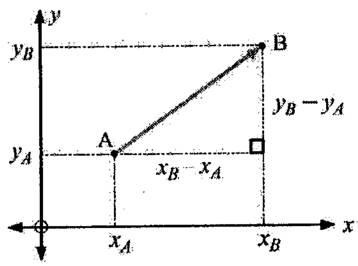
[07-11-13 - HH-Phy-B]
RT

G VECTORS IN COORDINATE GEOMETRY

VECTORS BETWEEN TWO POINTS



In 2-D: consider points $A(x_A, y_A)$ and $B(x_B, y_B)$

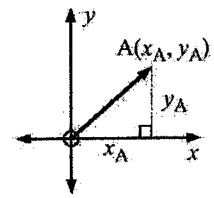
In going from A to B,

$x_B - x_A$ is the x-**step**, and

$y_B - y_A$ is the y-**step**.

Consequently

$$\overrightarrow{AB} = \begin{bmatrix} x_B - x_A \\ y_B - y_A \end{bmatrix}.$$



Notice that if O is (0, 0) and A is (x_A, y_A) then $\overrightarrow{OA}$ is $\begin{bmatrix} x_A \\ y_A \end{bmatrix}$.

In 3-D: if the points are $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$

$$\overrightarrow{AB} = \begin{bmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{bmatrix} \quad \overrightarrow{OA} = \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} \quad \text{and note} \quad \begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= \mathbf{b} - \mathbf{a} \end{aligned}$$

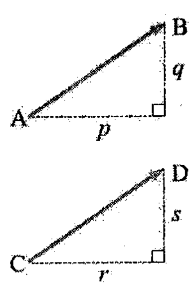
where **a**, **b** are the position vectors of A and B respectively.

DISTANCE BETWEEN TWO POINTS

The distance between two points A and B is the length of vector $\overrightarrow{AB}$ (or $\overrightarrow{BA}$), given by $|\overrightarrow{AB}|$.

Hence, the distance between points A and B is the length of vector $\overrightarrow{AB}$, given by $|\overrightarrow{AB}|$.

VECTOR EQUALITY



Two vectors are **equal** if they have the same length and direction.

Consequently, in 2-D their x-steps are equal i.e., $p = r$
and their y-steps are equal i.e., $q = s$

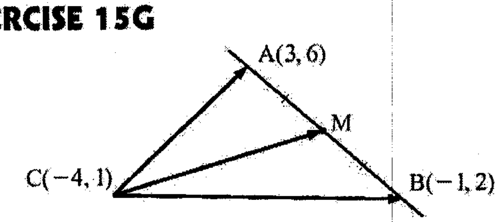
i.e., $\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix} \Leftrightarrow p = r$ and $q = s$.

(where $\Leftrightarrow$ reads "if and only if")

In 3-D, $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \Leftrightarrow a = p,\ b = q$ and $c = r$.

EXERCISE 15G

X **1**



Find:

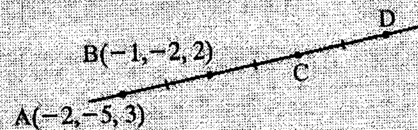
a the coordinates of M

b vectors $\overrightarrow{CA}$, $\overrightarrow{CM}$ and $\overrightarrow{CB}$.

c Verify that $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{CB}$.

Example 26

Find the coordinates of C and D in:



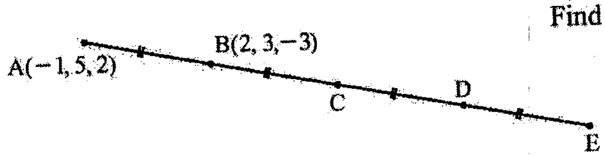
$\overrightarrow{AB} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$
$= \overrightarrow{OA} + 2\overrightarrow{AB}$
$= \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}$
$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
C is (0, 1, 1)

$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$
$= \overrightarrow{OA} + 3\overrightarrow{AB}$
$= \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 9 \\ -3 \end{bmatrix}$
$= \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$
$\therefore$ D is (1, 4, 0)

2 Find B if C is the centre of a circle with diameter AB:
- a A is (3, −2) and C(1, 4)
- b A is (0, 5) and C(−1, −2)
- c A is (−1, −4) and C(3, 0)

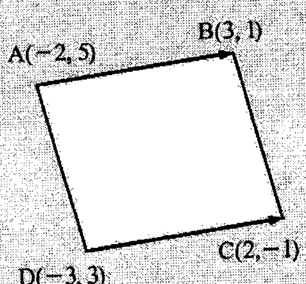
3



Find the coordinates of C, D and E.

Example 27

Use vectors to show that ABCD is a parallelogram where A is (−2, 5), B(3, 1), C(2, −1) and D is (−3, 3).



$\overrightarrow{AB} = \begin{bmatrix} 3-(-2) \\ 1-5 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$

$\overrightarrow{DC} = \begin{bmatrix} 2-(-3) \\ -1-3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$

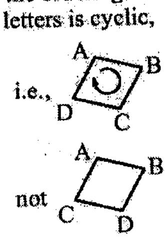
i.e., $\overrightarrow{AB} = \overrightarrow{DC}$

$\therefore$ side AB is parallel to side DC and is equal in length (magnitude) to side DC.

Hence ABCD is a parallelogram.

Given ABCD, the ordering of letters is cyclic,

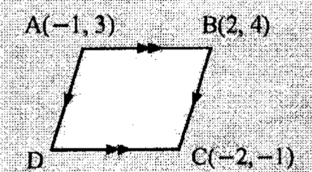
i.e.,



4 Use vectors to find whether or not ABCD is a parallelogram:
- a A(3, −1), B(4, 2), C(−1, 4) and D(−2, 1)
- b A(5, 0, 3), B(−1, 2, 4), C(4, −3, 6) and D(10, −5, 5)
- c A(2, −3, 2), B(1, 4, −1), C(−2, 6, −2) and D(−1, −1, 2).

Example 28

Use vector methods to find the remaining vertex of:



If D is (a, b) then

$$\overrightarrow{CD} = \begin{bmatrix} a-(-2) \\ b-(-1) \end{bmatrix} = \begin{bmatrix} a+2 \\ b+1 \end{bmatrix}$$

But $\overrightarrow{CD} = \overrightarrow{BA}$

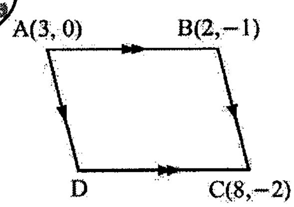
$$\therefore \begin{bmatrix} a+2 \\ b+1 \end{bmatrix} = \begin{bmatrix} -1-2 \\ 3-4 \end{bmatrix}$$

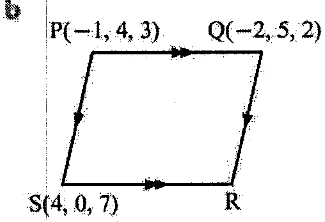
$\therefore\ a+2=-3$ and $b+1=-1$

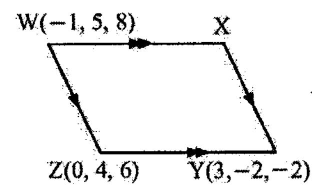
$\therefore\ a=-5$ and $b=-2$

So, D is $(-5, -2)$.

5 Use vector methods to find the remaining vertex of:

a


b


c


6 Find scalars r and s such that:

a $r\begin{bmatrix} 1 \\ -1 \end{bmatrix} + s\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -8 \\ -27 \end{bmatrix}$

b $r\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -19 \\ 2 \end{bmatrix}$

Three or more points are said to be **collinear** if they lie on the same straight line.

Notice that, A, B and C are collinear if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some scalar k.

Example 29

Prove that A(−1, 2, 3), B(4, 0, −1) and C(14, −4, −9) are collinear and hence find the ratio in which B divides CA.

$$\overrightarrow{AB} = \begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix} \quad \overrightarrow{BC} = \begin{bmatrix} 10 \\ -4 \\ -8 \end{bmatrix} = 2\begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix} \quad \therefore\ \overrightarrow{BC} = 2\overrightarrow{AB}$$

$\therefore$ BC is parallel to AB and since B is common to both, A, B and C are collinear.

To find the ratio in which B divides CA, we find

$$\overrightarrow{CB} : \overrightarrow{BA} = -2\begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix} : -\begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix} = 2:1$$

$\therefore$ B divides CA internally in the ratio 2 : 1.

7 a Prove that A(−2, 1, 4), B(4, 3, 0) and C(19, 8, −10) are collinear and hence find the ratio in which A divides CB.

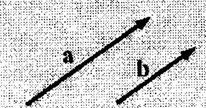
b Prove that P(2, 1, 1), Q(5, −5, −2) and R(−1, 7, 4) are collinear and hence find the ratio in which Q divides PR.

8 **a** A(2, −3, 4), B(11, −9, 7) and C(−13, a, b) are collinear. Find a and b.
b K(1, −1, 0), L(4, −3, 7) and M(a, 2, b) are collinear. Find a and b.

H PARALLELISM

PARALLELISM

If two non-zero vectors are **parallel**, then one is a scalar multiple of the other and vice versa.



Note:
- If **a** is parallel to **b**, then there exists a scalar, k say, such that $\mathbf{a} = k\mathbf{b}$.
- If $\mathbf{a} = k\mathbf{b}$ for some scalar k, then
  - ► **a** is parallel to **b**, and
  - ► $|\mathbf{a}| = |k|\,|\mathbf{b}|$.

Notice that $\mathbf{a} = \begin{bmatrix} 2 \\ 6 \\ -4 \end{bmatrix}$ is parallel to $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 4 \\ 12 \\ -8 \end{bmatrix}$ as $\mathbf{a} = 2\mathbf{b}$ and $\mathbf{a} = \frac{1}{2}\mathbf{c}$.

Also $\mathbf{a} = \begin{bmatrix} 2 \\ 6 \\ -4 \end{bmatrix}$ is parallel to $\mathbf{d} = \begin{bmatrix} -3 \\ -9 \\ 6 \end{bmatrix}$ as $\mathbf{a} = -\frac{2}{3}\mathbf{d}$.

Example 30

Find r and s given that $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ r \end{bmatrix}$ is parallel to $\mathbf{b} = \begin{bmatrix} s \\ 2 \\ -3 \end{bmatrix}$.

Since **a** and **b** are parallel, then $\mathbf{a} = k\mathbf{b}$ for some scalar k

$$\therefore \begin{bmatrix} 2 \\ -1 \\ r \end{bmatrix} = k\begin{bmatrix} s \\ 2 \\ -3 \end{bmatrix}$$

$$\therefore 2 = ks, \quad -1 = 2k \quad \text{and} \quad r = -3k$$

Consequently, $k = -\frac{1}{2}$ and $\therefore 2 = -\frac{1}{2}s$ and $r = -3\left(\frac{-1}{2}\right)$

$$\therefore r = \frac{3}{2} \text{ and } s = -4$$

EXERCISE 15H

1 $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -6 \\ r \\ s \end{bmatrix}$ are parallel. Find r and s.

2 Find scalars a and b, given that $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} a \\ 2 \\ b \end{bmatrix}$ are parallel.

3 **a** Find a vector of length 1 unit which is parallel to $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$.
(**Hint:** Let the vector be $k\mathbf{a}$.)

b Find a vector of length 2 units which is parallel to $\mathbf{b} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$.

4 What can be deduced from the following?

a $\overrightarrow{AB} = 3\overrightarrow{CD}$ b $\overrightarrow{RS} = -\frac{1}{2}\overrightarrow{KL}$ c $\overrightarrow{AB} = 2\overrightarrow{BC}$ d $\overrightarrow{BC} = \frac{1}{3}\overrightarrow{AC}$

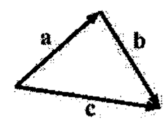
5 The position vectors of P, Q, R and S from O are $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$ respectively.

a Deduce that PR and QS are parallel.

b What is the relationship between the lengths of PR and QS?

TRIANGLE INEQUALITY

In any triangle, the sum of any two sides must always be greater than the third side. This is based on the well known result "the shortest distance between two points is a straight line".



6 Prove that $|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|$ using a geometrical argument.

[**Hint:** Consider a **a** is not parallel to **b** and use the triangle inequality
b **a** and **b** parallel c any other cases.]

I UNIT VECTORS

A **unit vector** is any vector which has a length of one unit.

For example,

- $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a unit vector as its length is $\sqrt{1^2 + 0^2 + 0^2} = 1$
- $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ is a unit vector as its length is $\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = 1$
- $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are special unit vectors in the direction of the positive X, Y and Z-axes respectively.

Notice that $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \Leftrightarrow \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.

component form — unit vector form

Thus, $\mathbf{a} = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$ can be written as $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ and vice versa.

We call **i**, **j** and **k** the **base** vectors as any vector can be written as a linear combination of the vectors **i**, **j** and **k**.

EXERCISE 15I

1 Express the following vectors in component form and find their length:

a $\mathbf{i} - \mathbf{j} + \mathbf{k}$ b $3\mathbf{i} - \mathbf{j} + \mathbf{k}$ c $\mathbf{i} - 5\mathbf{k}$ d $\frac{1}{2}(\mathbf{j} + \mathbf{k})$

2 Find k for the unit vectors:

a $\begin{bmatrix} 0 \\ k \end{bmatrix}$ b $\begin{bmatrix} k \\ 0 \end{bmatrix}$ c $\begin{bmatrix} k \\ 1 \end{bmatrix}$ d $\begin{bmatrix} -\frac{1}{2} \\ k \\ \frac{1}{4} \end{bmatrix}$ e $\begin{bmatrix} k \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$

Example 31

Find the length of $2\mathbf{i} - 5\mathbf{j}$.

As $2\mathbf{i} - 5\mathbf{j} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, its length is

$\sqrt{2^2 + (-5)^2}$
$= \sqrt{29}$ units

3 Find the length of the vectors:

a $3\mathbf{i} + 4\mathbf{j}$ b $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ c $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ d $-2.36\mathbf{i} + 5.65\mathbf{j}$

4 Find the unit vector in the direction of: a $\mathbf{i} + 2\mathbf{j}$ b $2\mathbf{i} - 3\mathbf{k}$ c $-2\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$

Example 32

Find a vector **b** of length 7 in the opposite direction to the vector $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

The unit vector in the direction of **a** is $\dfrac{1}{\sqrt{4+1+1}}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \dfrac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

We now multiply this unit vector by -7. Thus $\mathbf{b} = -\dfrac{7}{\sqrt{6}}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

5 Find a vector **b** if:

a it has the same direction as $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and has length 3 units

b it has opposite direction to $\begin{bmatrix} -1 \\ -4 \end{bmatrix}$ and has length 2 units

c it has the same direction as $\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$ and has length 6 units

d it has opposite direction to $\begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}$ and has length 5 units

Note:

- vector **b** of length k, $k > 0$ in the same direction as **a** is $\mathbf{b} = \dfrac{k}{|\mathbf{a}|}\mathbf{a}$
- vector **b** of length k, $k > 0$ in the opposite direction to **a** is $\mathbf{b} = -\dfrac{k}{|\mathbf{a}|}\mathbf{a}$
- vector **b** of length k, $k > 0$ which is parallel to **a** is $\mathbf{b} = \pm\dfrac{k}{|\mathbf{a}|}\mathbf{a}$

J THE SCALAR PRODUCT OF TWO VECTORS

Up to now, we have learned how to add, subtract and multiply vectors by a scalar. These operations have all been demonstrated to have practical uses, for example, **scalar multiplication** is used in the concept of **parallelism** and **finding unit** vectors.

We will now learn how to find the **product of two vectors** with practical applications being the reason for the definitions given.

Notation: With ordinary numbers numbers a and b we can write the product of a and b as ab or $a \times b$.

Consequently, a^2, a^3 etc. makes sense as, for example, $a^3 = aaa$ or $a \times a \times a$. There is only one meaning for product.

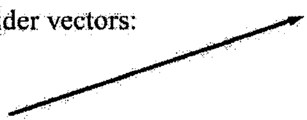
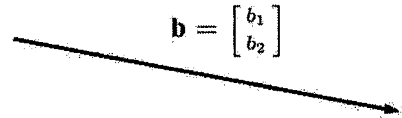
With vectors, there are *two useful ways* for finding the product of two vectors that will be defined later. These are:

- The **scalar product** of 2 vectors which results in a **scalar** answer, and has the notation $\mathbf{a} \bullet \mathbf{b}$ (read **a** dot **b**).
- The **vector product** of 2 vectors which results in a **vector** answer, and has the notation $\mathbf{a} \times \mathbf{b}$ (read **a** cross **b**).

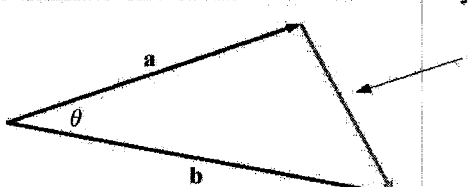
Consequently, for vector **a**, $\mathbf{a}^2$ or $(\mathbf{a})^2$ has no meaning, as it not clear whether we mean $\mathbf{a} \bullet \mathbf{a}$ (a scalar answer) *or* $\mathbf{a} \times \mathbf{a}$ (a vector answer).

So we should **never** write $\mathbf{a}^n$ or $(\mathbf{a})^n$.

ANGLE BETWEEN VECTORS

Consider vectors:  $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and  $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

We translate one of the vectors so that they both emanate from the same point.



This vector is $-\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$ and has length $|\mathbf{b} - \mathbf{a}|$.

Using the cosine rule, $|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta$

But $\mathbf{b} - \mathbf{a} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \end{bmatrix}$

So $(b_1 - a_1)^2 + (b_2 - a_2)^2 = {a_1}^2 + {a_2}^2 + {b_1}^2 + {b_2}^2 - 2\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta$

which simplifies to $a_1 b_1 + a_2 b_2 = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$

So, $\cos\theta = \dfrac{a_1 b_1 + a_2 b_2}{|\mathbf{a}|\,|\mathbf{b}|}$ can be used to find the angle between two vectors **a** and **b**.

In 3-D, it can easily be shown that $\cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{|\mathbf{a}|\,|\mathbf{b}|}$, where $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ can be used to find the angle between two vectors **a** and **b**.

Definition: If $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$,

the **scalar product** of **a** and **b** (also known as the **dot product** or **inner product**) is defined as $\mathbf{a} \bullet \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

This definition is simply an extension of the 2-dimensional definition, adding the Z-component.

ALGEBRAIC PROPERTIES OF THE SCALAR PRODUCT

Dot product has the same algebraic properties for 3-D vectors as it has for its 2-D counterparts.

- $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$
- $\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|^2$
- $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$ and
  $(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{c} + \mathbf{d}) = \mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d}$

These properties are proven in general by using vectors such as $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, etc.

Be careful not to confuse the **scalar product**, which is the product of two vectors to give a scalar answer, with **scalar multiplication**, which is the product of a scalar and a vector to give a parallel vector. They are both quite different.

GEOMETRIC PROPERTIES OF THE SCALAR PRODUCT

- If θ is the angle between vectors **a** and **b** then: $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$
- For non-zero vectors **a** and **b**: $\mathbf{a} \bullet \mathbf{b} = 0 \;\Leftrightarrow\;$ **a** and **b** are **perpendicular**.
  $\mathbf{a} \bullet \mathbf{b} = \pm|\mathbf{a}|\,|\mathbf{b}| \;\Leftrightarrow\;$ **a** and **b** are non-zero **parallel vectors**

The proofs of these results for 3-dimensions are identical for those in 2-dimensions.

Example 33

If $\mathbf{p} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, find: **a** $\mathbf{p} \bullet \mathbf{q}$ **b** the angle between **p** and **q**.

a $\mathbf{p} \bullet \mathbf{q}$

$= \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$= 2(-1) + 3(0) + (-1)2$

$= -2 + 0 - 2$

$= -4$

b $\mathbf{p} \bullet \mathbf{q} = |\mathbf{p}|\,|\mathbf{q}|\cos\theta$

$\therefore\; -4 = \sqrt{4+9+1}\sqrt{1+0+4}\cos\theta$

$\therefore\; -4 = \sqrt{14}\sqrt{5}\cos\theta$

$\therefore\; -4 = \sqrt{70}\cos\theta$

$\therefore\; \cos\theta = -\frac{4}{\sqrt{70}}$

$\therefore\; \theta = \arccos\left(-\frac{4}{\sqrt{70}}\right) \doteq 118.56^\circ$

EXERCISE 15J.1

1 For $\mathbf{p} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, find:

a $\mathbf{q} \bullet \mathbf{p}$ **b** $\mathbf{q} \bullet \mathbf{r}$ **c** $\mathbf{q} \bullet (\mathbf{p} + \mathbf{r})$ **d** $3\mathbf{r} \bullet \mathbf{q}$
e $2\mathbf{p} \bullet 2\mathbf{p}$ **f** $\mathbf{i} \bullet \mathbf{p}$ **g** $\mathbf{q} \bullet \mathbf{j}$ **h** $\mathbf{i} \bullet \mathbf{i}$

2 For $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ find:

a $\mathbf{a} \bullet \mathbf{b}$ **b** $\mathbf{b} \bullet \mathbf{a}$ **c** $|\mathbf{a}|^2$
d $\mathbf{a} \bullet \mathbf{a}$ **e** $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c})$ **f** $\mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$

3 Find: **a** $(\mathbf{i} + \mathbf{j} - \mathbf{k}) \bullet (2\mathbf{j} + \mathbf{k})$ **b** $\mathbf{i} \bullet \mathbf{i}$ **c** $\mathbf{i} \bullet \mathbf{j}$

4 Using $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ prove that $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$.

Hence, prove that $(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{c} + \mathbf{d}) = \mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d}$.

Example 34

Find t such that $\mathbf{a} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ t \end{bmatrix}$ are perpendicular.

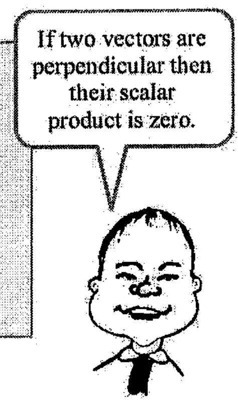
Since $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, $\mathbf{a} \bullet \mathbf{b} = 0$

$\therefore \begin{bmatrix} -1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ t \end{bmatrix} = 0$

$\therefore (-1)(2) + 5t = 0$

$\therefore -2 + 5t = 0$

$\therefore 5t = 2$ and so $t = \frac{2}{5}$



5 Find t given that these vectors are perpendicular:

a $\mathbf{p} = \begin{bmatrix} 3 \\ t \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ **b** $\mathbf{r} = \begin{bmatrix} t \\ t+2 \end{bmatrix}$ and $\mathbf{s} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

c $\mathbf{a} = \begin{bmatrix} t \\ t+2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2-3t \\ t \end{bmatrix}$ **d** $\mathbf{a} = \begin{bmatrix} 3 \\ -1 \\ t \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2t \\ -3 \\ -4 \end{bmatrix}$

6 For question **5** find where possible the value(s) of t for which the given vectors are *parallel*. Explain why sometimes the vectors can never be parallel.

7 Show that $\mathbf{a} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix}$ are mutually perpendicular.

8 **a** Show that $\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ are perpendicular.

b Find t if $\begin{bmatrix} 3 \\ t \\ -2 \end{bmatrix}$ is perpendicular to $\begin{bmatrix} 1-t \\ -3 \\ 4 \end{bmatrix}$.

9 Consider triangle ABC in which A is (5, 1, 2), B(6, −1, 0) and C(3, 2, 0). Using scalar product only, show that the triangle is right angled.

10 A(2, 4, 2), B(−1, 2, 3), C(−3, 3, 6) and D(0, 5, 5) are vertices of a quadrilateral.

a Prove that ABCD is a parallelogram.

b Find $|\overrightarrow{AB}|$ and $|\overrightarrow{BC}|$. What can be said about ABCD?

c Find $\overrightarrow{AC} \bullet \overrightarrow{BD}$. What property of figure ABCD has been found to be valid?

Example 35

Find the measure of the angle between the lines $2x + y = 5$ and $3x - 2y = 8$.

$2x + y = 5$ has slope $-\frac{2}{1}$ and $\therefore$ has direction vector $\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{a}$, say.

$3x - 2y = 8$ has slope $\frac{3}{2}$ and $\therefore$ has direction vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \mathbf{b}$, say.

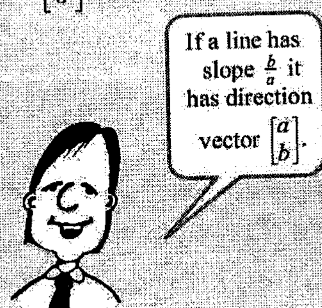
If the angle between the lines is θ, then

$$\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{(1 \times 2) + (-2 \times 3)}{\sqrt{1+4}\sqrt{4+9}}$$

$$= \frac{-4}{\sqrt{5}\sqrt{13}}$$

$$\therefore \quad \theta = \arccos\left(\frac{-4}{\sqrt{65}}\right) \doteq 119.7^\circ$$

$\therefore$ the angle is 119.7° or 60.3°.



11 Find the measure of the angle between the lines:

a $x - y = 3$ and $3x + 2y = 11$

b $y = x + 2$ and $y = 1 - 3x$

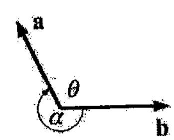
c $y + x = 7$ and $x - 3y + 2 = 0$

d $y = 2 - x$ and $x - 2y = 7$

Notice that, as $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$:

if θ is acute, $\cos\theta > 0$ and $\therefore$ $\mathbf{a} \bullet \mathbf{b} > 0$
if θ is obtuse, $\cos\theta < 0$ and $\therefore$ $\mathbf{a} \bullet \mathbf{b} < 0$.

Can you explain why?



Note: Two vectors form two angles as in the diagram drawn, i.e., θ and α. The angle between two vectors is always taken as the smaller angle, so we take θ to be the angle between the two vectors with $0 \leqslant \theta \leqslant 180^\circ$.

12 Find $\mathbf{p} \bullet \mathbf{q}$ for: **a** $|\mathbf{p}| = 2$, $|\mathbf{q}| = 5$ and $\theta = 60^\circ$ **b** $|\mathbf{p}| = 6$, $|\mathbf{q}| = 3$ and $\theta = 120^\circ$

Example 36

Find the form of all vectors which are perpendicular to $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} -4 \\ 3 \end{bmatrix} = -12 + 12 = 0.$

So, $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is one such vector

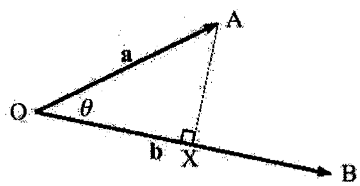
$\therefore$ required vectors have form $k\begin{bmatrix} -4 \\ 3 \end{bmatrix}$, $k \neq 0$

13 Find the form of all vectors which are perpendicular to:

a $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ **b** $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$ **c** $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ **d** $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ **e** $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

PROJECTION (EXTENSION)

If **a** and **b** are two vectors then we say the projection of **a** on **b** is the length of the projection vector of **a** on **b**, i.e., the length of XO in the given diagram.



the **projection** of **a** on **b** $= |\mathbf{a}|\cos\theta$

$$= |\mathbf{a}| \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$

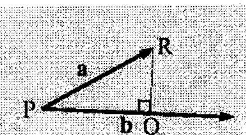
$$= \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|}$$

If θ is obtuse, then the projection of **a** on **b** is given by $\text{OX} = -\frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|}$. Prove this.

Hence,

- the **projection** of **a** on **b** is $\frac{|\mathbf{a} \bullet \mathbf{b}|}{|\mathbf{b}|}$
- the **projection vector** of **a** on **b** is $\left(\frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|}\right) \frac{1}{|\mathbf{b}|}\mathbf{b}$

the length and direction of the projection vector — unit vector in the direction of **b**



Example 37

If $\mathbf{a} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}$, find

a $\mathbf{a} \bullet \mathbf{b}$ **b** the projection **a** and **b** **c** the projection vector of **b** on **a**.

a $\mathbf{a} \bullet \mathbf{b}$
$= 2(-1) + 3(0) + 1(-3)$
$= -2 + 0 - 3$
$= -5$

b the projection **a** and **b** $= \frac{|\mathbf{a} \bullet \mathbf{b}|}{|\mathbf{b}|}$
$= \frac{|-5|}{\sqrt{1+0+9}}$
$= \frac{5}{\sqrt{10}}$ units

c the projection vector of **b** on **a** $= \left(\frac{\mathbf{b} \bullet \mathbf{a}}{|\mathbf{a}|}\right) \frac{1}{|\mathbf{a}|}\mathbf{a}$
$= \frac{-5}{\sqrt{14}} \frac{1}{\sqrt{14}}\mathbf{a}$
$= -\frac{5}{14}\mathbf{a}$ which is $\begin{bmatrix} -\frac{5}{7} \\ -\frac{15}{14} \\ -\frac{5}{14} \end{bmatrix}$

EXERCISE 15C.3

1 **a** $\begin{bmatrix}-3\\-15\end{bmatrix}$ **b** $\begin{bmatrix}-1\\2\end{bmatrix}$ **c** $\begin{bmatrix}0\\14\end{bmatrix}$ **d** $\begin{bmatrix}5\\-3\end{bmatrix}$ **e** $\begin{bmatrix}\frac{5}{2}\\\frac{11}{2}\end{bmatrix}$ **f** $\begin{bmatrix}-7\\7\end{bmatrix}$

g $\begin{bmatrix}5\\11\end{bmatrix}$ **h** $\begin{bmatrix}3\\\frac{17}{3}\end{bmatrix}$ **2** **a** $\begin{bmatrix}8\\-1\end{bmatrix}$ **b** $\begin{bmatrix}8\\-1\end{bmatrix}$ **c** $\begin{bmatrix}8\\-1\end{bmatrix}$

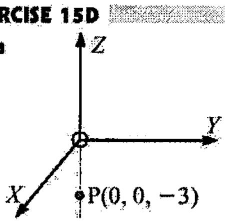
EXERCISE 15C.4

1 **a** $\sqrt{13}$ units **b** $\sqrt{17}$ units **c** $5\sqrt{2}$ units **d** $\sqrt{10}$ units
e $\sqrt{29}$ units

2 **a** $\sqrt{10}$ units **b** $2\sqrt{10}$ units **c** $2\sqrt{10}$ units **d** $3\sqrt{10}$ units
e $3\sqrt{10}$ units **f** $2\sqrt{5}$ units **g** $8\sqrt{5}$ units **h** $8\sqrt{5}$ units
i $\sqrt{5}$ units **j** $\sqrt{5}$ units

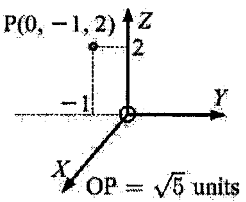
EXERCISE 15D

1 **a**



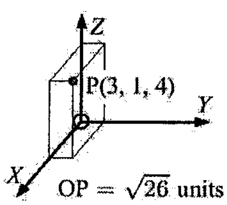
OP = 3 units

b



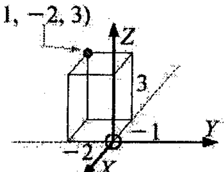
OP = $\sqrt{5}$ units

c



OP = $\sqrt{26}$ units

d P(−1, −2, 3)



OP = $\sqrt{14}$ units

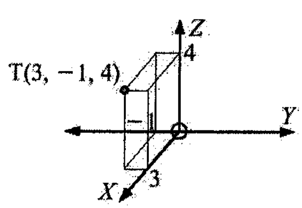
2 **a** **i** $\sqrt{14}$ units **ii** $(-\frac{1}{2}, \frac{1}{2}, 2)$ **b** **i** $\sqrt{14}$ units **ii** $(1, -\frac{1}{2}, \frac{3}{2})$
c **i** $\sqrt{21}$ units **ii** $(1, -\frac{1}{2}, 0)$ **d** **i** $\sqrt{14}$ units **ii** $(1, \frac{1}{2}, -\frac{3}{2})$

4 **a** isosceles **b** right angled **c** right angled
d straight line **5** $(0, 3, 5)$, $r = \sqrt{3}$ units

6 **a** $(0, y, 0)$ **b** $(0, 2, 0)$ and $(0, -4, 0)$

EXERCISE 15E.1

1 **a**



b $\overrightarrow{OT} = \begin{bmatrix}3\\-1\\4\end{bmatrix}$

c OT = $\sqrt{26}$ units

2 **a** $\overrightarrow{AB} = \begin{bmatrix}4\\-1\\-3\end{bmatrix}$, $\overrightarrow{BA} = \begin{bmatrix}-4\\1\\3\end{bmatrix}$ **b** AB = $\sqrt{26}$ units
BA = $\sqrt{26}$ units

3 $\overrightarrow{OA} = \begin{bmatrix}3\\1\\0\end{bmatrix}$, $\overrightarrow{OB} = \begin{bmatrix}-1\\1\\2\end{bmatrix}$, $\overrightarrow{AB} = \begin{bmatrix}-4\\0\\2\end{bmatrix}$

4 **a** $\overrightarrow{NM} = \begin{bmatrix}5\\-4\\-1\end{bmatrix}$ **b** $\overrightarrow{MN} = \begin{bmatrix}-5\\4\\1\end{bmatrix}$ **c** MN = $\sqrt{42}$ units

5 **a** $\overrightarrow{OA} = \begin{bmatrix}-1\\2\\5\end{bmatrix}$, OA = $\sqrt{30}$ units

b $\overrightarrow{AC} = \begin{bmatrix}-2\\-1\\-5\end{bmatrix}$, AC = $\sqrt{30}$ units

c $\overrightarrow{CB} = \begin{bmatrix}5\\-1\\3\end{bmatrix}$, CB = $\sqrt{35}$ units

6 **a** $\sqrt{13}$ units **b** $\sqrt{14}$ units **c** 3 units

EXERCISE 15E.2

1 **a** $a = 5$, $b = 6$, $c = -6$ **b** $a = 4$, $b = 2$, $c = 1$

2 **a** $a = \frac{1}{3}$, $b = 2$, $c = 1$ **b** $a = 1$, $b = 2$
c $a = 1$, $b = -1$, $c = 2$

3 **a** $r = 2$, $s = 4$, $t = -7$ **b** $r = -4$, $s = 0$, $t = 3$

4 **a** $\overrightarrow{AB} = \begin{bmatrix}2\\-5\\-1\end{bmatrix}$, $\overrightarrow{DC} = \begin{bmatrix}2\\-5\\-1\end{bmatrix}$

b ABCD is a parallelogram **5** **a** S = (−2, 8, −3)

EXERCISE 15F

1 **a** $\mathbf{x} = \frac{1}{2}\mathbf{q}$ **b** $\mathbf{x} = 2\mathbf{n}$ **c** $\mathbf{x} = -\frac{1}{3}\mathbf{p}$ **d** $\mathbf{x} = \frac{1}{2}(\mathbf{r} - \mathbf{q})$
e $\mathbf{x} = \frac{1}{5}(4\mathbf{s} - \mathbf{t})$ **f** $\mathbf{x} = 3(4\mathbf{m} - \mathbf{n})$

2 **a** $\mathbf{y} = \begin{bmatrix}-1\\\frac{3}{2}\end{bmatrix}$ **b** $\mathbf{y} = \begin{bmatrix}2\\4\end{bmatrix}$ **c** $\mathbf{y} = \begin{bmatrix}\frac{3}{2}\\-\frac{1}{2}\end{bmatrix}$ **d** $\mathbf{y} = \begin{bmatrix}\frac{5}{4}\\\frac{3}{4}\end{bmatrix}$

4 **a** $\mathbf{x} = \begin{bmatrix}4\\-6\\-5\end{bmatrix}$ **b** $\mathbf{x} = \begin{bmatrix}1\\-\frac{2}{3}\\\frac{5}{3}\end{bmatrix}$ **c** $\mathbf{x} = \begin{bmatrix}\frac{3}{2}\\-1\\\frac{5}{2}\end{bmatrix}$

5 $\overrightarrow{AB} = \begin{bmatrix}3\\4\\-2\end{bmatrix}$, AB = $\sqrt{29}$ units

7 **a** $\overrightarrow{BD} = \frac{1}{2}\mathbf{a}$ **b** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ **c** $\overrightarrow{BA} = -\mathbf{b} + \mathbf{a}$
d $\overrightarrow{OD} = \mathbf{b} + \frac{1}{2}\mathbf{a}$ **e** $\overrightarrow{AD} = \mathbf{b} - \frac{1}{2}\mathbf{a}$ **f** $\overrightarrow{DA} = \frac{1}{2}\mathbf{a} - \mathbf{b}$

8 **a** $\begin{bmatrix}-1\\5\\-1\end{bmatrix}$ **b** $\begin{bmatrix}-3\\4\\-2\end{bmatrix}$ **c** $\begin{bmatrix}-3\\6\\-5\end{bmatrix}$

9 **a** $\begin{bmatrix}3\\1\\-2\end{bmatrix}$ **b** $\begin{bmatrix}1\\-3\\4\end{bmatrix}$ **c** $\begin{bmatrix}1\\4\\-9\end{bmatrix}$ **d** $\begin{bmatrix}2\\-4\\10\end{bmatrix}$ **e** $\begin{bmatrix}3\\2\\-5\end{bmatrix}$

f $\begin{bmatrix}-1\\\frac{3}{2}\\-\frac{7}{2}\end{bmatrix}$ **g** $\begin{bmatrix}1\\-4\\7\end{bmatrix}$ **h** $\begin{bmatrix}4\\2\\-2\end{bmatrix}$

10 **a** $\sqrt{11}$ units **b** $\sqrt{14}$ units **e** $\begin{bmatrix}\sqrt{11}\\-3\sqrt{11}\\2\sqrt{11}\end{bmatrix}$ **f** $\begin{bmatrix}-\frac{1}{\sqrt{11}}\\\frac{1}{\sqrt{11}}\\\frac{3}{\sqrt{11}}\end{bmatrix}$
c $\sqrt{38}$ units **d** $\sqrt{3}$ units

EXERCISE 15G

1 **a** M(1, 4) **b** $\overrightarrow{CA} = \begin{bmatrix}7\\5\end{bmatrix}$, $\overrightarrow{CM} = \begin{bmatrix}5\\3\end{bmatrix}$, $\overrightarrow{CB} = \begin{bmatrix}3\\1\end{bmatrix}$

2 **a** B(−1, 10) **b** B(−2, −9) **c** B(7, 4)

3 C(5, 1, −8), D(8, −1, −13), E(11, −3, −18)

4 **a** parallelogram **b** parallelogram **c** not parallelogram

5 **a** D(9, −1) **b** R(3, 1, 6) **c** X(2, −1, 0)

6 **a** $r = 2$, $s = -5$ **b** $r = 4$, $s = -1$

7 **a** $-7 : 2$ **b** $-1 : 2$

8 **a** $a = 7$, $b = -1$ **b** $a = -\frac{7}{2}$, $b = -\frac{21}{2}$

EXERCISE 15H

1 $r = 3$, $s = -9$ **3** **a** $\begin{bmatrix}\frac{2}{3}\\-\frac{1}{3}\\-\frac{2}{3}\end{bmatrix}$ **b** $\begin{bmatrix}-\frac{1}{3}\\-\frac{2}{3}\\\frac{4}{3}\end{bmatrix}$

2 $a = -6$, $b = -4$

4 **a** $\overrightarrow{AB} \parallel \overrightarrow{CD}$, AB = 3 CD
b $\overrightarrow{RS} \parallel \overrightarrow{KL}$, RS = $\frac{1}{2}$KL opposite direction

c A, B and C are collinear and AB = 2 BC
d A, B and C are collinear and AC = 3 BC

5 **a** $\overrightarrow{PR} = \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}$, $\overrightarrow{QS} = \begin{bmatrix} -2 \\ -6 \\ 6 \end{bmatrix}$ **b** PR = $\frac{1}{2}$ QS

EXERCISE 15I

1 **a** $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\sqrt{3}$ units **b** $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$, $\sqrt{11}$ units

c $\begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$, $\sqrt{26}$ units **d** $\begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$, $\frac{1}{\sqrt{2}}$ units

2 **a** $k = \pm 1$ **b** $k = \pm 1$ **c** $k = 0$ **d** $k = \pm\frac{\sqrt{11}}{4}$ **e** $k = \pm\frac{2}{3}$

3 **a** 5 units **b** $\sqrt{6}$ units **c** 3 units **d** $\doteq$ 6.12 units

4 **a** $\frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$ **b** $\frac{1}{\sqrt{13}}(2\mathbf{i} - 3\mathbf{k})$ **c** $\frac{1}{\sqrt{33}}(-2\mathbf{i} - 5\mathbf{j} - 2\mathbf{k})$

5 **a** $\frac{3}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ **b** $-\frac{2}{\sqrt{17}}\begin{bmatrix} -1 \\ -4 \end{bmatrix}$ **c** $\frac{6}{\sqrt{18}}\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$ **d** $-\frac{5}{3}\begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}$

EXERCISE 15J.1

1 **a** 7 **b** 22 **c** 29 **d** 66 **e** 52 **f** 3 **g** 5 **h** 1

2 **a** 2 **b** 2 **c** 14 **d** 14 **e** 4 **f** 4

3 **a** 1 **b** 1 **c** 0

5 **a** $t = 6$ **b** $t = -8$ **c** $t = 0$ or 2 **d** $t = -\frac{3}{2}$

6 **a** $t = -\frac{3}{2}$ **b** $t = -\frac{6}{7}$ **c** $t = \frac{-1 \pm \sqrt{5}}{2}$ **d** impossible

7 Show $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{c} = \mathbf{a} \bullet \mathbf{c} = 0$ **8** **b** $t = -\frac{5}{6}$

9 $\overrightarrow{AB} \bullet \overrightarrow{AC} = 0$, $\therefore$ $\angle$BAC is a right angle

10 **b** AB = $\sqrt{14}$ units, BC = $\sqrt{14}$ units, ABCD is a rhombus
c 0, the diagonals of a rhombus are perpendicular.

11 **a** 101.3° or 78.7° **b** 116.6° or 63.4°
c 63.4° or 116.6° **d** 71.6° or 108.4°

12 **a** 5 **b** −9

13 **a** $k\begin{bmatrix} -2 \\ 5 \end{bmatrix}$, $k \neq 0$ **b** $k\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $k \neq 0$ **c** $k\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $k \neq 0$

d $k\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $k \neq 0$ **e** $k\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $k \neq 0$

EXERCISE 15J.2

1 **a** −1 **b** 109.5° (acute 70.5°) **c** $\begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$ **d** $\frac{1}{\sqrt{3}}$

2 $\angle$ABC $\doteq$ 62.5°, the exterior angle 117.5°

3 **a** 54.7° **b** 60° **c** 35.3°

4 **a** 30.3° **b** 54.2° **5** **a** M$(\frac{3}{2}, \frac{5}{2}, \frac{3}{2})$ **b** 51.5°

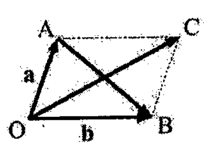
6 **a** $t = 0$ or −3 **b** $r = -2$, $s = 5$, $t = -4$

7 **a** 74.5° **b** 72.45°

8 $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ will do

$\mathbf{a} \bullet \mathbf{b} = \mathbf{a} \bullet \mathbf{c}$, but $\mathbf{b} \neq \mathbf{c}$

10 **a** **Hint:** Square both sides.
b Consider the parallelogram. Find $\overrightarrow{AB}$ and $\overrightarrow{OC}$, etc.



11 −7

12 $\mathbf{a} \bullet \mathbf{b}$ is a scalar and so $\mathbf{a} \bullet \mathbf{b} \bullet \mathbf{c}$ is a scalar 'dotted' with a vector which is meaningless.

EXERCISE 15K.1

1 **a** [2, 5, 11] **b** [2, 4, 1] **c** $-\mathbf{i} - \mathbf{j} - \mathbf{k}$ **d** $\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$

2 **a** $\mathbf{a} \times \mathbf{b} = [-11, -2, 5]$,
$\mathbf{a} \bullet (\mathbf{a} \times \mathbf{b}) = 0 = \mathbf{b} \bullet (\mathbf{a} \times \mathbf{b})$
$\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to both **a** and **b**

3 **a** $\mathbf{i} \times \mathbf{i} = \mathbf{0}$ $\mathbf{j} \times \mathbf{j} = \mathbf{0}$ $\mathbf{k} \times \mathbf{k} = \mathbf{0}$
b $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$ $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$
$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$ $\mathbf{k} \times \mathbf{i} = \mathbf{j}$
$\mathbf{a} \times \mathbf{a} = \mathbf{0}$ $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

5 **a** $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$ **b** 17 **c** 17

7 **a** $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ **b** $\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$ **c** $\begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$ **d** $\begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$

8 $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$

11 **a** $\mathbf{a} \times \mathbf{b}$ **b** 0 **c** 0

12 **a** $k\begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$ **b** $k\begin{bmatrix} 6 \\ 22 \\ -15 \end{bmatrix}$ **c** $(-\mathbf{i} + \mathbf{j} - 2\mathbf{k})n$

d $(5\mathbf{i} + \mathbf{j} + 4\mathbf{k})n$ $n, k \in \mathcal{R}$, $n, k \neq 0$

13 $k\begin{bmatrix} 4 \\ -5 \\ -7 \end{bmatrix}$, $k \neq 0$ $\frac{\sqrt{10}}{6}\begin{bmatrix} 4 \\ -5 \\ -7 \end{bmatrix}$ *or* $-\frac{\sqrt{10}}{6}\begin{bmatrix} 4 \\ -5 \\ -7 \end{bmatrix}$

14 **a** $\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ **b** $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

EXERCISE 15K.2

1 **a** $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

2 **a** $\mathbf{a} \bullet \mathbf{b} = -1$ $\mathbf{a} \times \mathbf{b} = [1, 5, 1]$
b $\cos\theta = -\frac{1}{\sqrt{28}}$ **c** $\sin\theta = \frac{\sqrt{27}}{\sqrt{28}}$ **d** $\sin\theta = \frac{\sqrt{27}}{\sqrt{28}}$

4 **a** $\overrightarrow{OA} = [2, 3, -1]$ $\overrightarrow{OB} = [-1, 1, 2]$
b $\overrightarrow{OA} \times \overrightarrow{OB} = [7, -3, 5]$ $|\overrightarrow{OA} \times \overrightarrow{OB}| = \sqrt{83}$
c Area ΔOAB $= \frac{1}{2}|\overrightarrow{OA}||\overrightarrow{OB}|\sin\theta$
$= \frac{1}{2}|\overrightarrow{OA} \times \overrightarrow{OB}| = \frac{\sqrt{83}}{2}$ units²

5 **a** $\overrightarrow{OC}$ is parallel to $\overrightarrow{AB}$ **b** $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c}$

EXERCISE 15K.3

1 **a** $\frac{\sqrt{101}}{2}$ units² **b** $\frac{\sqrt{133}}{2}$ units² **c** $\frac{\sqrt{69}}{2}$ units²

2 $8\sqrt{2}$ units² **3** **a** D(−4, 1, 3) **b** $\sqrt{307}$ units²

4 **a** 4 units³ **b** $(\sqrt{42} + 2\sqrt{3} + 3\sqrt{2} + 6)$ units²

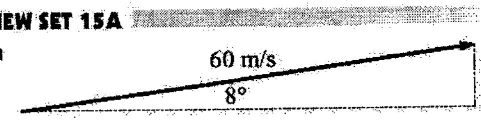
5 **a** (3, 1, 0), (1, 3, 3), (4, 2, 3), (4, 3, 3) **b** $\doteq$ 79.01°
c 9 units³ **6** $k = 2 \pm 2\sqrt{33}$

7 $S = \frac{1}{2}\{|\mathbf{a} \times \mathbf{b}| + |\mathbf{a} \times \mathbf{c}| + |\mathbf{b} \times \mathbf{c}| + |(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})|\}$

9 **a** Yes **b** No **10** $k = \frac{23}{10}$

REVIEW SET 15A

1 **a**



Scale: 1 cm ≡ 10 m/s

b