{DB}$, express $\overrightarrow{OX}$ and $\overrightarrow{XB}$ in terms of **a**, **b**, k, l and hence evaluate k and l. *(C)*



12. $ABCDEF$ is a regular hexagon, centre G. The position vectors of A, B and D relative to an origin O are **a**, **b** and **d** respectively. Find the position vectors of G, C, E and F in terms of **a**, **b** and **d**.

13. The position vectors of points A, B and C relative to an origin O are **p**, 2**p** + **q** and 4**p** − 5**q** respectively. Given that $ABCD$ is a parallelogram, find the position vector of D in terms of **p** and **q**. Given that T is a point whose position vector relative to O is 6**p** + λ**q** and $OBTC$ is a parallelogram, find the value of λ.

14. $ABCD$ is a parallelogram whose diagonals meet at M and the position vectors of A, B and C relative to an origin O are **a**, **b** and **c** respectively. Find the position vectors of D and M in terms of **a**, **b** and **c**.

23.4 VECTORS IN THE CARTESIAN PLANE

Let **i** and **j** be the unit vectors in the positive direction of the x-axis and the y-axis respectively. Then the position vector of the point $P(x, y)$ is

$$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j}$$

and its magnitude is

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2}.$$

The unit vector in the direction of $\overrightarrow{OP}$ is

$$\frac{\overrightarrow{OP}}{|\overrightarrow{OP}|} = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}.$$

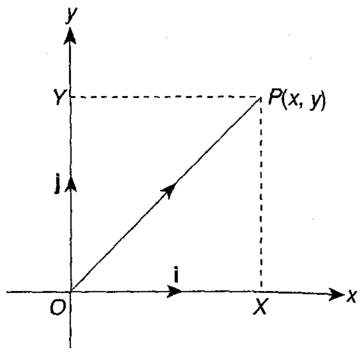


Fig. 23.26

In general:

If $\mathbf{a} = x\mathbf{i} + y\mathbf{j}$, $|x\mathbf{i} + y\mathbf{j}| = \sqrt{x^2 + y^2}$ and the unit vector in the direction of $\mathbf{a}$ is $\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}$.

Example 13 Given that $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$, find
(a) the magnitude of $\mathbf{a}$,
(b) the unit vector in the direction of $\mathbf{a}$,
(c) the vector $\mathbf{b}$ which has a magnitude of 20 units in the direction of $\mathbf{a}$.

Solution:

(a) $|\mathbf{a}| = \sqrt{3^2 + 4^2} = \underline{\underline{5}}$

(b) The unit vector in the direction of $\mathbf{a}$ is

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \underline{\underline{\frac{1}{5}(3\mathbf{i} + 4\mathbf{j})}}$$

(c) $\mathbf{b} = 20\hat{\mathbf{a}} = 20 \times \frac{1}{5}(3\mathbf{i} + 4\mathbf{j}) = \underline{\underline{4(3\mathbf{i} + 4\mathbf{j})}}$

Example 14 The position vectors of three points A, B and C relative to an origin are $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j}$, and $\mathbf{c} = m\mathbf{a} + (1 - m)\mathbf{b}$ respectively. Find
(a) the position vector of C in terms of m,
(b) the value of m if C lies on the y-axis.

Solution:

(a)
$$\begin{aligned}\mathbf{c} &= m\mathbf{a} + (1 - m)\mathbf{b} \\ &= m(2\mathbf{i} + \mathbf{j}) + (1 - m)(-\mathbf{i} + 2\mathbf{j}) \\ &= \underline{\underline{(3m - 1)\mathbf{i} + (2 - m)\mathbf{j}}}\end{aligned}$$

(b) If C lies on the y-axis, then its x-coordinate is 0,

i.e. $3m - 1 = 0$ and so, $m = \underline{\underline{\frac{1}{3}}}$.

Column Vectors

The vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ may be written in the column form

$$r = \begin{pmatrix} x \\ y \end{pmatrix}$$

which is called a **column** vector.

If $\mathbf{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then:

(a) $\mathbf{a} = \mathbf{b} \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$

(b) $k\mathbf{a} = k\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} kx_1 \\ ky_1 \end{pmatrix}$, where k is a scalar

(c) $\mathbf{a} + \mathbf{b} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$

(d) $\mathbf{a} - \mathbf{b} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}$

In general:

$$m\mathbf{a} + n\mathbf{b} = m\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + n\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} mx_1 + nx_2 \\ my_1 + ny_2 \end{pmatrix}$$

Example 15 Relative to an origin O, the position vectors of the points A, B and C are $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$ respectively. Find the value of m and of n such that $m\mathbf{a} + n\mathbf{b} = \mathbf{c}$. Show that $\overrightarrow{AC} = 2(\overrightarrow{OA} + \overrightarrow{OB})$.

Solution:

$$m\mathbf{a} + n\mathbf{b} = \mathbf{c}$$

$$m\begin{pmatrix} 2 \\ 1 \end{pmatrix} + n\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix} \Rightarrow 2m + n = 8$$

$$m + 3n = 9$$

Solving these equations, we have $m = \underline{\underline{3}}$ and $n = \underline{\underline{2}}$.

Then, $3\mathbf{a} + 2\mathbf{b} = \mathbf{c}$

$\Rightarrow \quad \mathbf{c} - \mathbf{a} = 2(\mathbf{a} + \mathbf{b})$

Hence,

$$\overrightarrow{AC} = 2(\overrightarrow{OA} + \overrightarrow{OB}).$$

Example 16 Relative to an origin O, the position vectors of two points A and B are $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 1 \end{pmatrix}$ respectively. Given that the point $P(t, t + 1)$ is on AB, find the value of t and the ratio $AP : PB$.

Solution: Since $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are parallel vectors,

$$\overrightarrow{AP} = k\overrightarrow{BP} \quad \text{for some scalar } k.$$

$$\overrightarrow{OP} - \overrightarrow{OA} = k(\overrightarrow{OP} - \overrightarrow{OB})$$

$$\begin{pmatrix} t \\ t+1 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} = k\left[\begin{pmatrix} t \\ t+1 \end{pmatrix} - \begin{pmatrix} 7 \\ 1 \end{pmatrix}\right]$$

$$\begin{pmatrix} t-1 \\ t-3 \end{pmatrix} = k\begin{pmatrix} t-7 \\ t \end{pmatrix}$$

which gives

$$t - 1 = k(t - 7) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$t - 3 = kt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\frac{t-1}{t-7} = \frac{t-3}{t} \quad (= k)$$

$$t^2 - t = t^2 - 10t + 21$$

$$t = \underline{\underline{\frac{7}{3}}}$$

From (2),

$$k = \frac{t-3}{t} = -\frac{2}{7}$$

$$\overrightarrow{AP} = -\frac{2}{7}\overrightarrow{BP} \Rightarrow AP : PB = \underline{\underline{2 : 7}}$$

Exercise 23.4 (Answers on p. 591)

1. Given that $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{c} = 4\mathbf{i} + \mathbf{j}$, find
   (a) $\mathbf{a} + \mathbf{b}$, (b) $3\mathbf{a} + 2\mathbf{b}$,
   (c) $2\mathbf{b} - \mathbf{c}$, (d) $2\mathbf{a} + \mathbf{b} - 2\mathbf{c}$.

2. Given that $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, find
   (a) $2\mathbf{a} + \mathbf{b}$, (b) $\mathbf{a} + \mathbf{b} + \mathbf{c}$,
   (c) $3\mathbf{a} + 2\mathbf{c}$, (d) $2\mathbf{a} + 3\mathbf{b} - 2\mathbf{c}$.

3. Find the magnitude of each of the following vectors. Write down the unit vector and the vector of magnitude 65 which are parallel to the given vector.
   (a) $3\mathbf{i} + 4\mathbf{j}$ (b) $5\mathbf{i} - 12\mathbf{j}$

4. Given that $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find the value of m and of n such that
   (a) $m\mathbf{a} + n\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, (b) $m\mathbf{a} + n\mathbf{b} = \begin{pmatrix} 4 \\ -7 \end{pmatrix}$.

5. The position vectors of the points A and B are $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ respectively. Given that

(a) $\overrightarrow{AE} = 3\overrightarrow{AB}$, find the position vector of E,

(b) $\overrightarrow{AP} = \lambda\overrightarrow{AB}$, express the position vector of P in terms of λ. Find the value of λ such that OP is parallel to $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

6. Relative to an origin O, the position vectors of the points P, Q and R are $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 5 \end{pmatrix}$ respectively. Find

(a) the unit vector parallel to $\overrightarrow{QR}$,

(b) the value of m and n such that $\overrightarrow{OP} + m\overrightarrow{OQ} = n\overrightarrow{OR}$ and hence deduce that $\overrightarrow{PR} = 3\overrightarrow{OQ}$.

7. The vector $\begin{pmatrix} p \\ q \end{pmatrix}$ is parallel to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find the relationship between p and q. Given that $p > 0$ and $\begin{pmatrix} p \\ q \end{pmatrix}$ has a magnitude of 5, find the value of p and of q.

8. Find the relationship between p and q if

(a) the vector $\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is parallel to $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$,

(b) the vector $\begin{pmatrix} p-1 \\ q \end{pmatrix}$ has a magnitude of 2.

9. The position vectors of the points A and B, relative to an origin O, are $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ respectively. Find the position vector of C if

(a) $\overrightarrow{AC} = 2\overrightarrow{AB}$, **(b)** $\overrightarrow{AC} = 3\overrightarrow{BA}$.

10. The position vectors of three points O, A and B are $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 6 \end{pmatrix}$ respectively. Given that $\overrightarrow{OC} = 3\overrightarrow{OA}$ and $\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OB}$, write down the position vectors of C and D. Given also that AB and CD intersect at E, and that $\overrightarrow{DE} = p\overrightarrow{DC}$ and $\overrightarrow{AE} = q\overrightarrow{AB}$, find the position vector of E in terms of

(a) p, **(b)** q.

Hence find the value of p and of q.

6. *Vectors in the Cartesian Plane*

(a) If P has coordinates (x, y) in a cartesian plane, then the position vec of P is

$$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the positive direction along the x-axis and the y-axis respectively.

(b) In column form, we write $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$.

(c) Unit vector in the direction of $\overrightarrow{OP}$ is

$$\frac{1}{\sqrt{x^2 + y^2}}(x\mathbf{i} + y\mathbf{j}) \quad \text{or} \quad \frac{1}{\sqrt{x^2 + y^2}}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Miscellaneous Exercise 23

(Answers on p. 591)

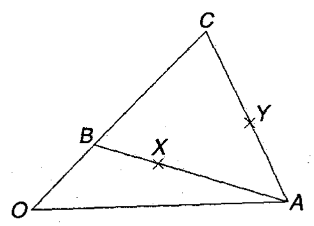
1. The position vectors of three collinear points P, Q and R relative to an origin O are $\mathbf{b} - 2\mathbf{a}$, $3\mathbf{a} - 2\mathbf{b}$ and $k\mathbf{a} + 6\mathbf{b}$ respectively.

(a) Find the value of k and state the ratio $PQ : QR$.

(b) Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of the point S such that $\overrightarrow{PS} = 2\overrightarrow{SQ}$.

2. The position vectors of three points A, B and C relative to an origin O are $2\mathbf{p} - \mathbf{q}$, $4\mathbf{q}$ and $14\mathbf{q} - 4\mathbf{p}$ respectively. Show that the points A, B and C lie on the same straight line and state the ratio $AB : BC$.
Given that $OABD$ is a parallelogram, find the position vector of D. P is the point on DB such that $\overrightarrow{DP} = k\overrightarrow{DB}$, find the position vector of P in terms of $\mathbf{p}$, $\mathbf{q}$ and k.
If O, P and C are collinear, find the value of k.

3. Given that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = 3\overrightarrow{OB}$, express $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
Given further that $\overrightarrow{AX} = 2\overrightarrow{XB}$ and $\overrightarrow{AY} = k\overrightarrow{YC}$, express $\overrightarrow{OX}$ and $\overrightarrow{OY}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and k.
If the points O, X and Y lie in a straight line, find k.



4. The position vectors of three points A, B and C relative to an origin O are $\mathbf{a}$, $\mathbf{b}$ and $5\mathbf{a}$ respectively. The point P lies on AB and is such that $AP = kAB$. The point Q lies on OP produced and is such that $OQ = \frac{1}{2k}OP$. Find the position vector of Q in terms of $\mathbf{a}$, $\mathbf{b}$ and k. If Q lies on BC, find the value of k and the ratio $BQ : QC$.

5. The position vectors of four points A, B, C and D relative to an origin O are $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 10 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ respectively. Two points P and Q are points on AB and CD respectively, $AP = \lambda AB$ and $CQ = \mu CD$. Show that the position vectors of P and Q are $\begin{pmatrix} 2 - \lambda \\ 1 + \lambda \end{pmatrix}$ and $\begin{pmatrix} 10 - 5\mu \\ 5 - \mu \end{pmatrix}$ respectively. Hence find the position vector of M, the point of intersection of AB and CD.

6. The position vectors of points A and B, relative to an origin O, are $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 9 \\ -3 \end{pmatrix}$ respectively. P, Q and R are points such that $\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OA}$, $\overrightarrow{AR} = k\overrightarrow{AB}$ and $\overrightarrow{OQ} = \frac{2}{3}\overrightarrow{OB}$. Write down the position vectors of P and Q. Express the position vector of R in terms of k. Given that P, Q and R are collinear, find the value of k.

7. Relative to an origin O, the position vectors of three points P, Q and R are $\mathbf{a} + \mathbf{b}$, $\mathbf{a}$ and $\mathbf{a} - 2\mathbf{b}$ respectively. The point L lies on OP and is such that $OL = \frac{3}{4}OP$. The point M lies on OR produced and is such that $OM = \lambda OR$. Find
   (a) in terms of $\mathbf{a}$, $\mathbf{b}$ and λ, the position vector of M,
   (b) the value of λ for which L, Q and M are collinear. Hence find the value of $\frac{QM}{LQ}$.

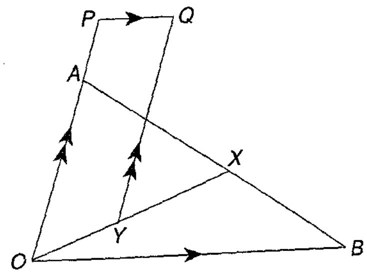
8. (a) Given that A, B and C are collinear points with position vectors $\mathbf{p}$, $\mathbf{p} + 2\mathbf{q}$, $\lambda(\mathbf{p} - 2\mathbf{q})$ respectively, find the value of
   (i) λ, (ii) $\frac{AB}{BC}$.
   (b) A, B and C are points with position vectors $2\mathbf{p} - 3\mathbf{q}$, $3\mathbf{p} - 2\mathbf{q}$ and $\mathbf{p} + 3\mathbf{q}$ respectively relative to an origin O. The point X is such that $\overrightarrow{OX} = k\overrightarrow{OA}$ and $\overrightarrow{CX} = m\overrightarrow{CB}$, express $\overrightarrow{OX}$ in terms of
   (i) k, $\mathbf{p}$ and $\mathbf{q}$, (ii) m, $\mathbf{p}$ and $\mathbf{q}$.
   Hence, evaluate k and m and write down the position vector of X in terms of $\mathbf{p}$ and $\mathbf{q}$.

9. **(a)** Find the unit vector **n** parallel to the vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$. Express the vector $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$ in terms of **n**.

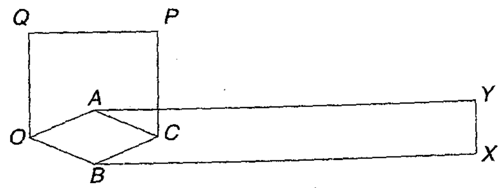
(b) Find the vector **p** which has a magnitude of 39 units and the same direction as $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$.

(c) The position vectors of A and B relative to an origin O are $\begin{pmatrix} 8 \\ 8 \end{pmatrix}$ and $\begin{pmatrix} 12 \\ 5 \end{pmatrix}$ respectively. Find $\overrightarrow{AB}$ and the unit vector parallel to $\overrightarrow{AB}$.

10. In the figure the points X and Y are such that $AX = \frac{1}{2}XB$ and $OY = YX$, while the point P is such that $OA = 3AP$. The lines YQ and PQ are parallel to OA and OB respectively.
Given that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, express $\overrightarrow{OP}$ and $\overrightarrow{OY}$ in terms of **a** and **b**.
Given that $\overrightarrow{YQ} = m\mathbf{a}$ and $\overrightarrow{PQ} = n\mathbf{b}$, find the values of m and n. Hence show that $OAQY$ is a parallelogram. *(C)*



11. $OACB$ is a parallelogram in which $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. $ABXY$ is a parallelogram in which $\overrightarrow{AY} = 3\overrightarrow{OC}$. $OCPQ$ is a parallelogram in which $\overrightarrow{CP} = 2\overrightarrow{BA}$. Find, in terms of **a** and **b**, the position vectors P, Q, X and Y relative to O.



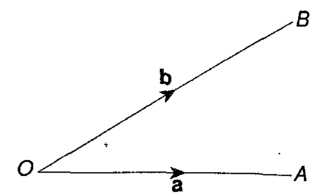
12. The position vectors of three points A, B and C, relative to an origin O, are **a**, **b** and k**a** respectively. The point P lies on AB and is such that $AP = 2PB$. The point Q lies on BC and is such that $CQ = 6QB$. Find, in terms of **a** and **b**, the position vectors of P and Q.
Given that OPQ is a straight line, find

(a) the value of k, **(b)** the ratio $\frac{OP}{PQ}$.

The position vector of a point R is $\frac{7}{3}\mathbf{a}$. Show that PR is parallel to BC. *(C)*

13. Relative to an origin O, the position vectors of three points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $-\mathbf{a} + 3\mathbf{b}$ respectively. The point P is such that $2\overrightarrow{AP} = 3\overrightarrow{BP}$. Find the position vector of P in terms of $\mathbf{a}$ and $\mathbf{b}$. The point Q is such that $\overrightarrow{OQ} = \overrightarrow{PO}$. State the position vector of Q in terms of $\mathbf{a}$ and $\mathbf{b}$, and hence show that Q lies on BC. State the ratio $BQ : QC$.

***14.** The figure shows points A and B with position vectors $\mathbf{a}$ and $\mathbf{b}$ relative to an origin O.
Draw the figure three times and illustrate the following:



(a) The positions of the points P and Q with position vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{a} + 2\mathbf{b}$ respectively, and hence, the locus of a point R with position vector $\mathbf{a} + k\mathbf{b}$ where k varies from 1 to 2.

(b) The locus of a point S with position vector $l(\mathbf{a} + 2\mathbf{b})$ where l varies from -1 to 3.

(c) The locus of a point T with position vector $m(2\mathbf{a} - \mathbf{b})$ where m varies from 0 to 2.

24 SCALAR PRODUCT

24.1 SCALAR PRODUCT

Angle Between Two Vectors

Consider the triangle OAB as shown in Fig. 24.1. The angle AOB, θ, is referred to as the angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$. This angle is also the angle between the vectors $\overrightarrow{AO}$ and $\overrightarrow{BO}$.

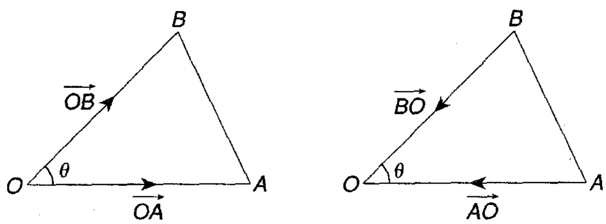


Fig. 24.1

Note that the angle θ is taken when the directions of the vectors are either both towards or away from their point of intersection.

Fig. 24.2 shows that the angle between $\overrightarrow{OA}$ and $\overrightarrow{BO}$ is $180° - \theta$.

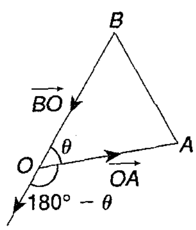


Fig. 24.2

In Fig. 24.3, the points A, B and C are vertices of an equilateral triangle.

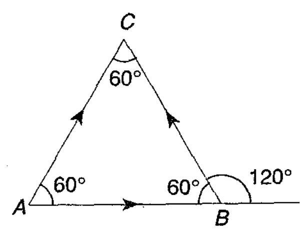


Fig. 24.3

The angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is 60°.
The angle between $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is 120°.
The angle between $\overrightarrow{AC}$ and $\overrightarrow{BC}$ is 60°.

Example 1 Fig. 24.4 shows a square $OABC$. Given that $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$, find the angle between

(a) $\mathbf{a}$ and $\mathbf{b}$,

(b) $\mathbf{a}$ and $-\mathbf{b}$,

(c) $\mathbf{a}$ and $\mathbf{b} - \mathbf{a}$.

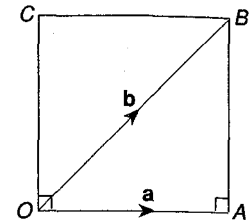


Fig. 24.4

Solution:

(a) $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$

Angle between $\mathbf{a}$ and $\mathbf{b} = \angle AOB$

$= \underline{\underline{45°}}$

(b) $\mathbf{a} = \overrightarrow{OA}$, $-\mathbf{b} = \overrightarrow{BO}$

Angle between $\mathbf{a}$ and $-\mathbf{b} = \angle AOB'$

$= \underline{\underline{135°}}$

(c) $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} - \mathbf{a} = \overrightarrow{AB}$

Angle between $\mathbf{a}$ and $\mathbf{b} - \mathbf{a} = \angle BAD$

$= \underline{\underline{90°}}$

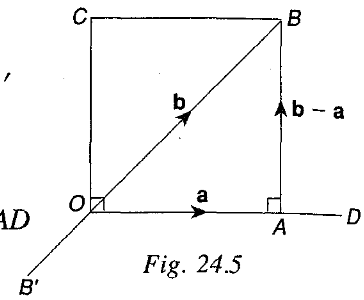


Fig. 24.5

The Scalar Product

Let θ be the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. The **scalar product** of $\mathbf{a}$ and $\mathbf{b}$ (which is a scalar) is defined by:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \theta$$

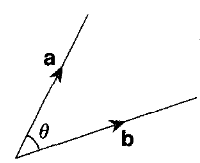


Fig. 24.6

We read $\mathbf{a} \cdot \mathbf{b}$ as '**a** dot **b**' and so the scalar product is also known as the **dot** product.

Example 2 Fig. 24.7 shows an equilateral triangle with vertices A, B and C. The midpoint of BC is D. Given that the length of each side of the triangle is l, find

(a) $\overrightarrow{AB} \cdot \overrightarrow{AC}$,

(b) $\overrightarrow{AB} \cdot \overrightarrow{BC}$,

(c) $\overrightarrow{AD} \cdot \overrightarrow{AB}$,

(d) $\overrightarrow{AD} \cdot \overrightarrow{BC}$.

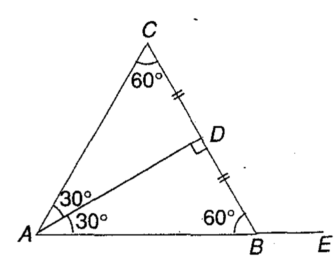


Fig. 24.7

Solution: (a) $\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}|\,|\overrightarrow{AC}| \cos \angle BAC = l^2 \cos 60° = \underline{\underline{\frac{1}{2}l^2}}$

(b) $\overrightarrow{AB} \cdot \overrightarrow{BC} = |\overrightarrow{AB}|\,|\overrightarrow{BC}| \cos \angle CBE = l^2 \cos 120° = \underline{\underline{-\frac{1}{2}l^2}}$

(c) $\overrightarrow{AD} \cdot \overrightarrow{AB} = |\overrightarrow{AD}|\,|\overrightarrow{AB}| \cos \angle DAB = (l \cos 30°)(l) \cos 30° = \underline{\underline{\frac{3}{4}l^2}}$

(d) $\overrightarrow{AD} \cdot \overrightarrow{BC} = |\overrightarrow{AD}|\,|\overrightarrow{BC}| \cos \angle ADB = |\overrightarrow{AD}|\,|\overrightarrow{BC}| \cos 90° = \underline{\underline{0}}$

Properties of the Scalar Product

Let θ be the angle between two non-zero vectors **a** and **b**.

1. $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \theta = |\mathbf{b}|\,|\mathbf{a}| \cos \theta = \mathbf{b} \cdot \mathbf{a}$ *(commutative law)*

   $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ *(distributive law)*

   $\lambda(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\lambda \mathbf{b}) = (\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda |\mathbf{a}|\,|\mathbf{b}| \cos \theta$

2. Cosine of the angle between the two vectors is given by:

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$

   (a) If $\mathbf{a} \cdot \mathbf{b} > 0$, then $\cos \theta > 0$ and so, θ is acute.
   (b) If $\mathbf{a} \cdot \mathbf{b} < 0$, then $\cos \theta < 0$ and so, θ is obtuse.

3. $\mathbf{a} \cdot \mathbf{b} = 0 \Leftrightarrow |\mathbf{a}|\,|\mathbf{b}| \cos \theta = 0$
   $\Leftrightarrow \theta = 90°$

$$\mathbf{a} \cdot \mathbf{b} = 0 \Leftrightarrow \mathbf{a} \text{ and } \mathbf{b} \text{ are perpendicular vectors}$$

   For the unit vectors **i** and **j** along the x-axis and y-axis respectively on the cartesian plane, we have:

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0$$

4. If $\theta = 0$, then $\cos \theta = 1$ and

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|\,|\mathbf{a}| \cos 0 = |\mathbf{a}|^2$$

   i.e. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ or $\mathbf{a} \cdot \mathbf{a} = a^2$

   For the unit vectors **i** and **j**,

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1.$$

   These properties enable us to compute scalar product of vectors.

Example 3 If **a**, **b** and **c** are non-zero vectors and **a** is perpendicular to **b** and **c**, show that **a** is perpendicular to $m\mathbf{b} + n\mathbf{c}$ where m and n are scalars.

Solution: Since **a** is perpendicular to **b** and **c**, $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = 0$.
Then,

$$\begin{aligned} \mathbf{a} \cdot (m\mathbf{b} + n\mathbf{c}) &= \mathbf{a} \cdot (m\mathbf{b}) + \mathbf{a} \cdot (n\mathbf{c}) \\ &= m(\mathbf{a} \cdot \mathbf{b}) + n(\mathbf{a} \cdot \mathbf{c}) \\ &= 0 \qquad (\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = 0) \end{aligned}$$

This shows that **a** is perpendicular to $m\mathbf{b} + n\mathbf{c}$.

Example 4 If $|\mathbf{a}| = 2$, $|\mathbf{b}| = 3$, $\mathbf{a} \cdot \mathbf{b} = 5$ and $\mathbf{c} = \mathbf{a} + \mathbf{b}$, evaluate

(a) cosine of the angle between **a** and **b**,

(b) $|\mathbf{c}|$.

Solution:

(a) Let θ be the angle between **a** and **b**.

$$\therefore \quad \cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \underline{\underline{\frac{5}{6}}}$$

(b)
$$\begin{aligned} |\mathbf{c}|^2 = \mathbf{c} \cdot \mathbf{c} &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} + 2\mathbf{a} \cdot \mathbf{b} \end{aligned}$$

$$\therefore \quad |\mathbf{c}|^2 = 2^2 + 3^2 + (2 \times 5) = 23$$

$$|\mathbf{c}| = \underline{\underline{\sqrt{23}}}$$

Example 5 The points A, B, C and D are the vertices of a rhombus. Show that the diagonals AC and BD are perpendicular.

Solution: Let $\overrightarrow{AB} = \overrightarrow{DC} = \mathbf{p}$ and $\overrightarrow{AD} = \overrightarrow{BC} = \mathbf{q}$.
Then,

$$\begin{aligned} \overrightarrow{AC} &= \overrightarrow{AB} + \overrightarrow{BC} \\ &= \mathbf{p} + \mathbf{q} \\ \overrightarrow{BD} &= \overrightarrow{AD} - \overrightarrow{AB} \\ &= \mathbf{q} - \mathbf{p} \end{aligned}$$

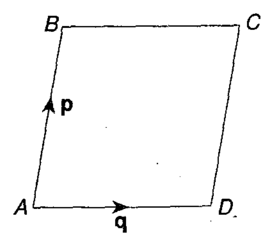


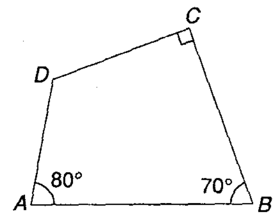
Fig. 24.8

$$\begin{aligned} \overrightarrow{AC} \cdot \overrightarrow{BD} &= (\mathbf{p} + \mathbf{q}) \cdot (\mathbf{q} - \mathbf{p}) \\ &= \mathbf{p} \cdot (\mathbf{q} - \mathbf{p}) + \mathbf{q} \cdot (\mathbf{q} - \mathbf{p}) \\ &= \mathbf{p} \cdot \mathbf{q} - \mathbf{p} \cdot \mathbf{p} + \mathbf{q} \cdot \mathbf{q} - \mathbf{q} \cdot \mathbf{p} \\ &= q^2 - p^2 \\ &= 0 \quad (\because p = q) \end{aligned}$$

$\overrightarrow{AC} \cdot \overrightarrow{BD} = 0 \Rightarrow$ diagonals AC and BD are perpendicular.

Exercise 24.1 (Answers on p. 591)

1. The points A, B, C and D are vertices of a quadrilateral as shown. In each case, find the angle between the two given vectors.



   (a) $\overrightarrow{AB}$ and $\overrightarrow{AD}$
   (b) $\overrightarrow{AB}$ and $\overrightarrow{BC}$
   (c) $\overrightarrow{AD}$ and $\overrightarrow{CD}$

2. $ABCDEF$ is a regular hexagon, centre O. For each of the following, find the angle between the two vectors.
   (a) $\overrightarrow{OA}$ and $\overrightarrow{OC}$
   (b) $\overrightarrow{OE}$ and $\overrightarrow{AB}$
   (c) $\overrightarrow{AE}$ and $\overrightarrow{FB}$
   (d) $\overrightarrow{AB}$ and $\overrightarrow{DE}$

3. $ABCD$ is a square of side $2a$. M and N are the midpoints of the sides BC and CD respectively. Find, in terms of a,
   (a) $\overrightarrow{AB} \cdot \overrightarrow{AD}$,
   (b) $\overrightarrow{AB} \cdot \overrightarrow{AM}$ and deduce the angle MAB,
   (c) $\overrightarrow{AB} \cdot \overrightarrow{MN}$.

4. Given that $\mathbf{a} \cdot \mathbf{b} = 3$ and $\mathbf{a} \cdot \mathbf{c} = 2$, evaluate the following:
   (a) $(2\mathbf{a}) \cdot (3\mathbf{b})$
   (b) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$
   (c) $\mathbf{a} \cdot (2\mathbf{b} - 3\mathbf{c})$

5. If $\mathbf{p} = 3\mathbf{a} + 2\mathbf{b}$ and $\mathbf{q} = 3\mathbf{a} - 2\mathbf{b}$, show that $\mathbf{p} \cdot \mathbf{q} = 9a^2 - 4b^2$.

6. If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors and the angle between them is 60°, show that $\mathbf{p} = 3\mathbf{u} - \mathbf{v}$ and $\mathbf{q} = 2\mathbf{u} - 10\mathbf{v}$ are perpendicular vectors.

7. Show that $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2$.

8. Given that $|\mathbf{p}| = 3$, $|\mathbf{q}| = 4$, $\mathbf{p} \cdot \mathbf{q} = 5$ and $\mathbf{r} = \mathbf{p} - \mathbf{q}$, evaluate
   (a) the cosine of the angle between $\mathbf{p}$ and $\mathbf{q}$,
   (b) $|\mathbf{r}|$,
   (c) the cosine of the angle between $\mathbf{p}$ and $\mathbf{r}$.

9. Given that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 2$ and $|\mathbf{a} - \mathbf{b}| = 4$, evaluate
   (a) $\mathbf{a} \cdot \mathbf{b}$,
   (b) $|\mathbf{a} + \mathbf{b}|$.

10. $ABCD$ is a parallelogram in which $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{AD} = \mathbf{q}$. Show that $\overrightarrow{AC} \cdot \overrightarrow{BD} = |\mathbf{q}|^2 - |\mathbf{p}|^2$. Deduce that if $ABCD$ is a rhombus, its diagonals are at right angles.

11. $ABCD$ is a rectangle in which $\overrightarrow{AC} = \mathbf{a}$ and $\overrightarrow{BD} = \mathbf{b}$, express $\overrightarrow{AB} \cdot \overrightarrow{AD}$ in terms of $|\mathbf{a}|$ and $|\mathbf{b}|$. Hence show that its diagonals are equal in length.

12. Relative to an origin O, the points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $-k\mathbf{a}$ $(k > 0)$ such that $|\mathbf{a}| = 2|\mathbf{b}|$ and $\mathbf{a} \cdot \mathbf{b} = 0$. By considering the scalar product $\overrightarrow{AB} \cdot \overrightarrow{CB}$, determine the value of k such that $\angle ABC = 90°$.

24.2 SCALAR PRODUCT OF VECTORS IN THE CARTESIAN PLANE

Let $\mathbf{a} = p\mathbf{i} + q\mathbf{j}$, $\mathbf{b} = r\mathbf{i} + s\mathbf{j}$.

Then,

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (p\mathbf{i} + q\mathbf{j}) \cdot (r\mathbf{i} + s\mathbf{j}) \\ &= p\mathbf{i} \cdot (r\mathbf{i} + s\mathbf{j}) + q\mathbf{j} \cdot (r\mathbf{i} + s\mathbf{j}) \\ &= p\mathbf{i} \cdot r\mathbf{i} + p\mathbf{i} \cdot s\mathbf{j} + q\mathbf{j} \cdot r\mathbf{i} + q\mathbf{j} \cdot s\mathbf{j} \\ &= pr + qs \qquad (\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = 1 \text{ and } \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = 0) \end{aligned}$$

$$\mathbf{a} \cdot \mathbf{b} = (p\mathbf{i} + q\mathbf{j}) \cdot (r\mathbf{i} + s\mathbf{j}) = pr + qs$$

Similarly, if we write $\mathbf{a} = \begin{pmatrix} p \\ q \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} r \\ s \end{pmatrix}$, then,

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} p \\ q \end{pmatrix} \cdot \begin{pmatrix} r \\ s \end{pmatrix} = pr + qs$$

Example 6 Given that $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, evaluate

(a) $\mathbf{a} \cdot \mathbf{b}$, **(b)** $\mathbf{a} \cdot \mathbf{c}$, **(c)** $\mathbf{a} \cdot (2\mathbf{b} - 3\mathbf{c})$.

Solution:

(a) $\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \end{pmatrix} = -3 + 6 = \underline{\underline{3}}$

(b) $\mathbf{a} \cdot \mathbf{c} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \end{pmatrix} = 6 - 10 = \underline{\underline{-4}}$

(c) $\mathbf{a} \cdot (2\mathbf{b} - 3\mathbf{c}) = 2\mathbf{a} \cdot \mathbf{b} - 3\mathbf{a} \cdot \mathbf{c} = 2 \times 3 - 3 \times (-4) = \underline{\underline{18}}$

Example 7 The position vectors of A and B are $2\mathbf{i} - \mathbf{j}$ and $3\mathbf{i} + \mathbf{j}$ respectively. Evaluate

(a) $\overrightarrow{OA} \cdot \overrightarrow{OB}$ and hence find $\angle AOB$,

(b) $\overrightarrow{OA} \cdot \overrightarrow{AB}$ and hence find $\angle OAB$.

Solution:

(a) $\overrightarrow{OA} = 2\mathbf{i} - \mathbf{j}, \quad \overrightarrow{OB} = 3\mathbf{i} + \mathbf{j}$

$$\overrightarrow{OA} \cdot \overrightarrow{OB} = (2\mathbf{i} - \mathbf{j}) \cdot (3\mathbf{i} + \mathbf{j}) = 6 - 1 = \underline{\underline{5}}$$

$$\cos \angle AOB = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OA}|\,|\overrightarrow{OB}|} = \frac{5}{\sqrt{5}\sqrt{10}} = \frac{1}{\sqrt{2}}$$

$$\angle AOB = \underline{\underline{45°}}$$

(b) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (3\mathbf{i} + \mathbf{j}) - (2\mathbf{i} - \mathbf{j}) = \mathbf{i} + 2\mathbf{j}$

$\overrightarrow{OA} \cdot \overrightarrow{AB} = (2\mathbf{i} - \mathbf{j}) \cdot (\mathbf{i} + 2\mathbf{j}) = 0$

$\Rightarrow \angle OAB$ is a right angle.

Example 8 If the vectors $\mathbf{a} = p\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - 6\mathbf{j}$ are perpendicular, find

(a) the value of p,

(b) $|\mathbf{a}|$ and $|3\mathbf{b} - \mathbf{a}|$,

(c) $\mathbf{a} \cdot (3\mathbf{b} - \mathbf{a})$ and hence, deduce the angle between $\mathbf{a}$ and $3\mathbf{b} - \mathbf{a}$.

Solution:

(a) Since $\mathbf{a}$ and $\mathbf{b}$ are perpendicular vectors,

$$\mathbf{a} \cdot \mathbf{b} = 0$$

$$2p + 3(-6) = 0$$

which gives $p = \underline{\underline{9}}$

(b) Now

$$\mathbf{a} = 3(3\mathbf{i} + \mathbf{j})$$

$$|\mathbf{a}| = 3|3\mathbf{i} + \mathbf{j}| = \underline{\underline{3\sqrt{10}}}$$

$$3\mathbf{b} - \mathbf{a} = 3(2\mathbf{i} - 6\mathbf{j}) - 3(3\mathbf{i} + \mathbf{j})$$

$$= 3(-\mathbf{i} - 7\mathbf{j})$$

$$|3\mathbf{b} - \mathbf{a}| = 3|-\mathbf{i} - 7\mathbf{j}|$$

$$= 3\sqrt{50} = \underline{\underline{15\sqrt{2}}}$$

(c) $\mathbf{a} \cdot (3\mathbf{b} - \mathbf{a}) = 3\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a} = 0 - |\mathbf{a}|^2 = -90$

Let α be the angle between $\mathbf{a}$ and $3\mathbf{b} - \mathbf{a}$.

$$\cos \alpha = \frac{\mathbf{a} \cdot (3\mathbf{b} - \mathbf{a})}{|\mathbf{a}|\,|3\mathbf{b} - \mathbf{a}|} = \frac{-90}{(3\sqrt{10})(3\sqrt{50})} = -\frac{1}{\sqrt{5}}$$

$$\alpha = \underline{\underline{116.6°}}$$

Example 9 Given that $\mathbf{a} = \begin{pmatrix} p \\ q \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, find the values of p and q such that $\mathbf{a}$ is perpendicular to $\mathbf{b}$ and $|\mathbf{a}| = |\mathbf{b}|$.

Solution: Since the vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular,

$$\mathbf{a} \cdot \mathbf{b} = 0$$

$$\begin{pmatrix} p \\ q \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 0$$

$$3p - q = 0 \quad \text{.......... (1)}$$

Next,

$$|\mathbf{a}| = |\mathbf{b}|$$

$$\sqrt{p^2 + q^2} = \sqrt{3^2 + (-1)^2}$$

i.e. $p^2 + q^2 = 10$ .... (2)

Solving (1) and (2),

$$p^2 + (3p)^2 = 10$$

$$\Rightarrow \quad p^2 = 1$$

$$p = 1 \text{ or } -1$$

Since $q = 3p$ (from (1)), we have

$$\underline{\underline{p = 1}}, \quad \underline{\underline{q = 3}} \quad \text{or} \quad p = -1, \quad \underline{\underline{q = -3}}$$

Example 10 Given that $\mathbf{a} + k\mathbf{b}$ and $\mathbf{a} - k\mathbf{b}$ are perpendicular, show that $k^2 = \dfrac{\mathbf{a} \cdot \mathbf{a}}{\mathbf{b} \cdot \mathbf{b}}$.

If $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, find the possible values of k.

Solution: $\mathbf{a} + k\mathbf{b}$ and $\mathbf{a} - k\mathbf{b}$ are perpendicular.

$$\Rightarrow (\mathbf{a} + k\mathbf{b}) \cdot (\mathbf{a} - k\mathbf{b}) = 0$$

$$\Rightarrow \quad \mathbf{a} \cdot \mathbf{a} - k^2\mathbf{b} \cdot \mathbf{b} = 0$$

of

$$\therefore \quad k^2 = \frac{\mathbf{a} \cdot \mathbf{a}}{\mathbf{b} \cdot \mathbf{b}}$$

Since $\mathbf{a} \cdot \mathbf{a} = 3^2 + 4^2 = 25$ and $\mathbf{b} \cdot \mathbf{b} = 1^2 + 3^2 = 10$,

$$k^2 = \frac{25}{10} \Rightarrow k = \underline{\underline{\pm\sqrt{\frac{5}{2}}}}$$

Example 11 Relative to an origin O, the position vectors of the points A, B, C and D are $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$, $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 8 \end{pmatrix}$ and $\begin{pmatrix} p \\ q \end{pmatrix}$ respectively. Find the relationship between p and q if

(a) A, B and D are collinear,

(b) CD is perpendicular to AB.

Find the value of p and of q, using the answers to (a) and (b).

Solution: **(a)** A, B and D are collinear

$$\Rightarrow \quad \overrightarrow{AD} = k\overrightarrow{AB}$$

$$\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \end{pmatrix} = k\left[\begin{pmatrix} 7 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \end{pmatrix}\right]$$

$$\begin{pmatrix} p-1 \\ q-5 \end{pmatrix} = k\begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

$p - 1 = 6k$ and $q - 5 = -3k$

Eliminating k, we have

$$\underline{\underline{p + 2q = 11}} \quad \text{.................................. (1)}$$

(b) CD is perpendicular to $\overrightarrow{AB}$.

$$\Rightarrow \quad \overrightarrow{CD} . \overrightarrow{AB} = 0$$

$$\left[\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} 5 \\ 8 \end{pmatrix}\right] \cdot \left[\begin{pmatrix} 7 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \end{pmatrix}\right] = 0$$

$$\begin{pmatrix} p-5 \\ q-8 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -3 \end{pmatrix} = 0$$

$$6(p-5) - 3(q-8) = 0$$

$$\underline{\underline{2p - q = 2}} \quad \text{........ (2)}$$

Solving (1) and (2), $\underline{\underline{p = 3}}$ and $\underline{\underline{q = 4}}$.

Note: The point $D(3, 4)$ is the foot of the perpendicular from C to AB as shown in Fig. 24.9.

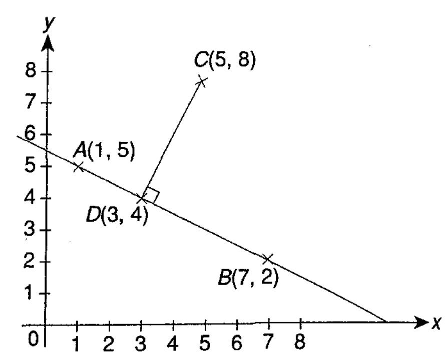


Fig. 24.9

Exercise 24.2 *(Answers on p. 592)*

1. Find the scalar products of the following pairs of vectors.
   (a) $2\mathbf{i} + 3\mathbf{j}$ and $4\mathbf{i} - 5\mathbf{j}$ (b) $\mathbf{i} - 2\mathbf{j}$ and $-\mathbf{i} + 3\mathbf{j}$
   (c) $3\mathbf{i} + 4\mathbf{j}$ and $3\mathbf{i} + 4\mathbf{j}$ (d) $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$
   (e) $\begin{pmatrix} 3 \\ t \end{pmatrix}$ and $\begin{pmatrix} t-1 \\ 2 \end{pmatrix}$

2. Find the angles between the following pairs of vectors.
(a) $2\mathbf{i} + \mathbf{j}$ and $3\mathbf{i} - 2\mathbf{j}$
(b) $\mathbf{i} - 3\mathbf{j}$ and $2\mathbf{i} - \mathbf{j}$
(c) $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$
(d) $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} b \\ -a \end{pmatrix}$

3. Relative to an origin O, the points A, B and C have position vectors $4\mathbf{i} + \mathbf{j}$, $3\mathbf{i} + 5\mathbf{j}$ and $-5\mathbf{i} + 3\mathbf{j}$ respectively. Evaluate $\overrightarrow{AB} \cdot \overrightarrow{BC}$ and hence deduce that the triangle ABC is right-angled at B.

4. The position vectors of A and B relative to an origin O are $2\mathbf{i} + 3\mathbf{j}$ and $-\mathbf{i} + 5\mathbf{j}$ respectively. Evaluate $\overrightarrow{OA} \cdot \overrightarrow{OB}$ and hence, find angle AOB.

5. The position vectors of P and Q relative to an origin O are $2\mathbf{i} - \mathbf{j}$ and $-3\mathbf{i} + 4\mathbf{j}$ respectively. Evaluate $\overrightarrow{OP} \cdot \overrightarrow{OQ}$ and hence, find angle POQ.

6. The position vectors of A and B relative to an origin O are $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 4 \end{pmatrix}$ respectively. Evaluate $\overrightarrow{OA} \cdot \overrightarrow{BA}$ and hence, find angle OAB.

7. The position vector of A relative to an origin O is $2\mathbf{i} + 5\mathbf{j}$. Given that $\overrightarrow{AB} = 3\mathbf{i} + t\mathbf{j}$, express $\overrightarrow{OA} \cdot \overrightarrow{OB}$ in terms of t and hence, find the value of t if the angle $AOB = \frac{\pi}{2}$.

8. Given that the position vectors of A, B and C relative to an origin O are $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$ respectively, evaluate $\overrightarrow{AB} \cdot \overrightarrow{AC}$ and hence, find angle BAC.

9. The position vectors of A, B and P relative to an origin O are $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix}$ respectively. Given that $\overrightarrow{AP} = k\overrightarrow{AB}$, express $\overrightarrow{OP} \cdot \overrightarrow{AB}$ in terms of k and hence, find the value of k for which $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{AB}$ and the value of x and of y.

10. The position vectors of A and B relative to an origin O are $\begin{pmatrix} 3 \\ t \end{pmatrix}$ and $\begin{pmatrix} -1 \\ t-2 \end{pmatrix}$ respectively. Express $\overrightarrow{OA} \cdot \overrightarrow{OB}$ in terms of t and hence find
(a) the possible values of t for which $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$,
(b) the cosine of $\angle AOB$ when $t = 4$.

11. Relative to an origin O, the position vectors of the points A and B are $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} p \\ 10 \end{pmatrix}$ respectively. Express $\overrightarrow{OA} \cdot \overrightarrow{OB}$ in terms of p. Given that $\cos \angle AOB = \frac{2}{\sqrt{5}}$, evaluate p.

12. Given that $\mathbf{a} = \lambda\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} - 3\mathbf{j}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 45°, find the value of λ.

13. Given that $\mathbf{p} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$,

(a) find the value of m and of n such that $m\mathbf{p} + n\mathbf{q} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$,

(b) find the values of k such that $k\mathbf{p} - \mathbf{q}$ is perpendicular to $\mathbf{p} + k\mathbf{q}$.

14. The position vectors of points A, B and C relative to an origin O are $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$ respectively. Write down the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$. Show that $AB = BC$ and calculate $\overrightarrow{AB} \cdot \overrightarrow{BC}$. State the special property of $\triangle ABC$ and deduce its area.

15. The position vectors, relative to an origin O, of three points A, B and C are $2\mathbf{i} + 2\mathbf{j}$, $5\mathbf{i} + 11\mathbf{j}$ and $11\mathbf{i} + 9\mathbf{j}$ respectively.

(a) Given that $\overrightarrow{OB} = m\overrightarrow{OA} + n\overrightarrow{OC}$, where m and n are scalar constants, find the value of m and of n.

(b) Evaluate $\overrightarrow{AB} \cdot \overrightarrow{BC}$ and state the deduction which can be made about $\angle ABC$.

(c) Evaluate $\overrightarrow{AB} \cdot \overrightarrow{AC}$ and hence find $\angle BAC$. *(C)*

16. Find the relationship between p and q if

(a) the vector $\begin{pmatrix} p \\ q \end{pmatrix}$ is perpendicular to the vector $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$,

(b) the vector $\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} 7 \\ 7 \end{pmatrix}$ is a scalar multiple of the vector $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

Relative to an origin O, the position vectors of the points A and B are $\begin{pmatrix} 7 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ respectively. The point C is the foot of the perpendicular from O to AB. Using the answers to (a) and (b), find the position vector of C relative to O. *(C)*

IMPORTANT NOTES AND MISCELLANEOUS EXAMPLES

IMPORTANT NOTES

1. ***Scalar Product of Two Non-Zero Vectors***

We have $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

(a) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ *(commutative law)*

$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ *(distributive law)*

$\lambda(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\lambda\mathbf{b}) = (\lambda\mathbf{a}) \cdot \mathbf{b} = \lambda|\mathbf{a}|\,|\mathbf{b}| \cos \theta$

(b) Cosine of the angle between **a** and **b** is given by

$$\cos\theta = \frac{\mathbf{a}\,.\,\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}.$$

(c) $\mathbf{a}\,.\,\mathbf{b} = 0 \;\Leftrightarrow\;$ **a** and **b** are perpendicular vectors

(d) $\mathbf{a}\,.\,\mathbf{a} = |\mathbf{a}|^2 = a^2$

2. ***Scalar Product of Vectors in the Cartesian Plane***

Suppose $\mathbf{a} = p\mathbf{i} + q\mathbf{j}$ and $\mathbf{b} = r\mathbf{i} + s\mathbf{j}$

or we write $\mathbf{a} = \begin{pmatrix} p \\ q \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} r \\ s \end{pmatrix}$, then,

(a) $\mathbf{a}\,.\,\mathbf{b} = pr + qs$

(b) $$\cos\theta = \frac{\mathbf{a}\,.\,\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{pr + qs}{\sqrt{p^2 + q^2}\,\sqrt{r^2 + s^2}}$$

(c) **a** and **b** are perpendicular $\Leftrightarrow \mathbf{a}\,.\,\mathbf{b} = 0$
$\Leftrightarrow pr + qs = 0$

(d) $\mathbf{a}\,.\,\mathbf{a} = p^2 + q^2$

MISCELLANEOUS EXAMPLES

Example 12 Given that the vectors $\mathbf{p} = \begin{pmatrix} 3 \\ t \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} t-1 \\ 2 \end{pmatrix}$, find the value(s) of t for which

(a) **p** and **q** are parallel vectors,

(b) **p** and **q** are perpendicular vectors.

Solution:

(a) **p** and **q** are parallel vectors $\Rightarrow$ $\mathbf{p} = k\mathbf{q}$ for some scalar k.

Then,

$$\begin{pmatrix} 3 \\ t \end{pmatrix} = k\begin{pmatrix} t-1 \\ 2 \end{pmatrix}$$

$\Rightarrow \quad 3 = k(t - 1)$ .... (1)

$t = 2k$ .... (2)

Eliminating k from (1) and (2), we have

$$t^2 - t - 6 = 0$$
$$(t + 2)(t - 3) = 0$$
$$t = \underline{\underline{-2 \text{ or } 3}}$$

(b) **p** and **q** are perpendicular vectors $\Rightarrow$ $\mathbf{p}\,.\,\mathbf{q} = 0$.
Then,

$$\begin{pmatrix} 3 \\ t \end{pmatrix} \cdot \begin{pmatrix} t-1 \\ 2 \end{pmatrix} = 0 \quad \Rightarrow \; 3(t-1) + 2t = 0$$

$$t = \underline{\underline{\tfrac{3}{5}}}$$

Example 13 Relative to an origin O, the position vectors of the points A and B are $\begin{pmatrix} p \\ q \end{pmatrix}$ and $\begin{pmatrix} q \\ p+1 \end{pmatrix}$ respectively where $q \neq 0$.
Find an equation connecting p and q in each of the following cases.
(a) O, A and B are collinear **(b)** $\angle OAB = 90°$

Solution:

(a) O, A and B are collinear $\Rightarrow \overrightarrow{OA} = k\overrightarrow{OB}$ for some scalar k.
Then,

$$\begin{pmatrix} p \\ q \end{pmatrix} = k\begin{pmatrix} q \\ p+1 \end{pmatrix}$$

$$\Rightarrow p = kq \quad \text{.. (1)}$$

$$q = k(p+1) \quad \text{.............................. (2)}$$

Eliminating k from (1) and (2), we have

$$q = \frac{p}{q}(p+1)$$

$$\underline{\underline{q^2 = p(p+1)}}$$

(b) $\angle OAB = 90° \;\Rightarrow \overrightarrow{OA}\,.\,\overrightarrow{AB} = 0$
Then,

$$\overrightarrow{OA}\,.\,(\overrightarrow{OB} - \overrightarrow{OA}) = 0$$

$$\overrightarrow{OA}\,.\,\overrightarrow{OB} = \overrightarrow{OA}\,.\,\overrightarrow{OA}$$

$$\begin{pmatrix} p \\ q \end{pmatrix} \cdot \begin{pmatrix} q \\ p+1 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix}$$

$$pq + q(p+1) = p^2 + q^2$$

$$q = p^2 + q^2 - 2pq$$

$$\underline{\underline{q = (p-q)^2}}$$

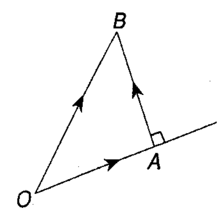


Fig. 24.10

Example 14 The vertices of the triangle ABC have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively relative to the origin O. State the type of triangle in each of the following cases.

(a) $|\mathbf{c} - \mathbf{a}| = |\mathbf{c} - \mathbf{b}|$ **(b)** $(\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{b}) = 0$

Solution:

(a) $|\mathbf{c} - \mathbf{a}| = |\mathbf{c} - \mathbf{b}| \Rightarrow |\overrightarrow{AC}| = |\overrightarrow{BC}|$

$\Rightarrow \triangle ABC$ is <u>isosceles.</u>

(b) $(\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{b}) = 0$

$\Rightarrow \overrightarrow{AC} \cdot \overrightarrow{BC} = 0$

$\Rightarrow \angle ACB = 90°$

$\Rightarrow \triangle ABC$ is <u>right-angled at C.</u>

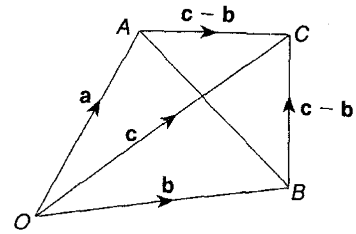


Fig. 24.11

Miscellaneous Exercise 24

(Answers on p. 592)

1. Find the relationship between p and q if

(a) the vector $\begin{pmatrix} p \\ 3 \end{pmatrix}$ is perpendicular to the vector $\begin{pmatrix} 4 \\ q \end{pmatrix}$,

(b) the vector $\begin{pmatrix} p \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ q \end{pmatrix}$ is perpendicular to the vector $\begin{pmatrix} p \\ 3 \end{pmatrix}$,

(c) the vector $\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is parallel to the vector $\begin{pmatrix} 2p \\ q \end{pmatrix}$,

(d) the vector $\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is perpendicular to the vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$,

(e) the vector $\begin{pmatrix} p \\ q \end{pmatrix}$ has a magnitude of 5.

2. Relative to an origin O, the points A, B and C have position vectors $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$ respectively.

(a) Evaluate $\overrightarrow{AB} \cdot \overrightarrow{AC}$ and $\overrightarrow{AB} \cdot \overrightarrow{BC}$.

(b) Find the angles BAC and ABC.

3. (a) Given that $\mathbf{p} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, find a scalar k such that $(\mathbf{q} - k\mathbf{p})$ is perpendicular to $\mathbf{q}$.

(b) Find the vector of unit magnitude in the same direction as the vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

(c) Find a unit vector perpendicular to the vector $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$.

4. Find the relationship between p and q if

(a) the vector $\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ is perpendicular to the vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$,

(b) the vector $\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ is parallel to the vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

Given that the position vectors of the points A, B and C are $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 9 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ respectively and the point D is the foot of the perpendicular from A to BC. Find, using the results in (a) and (b), the position vector of D.

5. Given that $\mathbf{p} = \begin{pmatrix} a \\ 8 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 8 \\ b \end{pmatrix}$, find

(a) the value of a and of n such that $\mathbf{p} + n\mathbf{q} = \begin{pmatrix} 2 \\ 16 \end{pmatrix}$,

(b) the value of b such that $\mathbf{r}$ is perpendicular to $\mathbf{q}$.

Using your value of a,

(c) find the value of k such that $|\mathbf{p}| = k(|\mathbf{p} + \mathbf{q}| - |\mathbf{q}|)$,

(d) evaluate $\mathbf{p} \cdot \mathbf{q}$ and hence find the angle between $\mathbf{p}$ and $\mathbf{q}$. *(C)*

6. Given that $\mathbf{a} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 11 \\ 17 \end{pmatrix}$.

(a) Show that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular vectors.

(b) Find the value of the constants m and n for which $m\mathbf{a} + n\mathbf{b} = \mathbf{c}$.

(c) Find the cosine of the angle between the direction of $\mathbf{a}$ and the direction of $\mathbf{b} - \mathbf{a}$.

7. The position vectors of two points A and B relative to an origin O are $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ respectively. Evaluate $\overrightarrow{OA} \cdot \overrightarrow{OB}$ and hence find $\angle AOB$. A point C has position vector $\overrightarrow{OC} = m\overrightarrow{OA} + n\overrightarrow{OB}$ where m and n are constants. Find the relationship between m and n if

(a) $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{OA}$, (b) $\overrightarrow{BC}$ is parallel to $\overrightarrow{OA}$.

8. **(a)** The projection of the vector $\overrightarrow{OP}$ on the vector $\overrightarrow{OA}$ is defined to be $|\overrightarrow{OP}| \cos \angle POA$. Given that $\overrightarrow{OP} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, evaluate $\overrightarrow{OP} \cdot \overrightarrow{OA}$ and hence find the length of the projection of $\overrightarrow{OP}$ on the vector $\overrightarrow{OA}$.

(b) Given the vectors $\mathbf{p} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 2 \\ t \end{pmatrix}$, find the value or values of t for which

(i) $\mathbf{p} + \mathbf{q}$ is parallel to $\mathbf{p} - \mathbf{q}$,

(ii) $\mathbf{p} - 2\mathbf{q}$ is perpendicular to $\mathbf{p} + 2\mathbf{q}$,

(iii) $|\mathbf{p} - \mathbf{q}| = |\mathbf{q}|$.

9. Relative to an origin O, the position vectors of two points A and B are $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 1 \end{pmatrix}$ respectively. The point C lies on AB, between A and B. Given that the position vector of C is $\begin{pmatrix} t^2 \\ t \end{pmatrix}$, find the value of t and the ratio $AC : CB$.

10. The three non-collinear points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively relative to an origin O. The point P is such that $\overrightarrow{AP} = k\overrightarrow{AB}$. Determine, in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and k,

(a) the position vector of P,

(b) the condition that the line segment CP is perpendicular to the line segment CA. Hence evaluate k when $\angle ACP = 90°$ and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are $\begin{pmatrix} 3 \\ 9 \end{pmatrix}$, $\begin{pmatrix} 8 \\ 8 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ respectively. *(C)*

***11.** The points A, B and P have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{p}$ respectively relative to an origin O and AOB is a right angle. Given that $\overrightarrow{AP} = k\overrightarrow{AB}$, show that $\cos \angle AOP = \dfrac{(1 - k)\,|\mathbf{a}|}{|\mathbf{p}|}$ and find a similar expression for $\cos \angle BOP$. If $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and $\angle AOP = \angle BOP$, find the value of k.

12. **(a)** Relative to an origin O, the position vectors of A and B are $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 12 \\ 5 \end{pmatrix}$ respectively. Find the unit vectors parallel to $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Hence, by using a vector method, find the cosine of the angle between the directions of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

(b) If $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$ and given that $OPRQ$ is a parallelogram, find in terms of $\mathbf{p}$ and $\mathbf{q}$, $\overrightarrow{OR}$ and $\overrightarrow{PQ}$. Evaluate $\overrightarrow{OR} \cdot \overrightarrow{PQ}$ and hence, write down, in terms of $\mathbf{p}$ and $\mathbf{q}$, a necessary and sufficient condition if $OPRQ$ is to be a rhombus.

13. The position vectors of two points A and B relative to an origin O are $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ respectively. Evaluate $\overrightarrow{OA} \, . \, \overrightarrow{OB}$ and deduce the acute angle between the line $2y = x$ and $y = 3x$.

***14.** The position vectors of the points A, B and R relative to an origin O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{r}$ respectively. State, in relation to the points A and B, the geometrical property of the position of R in each of the following cases.

(a) $|\mathbf{r} - \mathbf{a}| = |\mathbf{r} - \mathbf{b}|$

(b) $(\mathbf{r} - \mathbf{a}) \, . \, (\mathbf{r} - \mathbf{b}) = 0$

(c) $\mathbf{r} = \lambda(\mathbf{a} + \mathbf{b})$

15. If the vectors $\mathbf{a}$ and $\mathbf{a} + \mathbf{b}$ are perpendicular, show that

(a) $\mathbf{a} \, . \, \mathbf{b} = -|\mathbf{a}|^2$, (b) $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{b}|^2 - |\mathbf{a}|^2$.

***16.** The points P and R have position vectors $2\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - 3\mathbf{b}$ respectively, relative to an origin O. Given that $OPQR$ is a parallelogram, express the vectors $\overrightarrow{OQ}$ and $\overrightarrow{RP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. By evaluating two scalar products, show that if $OPQR$ is a square, then $|\mathbf{a}|^2 = 2|\mathbf{b}|^2$.

CHAPTER 23

Exercise 23.1 *(p. 481)*

1. **(a)** $\overrightarrow{AP}$ **(b)** $\overrightarrow{AB}$ **(c)** $\overrightarrow{AD}$ **3.** $\frac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA})$, $\frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$

4. **(a)** $\frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$, $-\frac{3}{2}\overrightarrow{OA} + \overrightarrow{OB}$, $-2\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$

(b) $\frac{1}{2}\overrightarrow{OC} + 2\overrightarrow{OA}$, $\overrightarrow{OC} + \frac{3}{2}\overrightarrow{OA}$, $\frac{1}{2}\overrightarrow{OC} - \frac{1}{2}\overrightarrow{OA}$

5. $2\overrightarrow{OB}$ **6.** **(a)** $\mathbf{p} + \mathbf{q}$ **(b)** $\mathbf{p} - \mathbf{q}$ **(c)** $\mathbf{q} - \frac{1}{2}\mathbf{p}$; $\sqrt{2}$ units

8. $\mathbf{q} - \sqrt{2}\,\mathbf{p}$ **9.** **(a)** $\overrightarrow{AD}$ **(b)** $2\overrightarrow{AD}$ **(c)** $\frac{1}{2}\overrightarrow{AD}$ **(d)** $5\frac{1}{2}\overrightarrow{AD}$

10. **(a)** $4\mathbf{q} - 2\mathbf{p}$ **(b)** $3(\mathbf{q} - \mathbf{p})$

Exercise 23.2 *(p. 485)*

1. $\frac{1}{2}(\mathbf{a} + \mathbf{b})$, $\mathbf{b} - \frac{1}{2}\mathbf{a}$, $\mathbf{b} - \frac{3}{2}\mathbf{a}$ **2.** $\mathbf{p} + \mathbf{q}$, $2\mathbf{p} + \mathbf{q}$, $\mathbf{p} + 2\mathbf{q}$, $\mathbf{q} - \mathbf{p}$

3. $s = \frac{4}{7}$, $t = -\frac{5}{7}$ **4.** -1

5. **(a)** $\left(1 - \frac{1}{2}k\right)\mathbf{a} + k\mathbf{c}$ **(b)** $n\mathbf{c}$; $n = 2$, $k = 2$

6. **(a)** $(1 - k)\mathbf{a} + k\mathbf{b}$ **(b)** $n\mathbf{a} + 2n\mathbf{b}$; $n = \frac{1}{3}$, $k = \frac{2}{3}$

7. $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{PQ} = 2\mathbf{b} - \frac{2}{3}\mathbf{a}$

(a) $\frac{2}{3}(1 - n)\mathbf{a} + 2n\mathbf{b}$ **(b)** $(1 - k)\mathbf{a} + k\mathbf{b}$; $n = \frac{1}{4}$, $k = \frac{1}{2}$

Exercise 23.3 *(p. 490)*

1. $\frac{1}{3}(2\mathbf{a} + \mathbf{b})$, $\frac{1}{3}(\mathbf{a} + 2\mathbf{b})$ **2.** $m\mathbf{a} + \mathbf{b}$ **3.** $2(\mathbf{a} - \mathbf{b})$, $5(\mathbf{a} - \mathbf{b})$

4. $\overrightarrow{AB} = 2\mathbf{p} - 3\mathbf{q}$, $\overrightarrow{AC} = 5\mathbf{p} + (k - 1)\mathbf{q}$; $k = -\frac{13}{2}$, $\lambda = \frac{2}{5}$

6. $(\lambda - 1)\mathbf{p} + (\lambda + 1)\mathbf{q}$, $(\lambda + 1)\mathbf{q}$, $\lambda = 1$

7. **(a)** $\frac{1}{2}(\mathbf{p} + \mathbf{q})$ **(b)** $\mathbf{q} - \mathbf{p}$ **(c)** $\frac{1}{2}\mathbf{p} - \mathbf{q}$

8. **(a)** $(1 - n)k\mathbf{a} + \frac{1}{2}n\mathbf{b}$ **(b)** $\frac{n - 2}{2(n - 1)}$

9. **(a)** $(m + 1)\mathbf{q} - m\mathbf{p}$ **(b)** $(3 + n)\mathbf{q} - 9\mathbf{p}$; $m = 9$, $n = 7$, $10\mathbf{q} - 9\mathbf{p}$

10. $\overrightarrow{AQ} = 2\mathbf{b} - \mathbf{a}$, $\overrightarrow{BP} = 3\mathbf{a} - \mathbf{b}$

(a) $(1 - \lambda)\mathbf{a} + 2\lambda\mathbf{b}$ **(b)** $3\mu\mathbf{a} + (1 - \mu)\mathbf{b}$; $\lambda = \frac{2}{5}$, $\mu = \frac{1}{5}$; $\frac{3}{5}\mathbf{a} + \frac{4}{5}\mathbf{b}$

11. $\frac{2}{5}(1 - l)\mathbf{a} + \frac{2}{5}l\,\mathbf{b}$, $\frac{4}{5}(\mathbf{b} - k\mathbf{a})$; $k = \frac{1}{4}$, $l = \frac{1}{2}$

12. $\overrightarrow{OG} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{d}$, $\overrightarrow{OC} = -\frac{1}{2}\mathbf{a} + \mathbf{b} + \frac{1}{2}\mathbf{d}$, $\overrightarrow{OE} = \mathbf{a} - \mathbf{b} + \mathbf{d}$, $\overrightarrow{OF} = \frac{3}{2}\mathbf{a} - \mathbf{b} + \frac{1}{2}\mathbf{d}$

13. $\overrightarrow{OD} = 3(\mathbf{p} - 2\mathbf{q})$, $\lambda = -4$ **14.** $\overrightarrow{OD} = \mathbf{a} + \mathbf{c} - \mathbf{b}$, $\overrightarrow{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$

Exercise 23.4 *(p. 494)*

1. (a) $5\mathbf{i}+\mathbf{j}$ (b) $13\mathbf{i}+4\mathbf{j}$ (c) $-3\mathbf{j}$ (d) $\mathbf{j}$
2. (a) $\begin{pmatrix}3\\4\end{pmatrix}$ (b) $\begin{pmatrix}4\\1\end{pmatrix}$ (c) $\begin{pmatrix}12\\-1\end{pmatrix}$ (d) $\begin{pmatrix}-5\\12\end{pmatrix}$
3. (a) $5,\ \frac{1}{5}(3\mathbf{i}+4\mathbf{j}),\ 13(3\mathbf{i}-4\mathbf{j})$ (b) $13,\ \frac{1}{13}(5\mathbf{i}-12\mathbf{j}),\ 5(5\mathbf{i}-12\mathbf{j})$
4. (a) $m=\frac{2}{5},\ n=\frac{11}{5}$ (b) $m=3,\ n=-2$
5. (a) $\begin{pmatrix}-7\\8\end{pmatrix}$ (b) $\begin{pmatrix}2-3\lambda\\3\lambda-1\end{pmatrix},\ \lambda=\frac{7}{15}$
6. (a) $\frac{1}{\sqrt{41}}\begin{pmatrix}5\\4\end{pmatrix}$ (b) $m=3,\ n=1$
7. $q=2p,\ p=\sqrt{5},\ q=2\sqrt{5}$
8. (a) $3q-p=12$ (b) $p^2+q^2-2p=3$
9. (a) $\begin{pmatrix}5\\1\end{pmatrix}$ (b) $\begin{pmatrix}-5\\6\end{pmatrix}$
10. $\begin{pmatrix}3\\6\end{pmatrix},\begin{pmatrix}-1\\2\end{pmatrix}$ (a) $\begin{pmatrix}4p-1\\4p+2\end{pmatrix}$ (b) $\begin{pmatrix}1-4q\\2+4q\end{pmatrix};\ p=\frac{1}{4},\ q=\frac{1}{4}$
11. $\hat{\mathbf{a}}=\begin{pmatrix}1\\0\end{pmatrix},\ \hat{\mathbf{b}}=\frac{1}{5}\begin{pmatrix}3\\4\end{pmatrix};\ \lambda=\frac{15}{4}$

Miscellaneous Exercise 23 *(p. 497)*

1. (a) $k=-\frac{31}{3},\ 3:8$ (b) $\frac{1}{3}(4\mathbf{a}-3\mathbf{b})$
2. $1:2,\ \overrightarrow{OD}=5\mathbf{q}-2\mathbf{p},\ \overrightarrow{OP}=2(k-1)\mathbf{p}+(5-k)\mathbf{q},\ k=\frac{1}{3}$
3. $\overrightarrow{AB}=\mathbf{b}-\mathbf{a},\ \overrightarrow{AC}=3\mathbf{b}-\mathbf{a};\ \overrightarrow{OX}=\frac{\mathbf{a}+2\mathbf{b}}{3},\ \overrightarrow{OY}=\frac{\mathbf{a}+3k\mathbf{b}}{1+k};\ k=\frac{2}{3}$
4. $\frac{1-k}{2k}\mathbf{a}+\frac{1}{2}\mathbf{b},\ k=\frac{1}{6},\ 1:1$
5. $\begin{pmatrix}0\\3\end{pmatrix}$
6. $\overrightarrow{OP}=\begin{pmatrix}2\\1\end{pmatrix},\ \overrightarrow{OQ}=\begin{pmatrix}6\\-2\end{pmatrix},\ \overrightarrow{OR}=\begin{pmatrix}4+5k\\2-5k\end{pmatrix},\ k=2$
7. $\overrightarrow{OM}=\lambda(\mathbf{a}-2\mathbf{b}),\ \lambda=3;\ 8$
8. (a) (i) 1 (ii) $\frac{1}{2}$
   (b) (i) $2k\mathbf{p}-3k\mathbf{q}$ (ii) $(2m+1)\mathbf{p}+(3-5m)\mathbf{q};\ k=\frac{11}{4},\ m=\frac{9}{4},\ \overrightarrow{OX}=\frac{11}{2}\mathbf{p}-\frac{33}{4}\mathbf{q}$
9. (a) $\frac{1}{5}\begin{pmatrix}3\\-4\end{pmatrix}$, $10\mathbf{n}$ (b) $\begin{pmatrix}15\\36\end{pmatrix}$ (c) $\begin{pmatrix}4\\-3\end{pmatrix},\ \frac{1}{5}\begin{pmatrix}4\\-3\end{pmatrix}$
10. $\frac{4}{3}\mathbf{a},\ \frac{1}{6}(2\mathbf{a}+\mathbf{b}),\ m=1,\ n=\frac{1}{6}$
11. $\overrightarrow{OP}=3\mathbf{a}-\mathbf{b},\ \overrightarrow{OQ}=2(\mathbf{a}-\mathbf{b}),\ \overrightarrow{OX}=3\mathbf{a}+4\mathbf{b},\ \overrightarrow{OY}=4\mathbf{a}+3\mathbf{b}$
12. $\frac{1}{3}(\mathbf{a}+2\mathbf{b}),\ \frac{1}{7}(k\mathbf{a}+6\mathbf{b})$ (a) 3 (b) $\frac{7}{2}$
13. $\overrightarrow{OP}=3\mathbf{b}-2\mathbf{a},\ \overrightarrow{OQ}=2\mathbf{a}-3\mathbf{b},\ 2:3$

CHAPTER 24

Exercise 24.1 *(p. 505)*

1. (a) 80° (b) 110° (c) 120°
2. (a) 120° (b) 120° (c) 120° (d) 180°
3. (a) 0 (b) $4a^2$, 26.57° (c) $-2a^2$
4. (a) 18 (b) 5 (c) 0

8. **(a)** $\frac{5}{12}$ **(b)** $\sqrt{15}$ **(c)** $\frac{4}{3\sqrt{15}}$ **9.** **(a)** $-\frac{3}{2}$ **(b)** $\sqrt{10}$

11. $\frac{1}{4}(|\mathbf{a}|^2 - |\mathbf{b}|^2)$ **12.** $\frac{1}{4}$

Exercise 24.2 *(p. 509)*

1. **(a)** -7 **(b)** -7 **(c)** 25 **(d)** 0 **(e)** $5t - 3$
2. **(a)** $60.26°$ **(b)** $45°$ **(c)** $90°$ **(d)** $90°$ **3.** 0 **4.** $13, 45°$
5. $-10, 153.4°$ **6.** $8, 119.7°$ **7.** $35 + 5t, t = -7$ **8.** $4, 78.7°$
9. $10k - 5, k = \frac{1}{2}, x = -\frac{1}{2}, y = \frac{3}{2}$ **10.** $t^2 - 2t - 3$; **(a)** $-1, 3$ **(b)** $\frac{1}{\sqrt{5}}$
11. $4p + 30; p = 5, 55$ **12.** $\frac{1}{7}$ **13.** **(a)** $m = 3, n = -1$ **(b)** $-1 \pm \sqrt{2}$
14. $\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \end{pmatrix}, 0, 12.5$ **15.** **(a)** $m = 19, n = -3$ **(b)** $0, 90°$ **(c)** $90, 33.69°$
16. **(a)** $2p + q = 0$ **(b)** $p - 2q + 7 = 0; \begin{pmatrix} -1.4 \\ 2.8 \end{pmatrix}$

Miscellaneous Exercise 24 *(p. 514)*

1. **(a)** $4p + 3q = 0$ **(b)** $p^2 - 4p - 3q + 9 = 0$ **(c)** $2p - 2q = pq$
(d) $3p - q = 8$ **(e)** $p^2 + q^2 = 25$
2. **(a)** $15, 5$ **(b)** $45°, 108.4°$
3. **(a)** 2 **(b)** $\frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **(c)** $\frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ or $-\frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$
4. **(a)** $4p + 3q + 2 = 0$ **(b)** $3p = 4q + 1$; $D(1, -2)$
5. **(a)** $a = 8, n = 2$ **(b)** 6 **(c)** $\sqrt{2}$ **(d)** $8, 81.87°$
6. **(b)** $m = 4, n = 5$ **(c)** $-\frac{1}{\sqrt{2}}$
7. $10, 45°$ **(a)** $m + n = 1$ **(b)** $n = 1$
8. **(a)** $30, 6$ **(b)** **(i)** $\frac{6}{5}$ **(ii)** $\pm\sqrt{\frac{9}{2}}$ **(iii)** $\frac{7}{3}$
9. $t = 2, 2:1$
10. **(a)** $(1 - k)\mathbf{a} + k\mathbf{b}$ **(b)** $\{(1 - k)\mathbf{a} + k\mathbf{b} - \mathbf{c}\} . (\mathbf{a} - \mathbf{c}) = 0, k = 2.5$
11. $\frac{k|\mathbf{b}|}{|\mathbf{p}|}, k = \frac{1}{3}$
12. **(a)** $\pm\frac{1}{5}\begin{pmatrix} 4 \\ -3 \end{pmatrix}, \pm\frac{1}{13}\begin{pmatrix} 12 \\ 5 \end{pmatrix}, \frac{33}{65}$ **(b)** $|\mathbf{q}|^2 - |\mathbf{p}|^2, |\mathbf{p}| = |\mathbf{q}|$ **13.** $5; 45°$
14. **(a)** R is equidistant from A and B. **(b)** $A\hat{R}B$ is $90°$
(c) R lies on the line passing through O and the midpoint of AB.
16. $3\mathbf{a} - 2\mathbf{b}, \mathbf{a} + 4\mathbf{b}$

Revision Exercise 17 *(p. 518)*

1. **(a)** $y = 2\sqrt{x} - x^2 + 10$ **(b)** **(i)** $\frac{1}{8}(2x - 1)^4 + c$ **(ii)** $-\frac{2}{3}(4 - x)^{\frac{3}{2}} + c$
(c) **(i)** $\frac{1}{2}e^{2x} + \frac{1}{2}x^2 + c$ **(ii)** $-\frac{1}{2}\cos 2x + \frac{1}{3}\sin 3x + c$
2. **(a)** **(i)** 39 **(ii)** 3.89 **(b)** $\frac{2}{\sqrt{5}}$
3. **(a)** $2\ \text{m s}^{-1}$ **(b)** $5.5\ \text{m s}^{-2}$ **(c)** 29.8 m
4. **(a)** $\frac{5}{6}$ sq. unit **(b)** $\frac{8}{15}\pi$ cu. units
5. **(a)** $-5, 99.2°$ **(b)** $2; 1:2$