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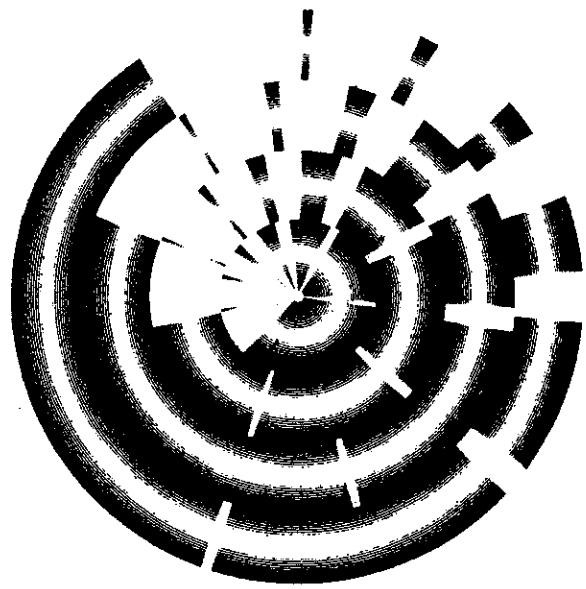
CHAPTER 3

PROGRESSIONS



SECTION 1. PROGRESSIONS
SECTION 2. MATHEMATICAL INDUCTION AND PROGRESSIONS

Mathematics is said to be "the study of infinity."

In studying progressions you will encounter the concept of infinity for the first time since elementary school. Infinite progressions will not be the major topic of this chapter, but the idea of "any finite number," which could be described as one step before infinity, will be considered.

Progressions are functions which relate real numbers or complex numbers to each natural number. A function which relates one point to each natural number is called a sequence of points.

If we regard a progression as a kind of function, an arithmetic progression is a linear function, and a geometric progression is an exponential function.

The study of progressions whose domain is the set of all natural numbers – the simplest infinite set – is the basis for the study of more general functions.

Example 2 If $a_n = \frac{1}{2^{n-1}}$, then progression $\{a_n\}$ is

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots .$$

Expressions in n, such as $4n - 3$ and $\frac{1}{2^{n-1}}$ in Examples 1 and 2, express the nth term of each progression.

In these expressions, if we take $n = 1$, we obtain the first term, $n = 2$ gives us the second term, $n = 3$ gives us the third term, and so on.

Such an expression in n, representing the nth term, is called the **general term** of the progression.

Problem Write the first five terms of the following progressions:

(1) $\{1 - 3n\}$ (2) $\{n^2 - n + 1\}$ (3) $\{(-1)^n\}$

A progression with a finite number of terms is called a **finite progression.** In a finite progression, we often make special reference to the **number of terms** and the **last term.**

For example,

$$3, 6, 9, 12, 15, 18, 21, 24, 27, 30$$

is a finite progression; the number of terms is 10 and the last term is 30.

2 Arithmetic Progressions

The progression

$$3, 8, 13, 18, 23, \dots$$

is created by adding 5 to the preceding number, starting with 3 as the first term.

A progression created by adding a fixed number to the preceding number over and over, starting from the first term, is called an **arithmetic progression**, and the fixed number is called the **common difference**.

The following relations hold between the nth term a_n and the $(n+1)$th term a_{n+1} of an arithmetic progression with common difference d.

$$a_{n+1} = a_n + d \quad \text{or} \quad a_{n+1} - a_n = d$$

Example The progression

$$3, 7, 11, 15, 19, 23, \ldots$$

is an arithmetic progression in which the first term is 3 and the common difference is 4.

A progression with a first term of 14 and a common difference of - 6 is

$$14, 8, 2, -4, -10, -16, \ldots .$$

The first several terms of a progression in which the first term is a and the common difference is d are

$$a_1 = a,\ a_2 = a + d,\ a_3 = a + 2d,\ a_4 = a + 3d,\ \ldots .$$

The General Term of an Arithmetic Progression

The nth term a_n of an arithmetic progression with a first term of a and a common difference of d is

$$a_n = a + (n-1)d.$$

Problem 1 (1) Find the 20th term of an arithmetic progression in which the first term is 4 and the common difference is 3.

(2) Find the 25th term of an arithmetic progression in which the first term is 7 and the common difference is $-\frac{1}{3}$.

Problem 2 Find the general term of the following arithmetic progressions:

(1) 23, 30, 37, 44, ... (2) $2, \frac{5}{4}, \frac{1}{2}, -\frac{1}{4}, \ldots$

Problem 3 Which term of an arithmetic progression with a first term of 8 and a common difference of -3 will be -37?

Demonstration 1 Find the first term and the common difference of an arithmetic progression in which the 4th term is 14 and the 9th term is 54.

[Solution] Take a as first term of this progression and d as the common difference, and then

$$a_4 = a + 3d = 14, \qquad a_9 = a + 8d = 54.$$

Solving these equalities, we obtain

$$a = -10, \qquad d = 8.$$

Therefore, the first term is -10 and the common difference is 8.

Problem 4 Find the general term of an arithmetic progression in which the third term is -4 and the 10th term is 38.

Problem 5 Find the common difference of an arithmetic progression in which the first term is 3, the last term is 94, and the number of terms is 15. Then find the 5th term and the 10th term.

Demonstration 2 Prove that a progression in which the nth term is expressed as $5n + 3$ is an arithmetic progression.

[Proof] Since

$$a_n = 5n + 3$$

and

$$a_{n+1} = 5(n + 1) + 3 = 5n + 8,$$

it follows that

$$a_{n+1} - a_n = 5.$$

Thus, this progression is an arithmetic progression with a common difference of 5.

Problem 6 Prove that the progression $\{pn + q\}$ is an arithmetic progression, provided that p and q are constants.

Problem 7 Prove that if two progressions $\{a_n\}$ and $\{b_n\}$ are arithmetic progressions, then the progression $\{a_n + b_n\}$ is also an arithmetic progression.

Next, let's consider the first n terms of an arithmetic progression.

Take S_n as the sum of the first n terms of an arithmetic progression in which the first term is a and the common difference is d.

$$S_n = a + (a + d) + (a + 2d) + \dots + \{a + (n - 1)d\} \quad (1)$$

If we take l as the last term of this progression and rewrite the right side by reversing the order of the terms, then

$$S_n = l + (l - d) + (l - 2d) + \dots + \{l - (n - 1)d\}. \quad (2)$$

Adding up the corresponding sides of (1) and (2), we obtain

$$2S_n = (a + l) + (a + l) + (a + l) + \dots + (a + l).$$

The right side of this expression is equivalent to n times $(a + l)$. Therefore,

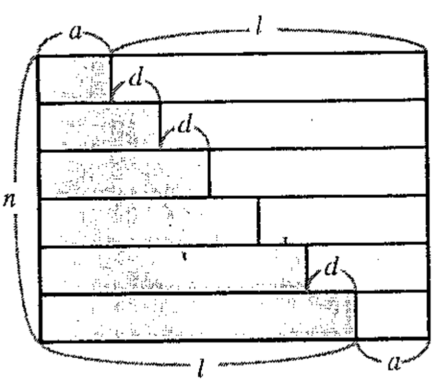
$$2S_n = n(a + l).$$

Thus,

$$S_n = \frac{n(a + l)}{2}.$$

Substituting $l = a + (n - 1)d$ into this expression, we obtain

$$S_n = \frac{n\{2a + (n - 1)d\}}{2}.$$



The Sum of an Arithmetic Progression

The sum S_n of an arithmetic progression with a first term of a, a last term of l, and a total of n terms is

$$S_n = \frac{n(a + l)}{2}.$$

The sum S_n of the first n terms of an arithmetic progression with a first term of a and a common difference of d is

$$S_n = \frac{n\{2a + (n-1)d\}}{2}.$$

Problem 8 Find the sum of each of the following arithmetic progressions:

(1) The first term is 7, the last term is 61, and there are 10 terms.

(2) The first term is -10, the common difference is 4, and there are 13 terms.

(3) The first term is 21, the common difference is -6, and the last term is -117.

Problem 9 Prove that the following equalities hold:

(1) $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

(2) $1 + 3 + 5 + ... + (2n - 1) = n^2$

Problem 10 (1) Find the sum of the natural numbers from 1 through 100.

(2) Find the sum of all the multiples of 3 among the natural numbers through 200.

Problem 11 If the sum of the first n terms of an arithmetic progression is 297, the first term is 45, and the common difference is -3, what is the value of n?

3 Geometric Progressions

A progression created by multiplying the preceding term by a fixed number again and again, starting from the first term, is called a **geometric progression**, and the fixed number is called the **common ratio.**

Therefore, the following relation holds between the nth term a_n and the $(n+1)$th term a_{n+1} of a geometric progression with a common ratio of r:

$$a_{n+1} = a_n r.$$

As a special case, when no term is equal to 0, we have

$$\frac{a_{n+1}}{a_n} = r.$$

Example The progression 1, 2, 4, 8, 16, ... is a geometric progression in which the first term is 1 and the common ratio is 2. A geometric progression with a first term of 27 and a common ratio of $-\frac{1}{3}$ is

$$27, -9, 3, -1, \frac{1}{3}, -\frac{1}{9}, \dots .$$

The first several terms of a geometric progression with a first term of a and a common ratio of r are

$$a_1 = a,\ a_2 = ar,\ a_3 = ar^2,\ a_4 = ar^3,\ \dots .$$

The General Term of a Geometric Progression

The nth term a_n of a geometric progression with a first term of a and a common ratio of r is

$$a_n = ar^{n-1}.$$

Problem 1 (1) Find the 6th term of a geometric progression in which the first term is 1 and the common ratio is 2.

(2) Find the 5th term of a geometric progression in which the first term is 3 and the common ratio is $-\frac{1}{3}$.

Problem 2 State the common ratio of the following geometric progressions. Then find the general term.

(1) $\sqrt{2}, 2, 2\sqrt{2}, 4, \ldots$ (2) $1, -3, 9, -27, 81, \ldots$

(3) $1, -1, 1, -1, 1, \ldots$

Problem 3 State the first term and the common ratio of a geometric progression in which the general term is $\frac{3^{n+1}}{2^n}$.

Problem 4 Find the first term and the common ratio of a geometric progression in which the third term is 4 and the 5th term is 36.

Next, let's consider the sum of the first n terms of a geometric progression.

If we take S_n as the sum of the first n terms of a geometric progression in which the first term is a and the common ratio is r, then

$$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}. \quad (1)$$

Multiplying both sides of this equation by r, we obtain

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n. \quad (2)$$

Subtracting the left side and the right side of (2) from the corresponding sides of (1), we get

$$(1-r)S_n = a - ar^n.$$

If $r \neq 1$, then

$$S_n = \frac{a(1-r^n)}{1-r}.$$

If $r = 1$, then from (1), we obtain

$$S_n = a + a + \ldots a = na.$$

The Sum of a Geometric Progression

The sum of the first n terms of a geometric progression with a first term of a and a common ratio of r is

for $r \neq 1$, $S_n = \dfrac{a(1-r^n)}{1-r}$.

for $r = 1$, $S_n = na$.

Problem 5 Find the sum of each of the following progressions:

(1) The first term is 3, the common ratio is 2, and there are 7 terms.

(2) The first term is 1, the common ratio is -3, and there are 6 terms.

(3) The first term is 5, the common ratio is $\frac{1}{2}$, and there are 5 terms.

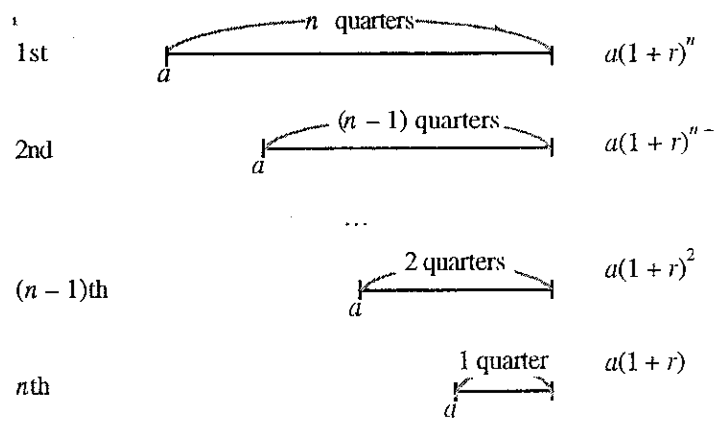
Problem 6 Find the sum of the first n terms of the following geometric progressions:

(1) 13, 52, 208, 832, ... (2) $\sqrt{3}$, -1, $\frac{1}{\sqrt{3}}$, $-\frac{1}{3}$, ...

Demonstration 1 If a fixed amount of a yen is added to savings at the beginning of each quarter, find the cumulative balance at the end of the nth quarter. Assume that the interest rate for a quarter is r, and that the interest is compounded quarterly.

[Solution] At the end of the nth quarter the amount deposited in the savings account for the first quarter, second quarter, third quarter, ..., nth quarter can be expressed in the following form:

$$a(1+r)^n \text{ yen}, a(1+r)^{n-1} \text{ yen}, a(1+r)^{n-2} \text{ yen}, \ldots, a(1+r) \text{ yen}.$$



If we take S yen as the sum of these amounts, then

$$S = a(1+r) + a(1+r)^2 + \ldots + a(1+r)^n.$$

This is the sum of a geometric progression in which the first term is $a(1+r)$, the common ratio is $(1+r)$, and the number of terms is n. The common ratio is not equal to 1, and therefore,

$$S = \frac{a(1+r)\{1-(1+r^n)\}}{1-(1+r)} = \frac{a(1+r)\{(1+r)^n-1\}}{r}.$$

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Problem 7 If we borrow A yen at the beginning of a certain year, then we must pay back a yen at the end of each year for the next n years in order to pay off the loan. Assuming an annual compound interest rate of r, express a in terms of A, r, and n.

Demonstration 2 Find the following sum, provided that $x \neq 1$.

$$1 + 2x + 3x^2 + \dots + nx^{n-1}$$

[Solution] Set

$$S_n = 1 + 2x + 3x^2 + \dots + (n-1)x^{n-2} + nx^{x-1}. \qquad (1)$$

Multiplying both sides by x, we obtain

$$xS_n = x + 2x^2 + 3x^3 + \dots + (n-1)x^{n-1} + nx^n. \qquad (2)$$

Subtracting (2) from (1), we get

$$(1-x)S_n = 1 + x + x^2 + \dots + x^{n-1} - nx^n.$$

Since $x \neq 1$, we can rearrange the right side to get

$$(1-x)S_n = \frac{1-x^n}{1-x} - nx^n = \frac{1-(n-1)x^n + nx^{n+1}}{1-x}.$$

Thus,

$$S_n = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}.$$

Problem 8 Find the sum of the first n terms of the following progression:

$$1 \cdot 2,\ 4 \cdot 2^2,\ 7 \cdot 2^3,\ 10 \cdot 2^4,\ 13 \cdot 2^5,\ \dots .$$

4 Various Types of Progressions

There are progressions besides arithmetic and geometric progressions in which the general term or the sum of the first n terms can easily be found. Let's consider some of these other progressions.

Demonstration 1 Find the following sum:

$$1^2 + 2^2 + 3^2 + ... + n^2.$$

[Solution] In the identity $(k + 1)^3 - k^3 = 3k^2 + 3k + 1$,

if $k = 1$, then $2^3 - 1^3 = 3 \cdot 1^2 + 3 \cdot 1 + 1$;

if $k = 2$, then $3^3 - 2^3 = 3 \cdot 2^2 + 3 \cdot 2 + 1$;

...

if $k = n$, then $(n + 1)^3 - n^3 = 3 \cdot n^2 + 3 \cdot n + 1$.

Adding the corresponding sides of these equalities, we obtain

$$\begin{aligned}(n + 1)^3 - 1^3 &= 3(1^2 + 2^2 + ... + n^2) + 3(1 + 2 + ... + n) + n \\ &= 3(1^2 + 2^2 + ... + n^2) + \frac{3n(n + 1)}{2} + n.\end{aligned}$$

Therefore,

$$\begin{aligned}3(1^2 + 2^2 + ... + n^2) &= (n + 1)^3 - \frac{3n(n + 1)}{2} - (n + 1) \\ &= \frac{n(n + 1)(2n + 1)}{2}.\end{aligned}$$

Thus,

$$1^2 + 2^2 + ... + n^2 = \frac{n(n + 1)(2n + 1)}{6}.$$

Problem 1 Find the following values using the result of Demonstration 1:

(1) $1^2 + 2^2 + 3^2 + \ldots + 10^2$

(2) $8^2 + 9^2 + 10^2 + \ldots + 15^2$.

Problem 2 Prove the following equality, using the identity $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$ as in Demonstration 1.

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Use this equality to find the following sum:

$$6^3 + 7^3 + 8^3 + \ldots + 13^3.$$

The Symbol $\sum$ for the Sum of a Progression

We can write the sum of the first n terms of a progression $a_1, a_2, a_3, \ldots, a_n \ldots$ using the symbol $\sum$* as

$$\sum_{k=1}^{n} a_k .$$

Thus,

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n.$$

Using this symbol, $1^2 + 2^2 + 3^2 + \ldots + n^2$ can be represented as $\sum_{k=1}^{n} k^2$. This expression can also be written using letters other than k such as $\sum_{i=1}^{n} i^2$ or $\sum_{j=1}^{n} j^2$.

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*$\sum$ is the Greek letter corresponding to S, the first letter of *Sum*, and is read "sigma" or, if you prefer, as "sum".

Moreover, $\sum_{k=3}^{6}(2k+3)$ expresses the sum 9 + 11 + 13 + 15 given by substituting 3, 4, 5, and 6 for k in $2k + 3$.

Furthermore, $\sum_{k=1}^{n} 1$ is the sum of the first n terms of the progression 1, 1, 1, ..., 1, ... Therefore,

$$\sum_{k=1}^{n} 1 = \underbrace{1 + 1 + 1 + \dots + 1}_{n \text{ times}} = n \ .$$

Analogously, if c is a constant, then

$$\sum_{k=1}^{n} c = cn.$$

Problem 3 Express the following sums using $\sum$.

(1) $1^3 + 2^3 + 3^3 + \dots + n^3$

(2) $5 + 9 + 13 + 17 + \dots + 41$

Problem 4 Rewrite the following sums as the sum of all the terms without using $\sum$.

(1) $\sum_{k=1}^{8}(3k+1)$ (2) $\sum_{i=0}^{4}\left(\frac{1}{2}\right)^i$ (3) $\sum_{j=1}^{n} j(j+1)$

We can use $\sum$ to formulate the following expressions for the sum of the given progressions:

The Sum of a Power

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \{\frac{n(n+1)}{2}\}^2$$

Given the two progressions

$$a_1, a_2, a_3, \ldots, a_n$$

$$b_1, b_2, b_3, \ldots, b_n$$

the following generalization holds:

$$(a_1 + b_1) + (a_2 + b_2) + \ldots + (a_n + b_n)$$

$$= (a_1 + a_2 + \ldots + a_n) + (b_1 + b_2 + \ldots + b_n).$$

Also, for a constant c,

$$ca_1 + ca_2 + \ldots + ca_n = c(a_1 + a_2 + \ldots + a_n).$$

If we express these formulas using $\sum$, we obtain the following formulas.

The Properties of $\sum$

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$$

Demonstration 2 Find the sum of the first n terms of a progression in which the nth term is expressed as $n^2 + 3n - 4$.

[Solution] Since the kth term of this progression is $k^2 + 3k - 4$, take S_n as the sum we want to find, and then

$$S_n = \sum_{k=1}^{n} (k^2 + 3k - 4) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 3k + \sum_{k=1}^{n} (-4)$$

$$= \sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k - 4\sum_{k=1}^{n} 1$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} - 4n$$

$$= \frac{1}{3} n(n-1)(n+7).$$

Problem 5 Find the following sums:

(1) $\sum_{k=1}^{n} (5k + 1)$ (2) $\sum_{i=1}^{n-1} (i+1)(i-2)$

Problem 6 Find the following sums:

(1) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + n(n+1)$

(2) $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + ... + n(n+1)(n+2)$

Problem 7 Find the sum of the first n terms of the following progressions:

(1) $1 \cdot 3,\ 2 \cdot 4,\ 3 \cdot 5,\ 4 \cdot 6,\ 5 \cdot 7, ...$

(2) $1^2 \cdot 2,\ 2^2 \cdot 5,\ 3^2 \cdot 8,\ 4^2 \cdot 11,\ 5^2 \cdot 14, ...$

We are given a progression $\{a_n\}$. The progression $\{b_n\}$ created by

$$b_n = a_{n+1} - a_n \quad (n = 1, 2, 3, \ldots).$$

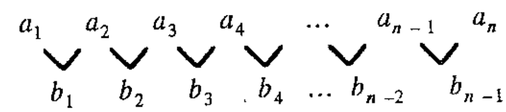
is called the **progression of differences** of the original progression.

$b_1 = a_2 - a_1$

$b_2 = a_3 - a_2$

$\ldots$

$b_{n-1} = a_n - a_{n-1}$



Adding up the corresponding sides of these equalities, we obtain

$$b_1 + b_2 + \ldots + b_{n-1} = a_n - a_1.$$

Therefore, for $n \geq 2$,

$$a_n = a_1 + (b_1 + b_2 + \ldots + b_{n-1})$$

$$= a_1 + \sum_{k=1}^{n-1} b_k.$$

Thus, for $n \geq 2$, a_n can be expressed in terms of the first term a_1 and the sum of the first $n - 1$ terms of the progression of differences.

Demonstration 3 Find the general term of the progression 1, 3, 7, 13, 21, 31, ...

[Solution] Take $\{a_n\}$ as the given progression and $\{b_n\}$ as the progression of differences.

$$2, 4, 6, 8, 10, \ldots$$

This is an arithmetic progression in which the first term is 2 and the common difference is 2. Therefore,

$$b_n = 2n.$$

Thus, for $n \geq 2$, a_n is

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k = 1 + \sum_{k=1}^{n-1} 2k$$

$$= 1 + n(n-1) = n^2 - n + 1.$$

When we substitute $n = 1$ into the expression $n^2 - n + 1$, it becomes 1, which is the same as a_1. Therefore, the general term a_n we want to find is

$$a_n = n^2 - n + 1.$$

Problem 8 Find the general term of each of the following progressions. Then find the sum of the first n terms.

(1) 1, 2, 5, 10, 17, 26, ...

(2) 3, 4, 1, 10, -17, 64, ...

Demonstration 4 Find the following sum:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)}.$$

[Solution] In general, since $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)}$$

$$= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \dots + (\frac{1}{n} - \frac{1}{n+1})$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Problem 9 Find the following sums:

(1) $\sum_{k=1}^{n} \frac{-1}{k(k+2)}$ (2) $\sum_{k=1}^{n} \frac{1}{(2k)^2 - 1}$

Problem 10 Use the identity

$$\frac{1}{k(k+1)(k+2)} = \frac{1}{2}\{\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\}$$

to find $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)}$.

Demonstration 5 Find the general term of a progression in which the sum of the first n terms is $n^3 - n$.

[Solution] Take a_n as the nth term of this progression and S_n as the sum of the first n terms.

Since for $n \geq 2$, $a_n = S_n - S_{n-1}$, we know that

$$a_n = (n^3 - n) - \{(n-1)^3 - (n-1)\}$$

$$= 3n(n-1). \qquad (1)$$

For $n = 1$, since $a_1 = S_1$, we obtain

$$a_1 = 1^3 - 1 = 0. \qquad (2)$$

From (1) and (2), for all natural numbers n

$$a_n = 3n(n-1).$$

Problem 11 Find the general term of a progression $\{a_n\}$ in which the sum S_n of the first n terms is $n^3 - n + 2$.

Exercises

1. Given a progression in which the first term is 5. For this progression the sum of the first 3 terms is equal to the sum of the first 5 terms. Find the common difference of the progression.

2. Given an arithmetic progression in which the 5th term is 108 and the 20th term is -237.

   (1) Find the first term and the common difference of this progression.

   (2) If the sum of the first n terms of this progression is a maximum, what is the value of n?

3. Given a geometric progression in which the third term is 12 and the 6th term is 96.

   (1) Find the sum of the squares of the first n terms.

   (2) Find the product of the first n terms.

4. Find the sum of the first n terms of the following progressions:

   (1) $1^2, 4^2, 7^2, 10^2, 13^2, \ldots$

   (2) $2 \cdot 3^2,\ 4 \cdot 4^2,\ 6 \cdot 5^2,\ 8 \cdot 6^2;\ 10 \cdot 7^2,\ \ldots$

5. Given a progression 2, 3, 9, 18, 28, 37, 43, 44, ... Find the general term of this progression by taking the progression of differences, and again taking the differences of the progression of differences.

6. Determine whether or not the progression in which the sum of the first n terms is expressed by a quadratic expression in n, $an^2 + bn + c$, is an arithmetic progression.