HW 7.6 1-13 odd

[7.6] Solve First Order Differential Equation (non separable)

$\frac{dy}{dx} + P(x)y = Q(x)$ It's first-order because highest derivative is first derivative. And linear because exactly this form.

eg: $\frac{dy}{dx} + \frac{2}{x}y = \frac{\sin 3x}{x^2}$ can't ever separate this.

So, solve $\underbrace{\frac{dy}{dx} + P(x)}_{\text{sum of derivatives}} = Q(x)$ we'll discover an "integrating factor"

strategy: write LHS as a product. so we can divide both sides by whatever is multiplying y.

Write LHS as deriv of a product. Recall $(uv)' = u'v + uv'$

then $\frac{d}{dx}(uy) = Q(x)$

$d(uy) = Q(x)\,dx$ mult both sides by dx

$\int d(uy) = \int Q(x)\,dx$

$uy = \int Q(x)\,dx$

$y = \frac{\int Q(x)\,dx}{u}$ wow.

So $\frac{dy}{dx} + P(x)y = Q(x)$

wish to mult both sides by a fn, so that LHS becomes the derivative of a product of two fns, one of them y.

$$\frac{dy}{dx} f + \underbrace{f\,P(x)}\,y = f\,Q(x)$$

a product, not composed

notice f instead of f(x) for simplicity

$$u'v + v'u = f\,Q(x)$$

use deriv of product $(uv)' = u'v + uv'$

clever. Now pick precisely the right fn for f.

Libby + "Mischa": "the deriv of f has to be the same as f times P(x)"

RT: "f must appear in its own derivative"

That would be

$$(e^u)' = e^u \frac{du}{dx} \quad !$$

$$\frac{dy}{dx} e^u + \underbrace{e^u P(x)}_{\text{deriv}}\,y = f\,Q(x)$$

And then where does P(x) come from?

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}$$

$$\Rightarrow \frac{du}{dx} = P(x)$$

well, $\int P(x)\,dx = P(x)$

special name for $e^{\int P(x)dx}$
is the integrating factor.

In general, we do this:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\frac{dy}{dx} e^{\int P(x)dx} + e^{\int P(x)dx} P(x)y = Q(x)e^{\int P(x)dx}$$

$$d\left(e^{\int P(x)dx} y\right) = Q(x)e^{\int P(x)dx}\,dx$$

$$y = \frac{\int Q(x)e^{\int P(x)dx}\,dx}{e^{\int P(x)dx}}$$

Example. Solve this: Solution is a family of infinitely many functions.

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{\sin 3x}{x^2}$$ doing what we did just now

$e^{\int \frac{2}{x} dx}$ is the integrating factor

$$= e^{2\ln x}$$
$$= e^{\ln x^2}$$ (arbitrary constant makes no difference)

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{\sin 3x}{x^2}$$

$$\frac{dy}{dx} e^{\ln x^2} + e^{\ln x^2} \frac{2}{x} y = e^{\ln x^2} \frac{\sin 3x}{x^2}$$

$$\frac{dy}{dx} x^2 + 2xy = \sin 3x$$

$$\frac{d}{dx}\left(x^2 y\right) = \sin 3x$$

$$d\left(x^2 y\right) = \sin 3x \, dx$$

$$\int d\left(x^2 y\right) = \int \sin 3x \, dx$$

$$x^2 y = -\frac{1}{3}\cos x + 3$$

This is known as the general solution. Includes constant as C.

$$y = -\frac{1}{3}\cos x \cdot x^{-2} + Cx^{-2}$$

An initial conditions problem, which will give particular solution.

Find y that makes this true:

$$\frac{dy}{dx} - 3y = xe^{3x} \quad \text{where } \left.\begin{matrix} x=0 \\ y=4 \end{matrix}\right] \text{initial conditions}$$

into exact form: $\frac{dy}{dx} + (-3)y = xe^{3x}$

$$I(x) = e^{\int -3dx}$$
$$= e^{-3x}$$

$$\frac{dy}{dx} \underset{f}{e^{-3x}} y + \underbrace{(-3)e^{-3x}}_{f'} y = xe^{3x}e^{-3x}$$

$$\frac{d}{dx}\left(e^{-3x}y\right) = x$$

$$\int d\left(e^{-3x}y\right) = \int x\,dx$$

$$e^{-3x}y = \frac{1}{2}x^2 + c$$
$$y = \frac{1}{2}e^{3x}x^2 + e^{3x}c$$

Then $4 = \frac{1}{2}e^{3(0)}(0)^2 + e^{3x}c$

$4 = c$

$\therefore\ y = \frac{1}{2}e^{3x}x^2 + 4e^{3x}$