[08-07-23-KC]

Limit proofs sequences

■ Prove the following

[1] Show that the limit of $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = \frac{2}{\sqrt{n}}$ is 0.

[2] Show that the limit of $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = \frac{2n+1}{1-3n}$ is $\frac{-2}{3}$.

[3] Show that the limit of $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = \frac{n^2+1}{n^2+100n}$ is 1.

[1] $\lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$

Pre. Find M s.t. for $n > M$, $\left|\frac{2}{\sqrt{n}} - 0\right| < \epsilon$, $\epsilon > 0$.

$$\left|\frac{2}{\sqrt{n}} - 0\right| = \frac{2}{\sqrt{n}} < \frac{2}{\sqrt{M}}$$

$$\frac{2}{\sqrt{M}} = \epsilon \Rightarrow M = \frac{4}{\epsilon^2}.$$

<u>Proof</u> Fix $\epsilon > 0$. Choose $M = \frac{4}{\epsilon^2}$. If $n > M$, then

$$\left|\frac{2}{\sqrt{n}} - 0\right| = \frac{2}{\sqrt{n}} < \frac{2}{\sqrt{M}} = \frac{2}{\sqrt{\frac{4}{\epsilon^2}}} = \epsilon.$$

$\square$

Prove $\displaystyle\lim_{n\to\infty} \frac{2n+1}{1-3n} = \frac{-2}{3}$

Pre for all $\epsilon > 0$, Find M such that if $n > M$,

$$\left| \frac{2n+1}{1-3n} + \frac{2}{3} \right| < \epsilon.$$

$$\left| \frac{2n+1}{1-3n} + \frac{2}{3} \right| = \left| \frac{6n+3+2-6n}{3-9n} \right| = \left| \frac{5}{3-9n} \right| = \left| \frac{5}{9n-3} \right|$$

$$= \frac{5}{9n-3} = \underbrace{\frac{5}{6n+3n-3} \leqslant \frac{5}{6n}}_{\substack{\text{Least } 3n-3 \text{ can} \\ \text{be is } 3(1)-3=0, \\ \text{for } n \geq 2, \text{ denom} \\ \text{is greater than } 6n}} \underbrace{< \frac{1}{n}}_{\substack{\frac{5}{6} \text{ of } \frac{1}{n} \text{ is less} \\ \text{than } \frac{1}{n}.}} < \frac{1}{M}$$

Now, $\frac{1}{M} = \epsilon \Rightarrow M = \frac{1}{\epsilon}$.

Proof Fix $\epsilon > 0$. Choose $M = \frac{1}{\epsilon}$. If $n > M$,

$$\left| \frac{2n+1}{1-3n} + \frac{2}{3} \right| = \left| \frac{5}{3-9n} \right| = \left| \frac{5}{9n-3} \right| = \frac{5}{9n-3}$$

$$= \frac{5}{6n+3n-3} \leqslant \frac{5}{6n} < \frac{1}{n} < \frac{1}{M} = \frac{1}{\frac{1}{\epsilon}} = \epsilon.$$

$\square$

[3] Prove $\lim_{n\to\infty} \frac{n^2+1}{n^2+100n} = 1$

Pre Find M s.t. for all $\epsilon > 0$, $n > M$ implies

$$\left| \frac{n^2+1}{n^2+100n} - 1 \right| < \epsilon .$$

$$\left| \frac{n^2+1}{n^2+100n} - 1 \right| = \left| \frac{n^2+1-n^2-100n}{n^2+100n} \right| = \left| \frac{1-100n}{n^2+100n} \right|$$

$$= \frac{|100n-1|}{|n^2+100n|} = \frac{100n-1}{n^2+100n} < \frac{100n}{n^2+100n} < \frac{100n}{n^2} = \frac{100}{n} < \frac{100}{M}$$

$$\frac{100}{M} = \epsilon \Rightarrow M = \frac{100}{\epsilon} .$$

Proof Let $\epsilon > 0$ be given. Choose $M = \frac{100}{\epsilon}$.

If $n > M$, then

$$\left| \frac{n^2+1}{n^2+100n} - 1 \right| = \left| \frac{|100n-1|}{|n^2+100|} \right| = \frac{100n-1}{n^2+100n} < \frac{100}{n^2} < \frac{100}{n}$$

$$< \frac{100}{M} = \frac{100}{\frac{100}{\epsilon}} = \epsilon .$$

$\square$