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Trigonometric identities

2.1 Identities and equations

You should recall from your study of algebra that mathematical expressions that involve an equals sign are called equations. An equation, actually a statement that two expressions are equal, is sometimes called a statement of equality. A statement of equality may be either true or false. Some examples of equations appear below.

$$(x-1)(x+1) = x^2 - 1. \qquad 3x - 4 = 5.$$

We are actually considering two types of algebraic equations here. The first equation is intended to indicate that if x represents any number for which $(x-1)(x+1)$ and $x^2 - 1$ are both meaningful expressions, then $(x-1)(x+1)$ and $x^2 - 1$ are different names for the same number. An equation of this type is called an **identity**.

The second equation expresses a different idea. The two expressions, $3x - 4$ and 5, may or may not be different names for the same number. Clearly, if $x = 3$, we have a true statement of equality and if $x \neq 3$, we have a false statement of equality. Equations of this type are called **conditional equations.**

The *replacement set* for an equation in one variable is a set of numbers any one of which may be used to replace the variable. Sometimes we agree before-

hand what set of numbers this will be. It may be the set of natural numbers, the set of integers, the set of complex numbers, or any other set of numbers. Often we assume that the replacement set for an equation is the set of all real numbers for which each expression in the equation is a meaningful expression.

A conditional equation may be defined more accurately as *an equation for which at least one element in the replacement set results in a false statement of equality*. The equation $3x - 4 = 5$ is a conditional equation, since 2 is in the replacement set for $3x - 4 = 5$, and $3 \cdot 2 - 4 = 5$ is a *false* statement of equality.

An identity may be defined as *an equation for which each element in the replacement set results in a true statement of equality.*

From these two definitions, you can see that every equation is either a conditional equation or an identity.

One should keep in mind that in algebra different replacement sets are used for the variables on different occasions. For example, if the replacement set for the equation $x^2 + 1 = 0$ were the set of real numbers, there would be no elements in the replacement set that result in a true statement of equality. If the replacement set for the equation $x^2 + 1 = 0$ is the set of complex numbers, both i and $-i$ result in true statements of equality. In each of these cases, the equation $x^2 + 1 = 0$ is a conditional equation, since, in each case, 2 is in the replacement set and 2 results in a false statement of equality.

In this section we shall extend the ideas of identities and conditiona equations to trigonometric equations. This chapter deals with techniques fo verifying that some trigonometric equations are identities, and Chapter 3 explores methods of solving conditional trigonometric equations.

In dealing with trigonometric expressions, we consider the replacemen set to be *all real numbers for which the expression is meaningful.* The replace ment set of a trigonometric equation is *the set of all real numbers for which eac member of the equation is a real number,* whether or not the resulting statemen of equality is true.

There are three ways in which substituting a particular real number fo the variable might fail to result in a meaningful expression:

1. It might leave a trigonometric function undefined.
2. It might create a situation which calls for dividing by 0.
3. It might create a situation which calls for taking an even root of a negative number.

The first of these three difficulties is illustrated in the equation $\sec x \sin x = \tan x$. You will remember that if x is a real number of the form $\pi/2 + n\pi$ where $n \in J$, both $\sec x$ and $\tan x$ are undefined. Thus, the replacement set fo $\sec x \sin x = \tan x$ is

$$R - \left\{x \mid x = \frac{\pi}{2} + n\pi, n \in J\right\}.$$

The second difficulty is illustrated in the equation

$$\sin x = \frac{1}{1 - \cos x}.$$

The right-hand member of the equation will not be defined if $\cos x = 1$, since this would involve division by 0. Since $\cos x = 1$ if $x = 2n\pi$, where $n \in J$, the replacement set for $\sin x = 1/(1 - \cos x)$ is the set

$$R - \{x \mid x = 2n\pi,\ n \in J\}.$$

We will use the symbol $\mathscr{D}$ to represent the replacement set for the expression or equation under consideration in any problem. Thus, in this last example we could have written

$$\mathscr{D} = R - \{x \mid x = 2n\pi,\ n \in J\}.$$

The discussion here has been basically algebraic in nature, and the student should recognize that the last two of the three difficulties listed can also occur in algebra. In practice, we must be alert to avoid the first two difficulties. The third will occur relatively infrequently in this book.

As indicated earlier, a conditional equation is an equation for which at least one number in the replacement set results in a false statement of equality. One way of proving that an equation is conditional is to demonstrate that such a number is in the replacement set. Take, for example, the equation $\sin x = 1/(1 - \cos x)$ for which the replacement set is $\mathscr{D} = R - \{x \mid x = 2n\pi,\ n \in J\}$. We note that

$$\pi \in \mathscr{D}, \quad \sin \pi = 0, \quad \frac{1}{1 - \cos \pi} = \frac{1}{1 - (-1)} = \frac{1}{2}, \quad \text{and} \quad 0 \neq \frac{1}{2}.$$

Thus, $\sin \pi \neq 1/(1 - \cos \pi)$, and $\sin x = 1/(1 - \cos x)$ is a conditional equation.

There are times when it is quite helpful to know whether a given trigonometric equation is an identity or a conditional equation. We have just demonstrated the basic method of proving that a trigonometric equation is a conditional equation—by showing that the use of a particular number in the replacement set results in a false statement of equality.

If we cannot find a number which proves that a trigonometric equation is a conditional equation, this does not necessarily mean that the equation in question is an identity. It may mean that we have not tried an appropriate number. If we suspect that a trigonometric equation is an identity, we can attempt to justify our suspicions by using techniques which are discussed throughout the remainder of this chapter. The equation $\sec x \sin x = \tan x$ is an identity, as you will be asked to verify in Exercises 2.2.

So far we have used only symbols with which the student is familiar to represent variables. In trigonometry it is common to use lowercase Greek letters to represent variables. We will use θ, ϕ, α, β, and γ as variables frequently. Their names are *theta*, *phi*, *alpha*, *beta*, and *gamma*, respectively.

Exercises 2.1

Prove that the equations are conditional equations. State the replacement set for each equation.

Example

$\sin x + \cos x = 1.$

Solution

$\mathscr{D} = R.$

$\frac{\pi}{4} \in \mathscr{D}, \sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \neq 1.$

1. $\cos x - \sin x = 1.$
2. $\sec x - \tan x = 1.$
3. $\cos x = \dfrac{1}{\sin x}.$
4. $\sin 2x = 2 \sin x.$
5. $\cos 2x = 2 \cos x.$
6. $\sin x - \cos x = \cos x - \sin x.$
7. $\sec x - \csc x = \tan x - \cot x.$
8. $\cos \dfrac{x}{2} = \dfrac{1}{2} \cos x.$
9. $\sin \dfrac{x}{2} = \dfrac{1}{2} \sin x.$
10. $\tan \dfrac{x}{2} = \dfrac{1}{2} \tan x.$
11. $\tan 2x = 2 \tan x.$
12. $\dfrac{\tan x + 1}{\sin x} = \sec x.$
13. $2 \sin x - \cos x + 1 = 0.$
14. $\cos x - 2 \tan x + \sec x = 0.$
15. $\csc x - \sec x = \sin x - \cos x.$

2.2 The fundamental identities

In Chapter 1 we defined the sine function and the cosine function. Th the other trigonometric functions were defined in terms of the sine and cosi functions. These four definitions give us four of the basic trigonomet identities.

1. $\tan \theta = \dfrac{\sin \theta}{\cos \theta}, \quad \mathscr{D} = R - \left\{\theta \mid \theta = \dfrac{\pi}{2} + n\pi,\ n \in J\right\}.$
2. $\cot \theta = \dfrac{\cos \theta}{\sin \theta}, \quad \mathscr{D} = R - \{\theta \mid \theta = n\pi,\ n \in J\}.$
3. $\sec \theta = \dfrac{1}{\cos \theta}, \quad \mathscr{D} = R - \left\{\theta \mid \theta = \dfrac{\pi}{2} + n\pi,\ n \in J\right\}.$
4. $\csc \theta = \dfrac{1}{\sin \theta}, \quad \mathscr{D} = R - \{\theta \mid \theta = n\pi,\ n \in J\}.$

There is nothing in the equation $\tan\theta = \sin\theta/\cos\theta$ to tell us that this is an identity rather than a conditional equation. We shall use the symbol $\equiv$ to represent the idea that one member of an equation is equal to the other member of the equation for each number in the replacement set for the equation. Thus, the four identities above could have been written

1. $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$, 2. $\cot\theta \equiv \dfrac{\cos\theta}{\sin\theta}$,

3. $\sec\theta \equiv \dfrac{1}{\cos\theta}$, 4. $\csc\theta \equiv \dfrac{1}{\sin\theta}$,

with the replacement set for each equation.

As indicated earlier, the replacement set for an equation is the set of all real numbers for which each member of the equation is a real number. If there is a single real number for which the two members of the equation are both real numbers, but not equal, then that equation is not an identity.

We shall use $\equiv$ in only two cases—when we have proved that an equation is an identity and when we state a problem in which you are to prove that the given equation is an identity.

Three very important identities that were developed in Chapter 1 appear below.

5. $\sin^2\theta + \cos^2\theta \equiv 1$, $\mathscr{D} = R$.

6. $\tan^2\theta + 1 \equiv \sec^2\theta$, $\mathscr{D} = R - \left\{\theta \mid \theta = \frac{\pi}{2} + n\pi,\ n \in J\right\}$.

7. $\cot^2\theta + 1 \equiv \csc^2\theta$, $\mathscr{D} = R - \{\theta \mid \theta = n\pi,\ n \in J\}$.

Several other identities were developed in the exercises. These are usually called reduction formulas, since, when we are trying to evaluate a trigonometric expression, we can use them to reduce the problem to the evaluation of a trigonometric expression where the variable represents a number between 0 and $\pi/2$. These reduction formulas appear below.

8. $\sin(-\theta) \equiv -\sin\theta$, $\mathscr{D} = R$.

9. $\cos(-\theta) \equiv \cos\theta$, $\mathscr{D} = R$.

10. $\tan(-\theta) \equiv -\tan\theta$, $\mathscr{D} = R - \left\{\theta \mid \theta = \frac{\pi}{2} + n\pi,\ n \in J\right\}$.

11. $\cot(-\theta) \equiv -\cot\theta$, $\mathscr{D} = R - \{\theta \mid \theta = n\pi,\ n \in J\}$.

12. $\sec(-\theta) \equiv \sec\theta$, $\mathscr{D} = R - \left\{\theta \mid \theta = \frac{\pi}{2} + n\pi,\ n \in J\right\}$.

13. $\csc(-\theta) \equiv -\csc\theta$, $\mathscr{D} = R - \{\theta \mid \theta = n\pi,\ n \in J\}$.

14. $\sin(\theta + \pi) \equiv -\sin\theta$, $\mathscr{D} = R$.

15. $\cos(\theta + \pi) \equiv -\cos\theta$, $\mathscr{D} = R$.

16. $\tan(\theta + \pi) \equiv \tan\theta$, $\mathscr{D} = R - \left\{\theta \mid \theta = \frac{\pi}{2} + n\pi,\ n \in J\right\}$.

17. $\cot(\theta + \pi) \equiv \cot\theta$, $\mathscr{D} = R - \{\theta \mid \theta = n\pi,\ n \in J\}$.

18. $\sec(\theta + \pi) \equiv -\sec\theta$, $\mathscr{D} = R - \left\{\theta \mid \theta = \frac{\pi}{2} + n\pi,\ n \in J\right\}$.

19. $\csc(\theta + \pi) \equiv -\csc\theta$, $\mathscr{D} = R - \{\theta \mid \theta = n\pi,\ n \in J\}$.

20. $\sin(\pi - \theta) \equiv \sin\theta$, $\mathscr{D} = R$.

21. $\cos(\pi - \theta) \equiv -\cos\theta$, $\mathscr{D} = R$.

22. $\tan(\pi - \theta) \equiv -\tan\theta$, $\mathscr{D} = R - \left\{\theta \mid \theta = \frac{\pi}{2} + n\pi,\ n \in J\right\}$.

23. $\cot(\pi - \theta) \equiv -\cot\theta$, $\mathscr{D} = R - \{\theta \mid \theta = n\pi,\ n \in J\}$.

24. $\sec(\pi - \theta) \equiv -\sec\theta$, $\mathscr{D} = R - \left\{\theta \mid \theta = \frac{\pi}{2} + n\pi,\ n \in J\right\}$.

25. $\csc(\pi - \theta) \equiv \csc\theta$, $\mathscr{D} = R - \{\theta \mid \theta = n\pi,\ n \in J\}$.

26. $\sin(\theta + 2n\pi) \equiv \sin\theta,\ n \in J$, $\mathscr{D} = R$.

27. $\cos(\theta + 2n\pi) \equiv \cos\theta,\ n \in J$, $\mathscr{D} = R$.

28. $\tan(\theta + n\pi) \equiv \tan\theta,\ n \in J$, $\mathscr{D} = R - \left\{\theta \mid \theta = \frac{\pi}{2} + n\pi,\ n \in J\right\}$.

29. $\cot(\theta + n\pi) \equiv \cot\theta,\ n \in J$, $\mathscr{D} = R - \{\theta \mid \theta = n\pi,\ n \in J\}$.

30. $\sec(\theta + 2n\pi) \equiv \sec\theta,\ n \in J$, $\mathscr{D} = R - \left\{\theta \mid \theta = \frac{\pi}{2} + n\pi,\ n \in J\right\}$.

31. $\csc(\theta + 2n\pi) \equiv \csc\theta,\ n \in J$, $\mathscr{D} = R - \{\theta \mid \theta = n\pi,\ n \in J\}$.

The student should learn all these identities. They seem like a lot to remember all at once, but in practice many of them can be remembered in other ways. The last six identities refer to the periodicity of the trigonometric functions. One way of remembering the others is to remember which functions are positive and which are negative in each quadrant.

2.3 Verifying trigonometric identities

The 31 basic identities considered in the last section were originally devel oped in several fashions. We proved the identity $\sin^2\theta + \cos^2\theta \equiv 1$ by usin the fact that we were considering a point on the unit circle. The first four iden tities were simply definitions. The last 24 in the list were developed primaril in the homework problems, through the consideration of the relative position of different points on the unit circle.

These techniques which we have used so far are helpful, but they will b of relatively little help in the development of further identities. As an aid t developing techniques of proving identities, let us consider two different way of proving the identity

$$\tan^2\theta + 1 \equiv \sec^2\theta.$$

We will consider the development given in the preceding chapter first.

(1) $\sin^2\theta + \cos^2 \equiv 1$

(2) $\dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{\cos^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta} \qquad \theta \neq \dfrac{\pi}{2} + n\pi,\ n \in J$

(3) $\left(\dfrac{\sin\theta}{\cos\theta}\right)^2 + 1 = \left(\dfrac{1}{\cos\theta}\right)^2$

(4) $\tan^2\theta + 1 = \sec^2\theta$

$\therefore \tan^2\theta + 1 \equiv \sec^2\theta, \quad \mathscr{D} = R - \left\{\theta \,|\, \theta = \dfrac{\pi}{2} + n\pi, n \in J\right\}.$

In this proof we started with a known identity. Then we divided both sides by the same quantity ($\cos^2\theta$) and indicated that those numbers which would cause division by 0 were not in the replacement set for the equation. The third step was simply an algebraic manipulation, so that we could apply the definitions of $\tan\theta$ and $\sec\theta$ in the fourth step. It is important to note here that we started with a known identity, not what we wanted to prove. One difficulty with this particular technique is that most students lack sufficient insight to see which known identity, if any, could be used to start proving that a given equation is an identity.

A second technique that can be used to verify identities consists of the following steps:

1. *State the desired identity,*
2. *Indicate the appropriate replacement set,*
3. *Start with one of the two sides of the equation,* and
4. *Attempt to transform that expression into the other side of the equation through a series of steps involving*
   a. *Algebraic manipulations* and
   b. *The use of previously proven identities.*

This technique is demonstrated below.

To prove: $\tan^2\theta + 1 \equiv \sec^2\theta, \quad \mathscr{D} = R - \left\{\theta \,|\, \theta = \dfrac{\pi}{2} + n\pi, n \in J\right\}.$

(1) $\tan^2\theta + 1 = \left(\dfrac{\sin\theta}{\cos\theta}\right)^2 + 1$

(2) $= \dfrac{\sin^2\theta}{\cos^2\theta} + 1$

(3) $= \dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{\cos^2\theta}{\cos^2\theta}$

(4) $= \dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta}$

(5) $= \dfrac{1}{\cos^2\theta}$

(6) $= \left(\dfrac{1}{\cos\theta}\right)^2$

(7) $= \sec^2\theta$

$\therefore \tan^2\theta + 1 \equiv \sec^2\theta, \quad \mathscr{D} = R - \left\{\theta \,|\, \theta = \dfrac{\pi}{2} + n\pi, n \in J\right\}.$

This proof has not combined any two steps into a single step. The first step applies the definition of the tangent function. The second, fourth, and sixth steps apply rules of algebra. The third step expresses 1 as $\cos^2\theta/\cos^2\theta$ so that we will have a common denominator for the addition of fractions. The fifth step applies a previously proven identity. The seventh step used the definition of $\sec\theta$.

This proof could be abbreviated quite satisfactorily as follows:

To prove: $\tan^2\theta + 1 \equiv \sec^2\theta, \quad \mathscr{D} = R - \left\{\theta \,|\, \theta = \dfrac{\pi}{2} + n\pi, n \in J\right\}.$

(1) $\tan^2\theta + 1 = \dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{\cos^2\theta}{\cos^2\theta}$

(2) $= \dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta}$

(3) $= \dfrac{1}{\cos^2\theta}$

(4) $= \sec^2\theta$

$\therefore \tan^2\theta + 1 \equiv \sec^2\theta, \quad \mathscr{D} = R - \left\{\theta \,|\, \theta = \dfrac{\pi}{2} + n\pi, n \in J\right\}.$

In practice, most students will prefer to combine steps as was done this last proof rather than giving each step separately, as was done in the fi proof. It should be noted that in using this technique we started with the mc complicated side of the equation and worked to obtain the less complicat side.

Let us consider another example.

To prove: $\dfrac{1 - \sin x}{\cos x} \equiv \dfrac{\cos x}{1 + \sin x}$. What is $\mathscr{D}$?

The two sides of the equation are of about the same complexity. One mi start by noting that the given equation will be an identity if and only if

$$(1 - \sin x)(1 + \sin x) = (\cos x)(\cos x).$$

Noticing that $(1 - \sin x)(1 + \sin x) = 1 - \sin^2 x$, by a rule of algebra, a that $1 - \sin^2 x = \cos^2 x$, we now have enough clues to start the following prc

(1) $1 = \sin^2 x + \cos^2 x$

(2) $1 - \sin^2 x = \cos^2 x$

(3) $(1 - \sin x)(1 + \sin x) = (\cos x)(\cos x)$

(4) $\dfrac{1-\sin x}{\cos x} = \dfrac{\cos x}{1+\sin x}$ (what restrictions here?)

$$\therefore \frac{\mathbf{1-\sin x}}{\mathbf{\cos x}} \equiv \frac{\mathbf{\cos x}}{\mathbf{1+\sin x}}, \quad \mathscr{D} = R - \left\{x \mid x = \frac{\pi}{2} + n\pi, n \in J\right\}.$$

The student should be able to provide justification for each step in the proof. In this particular proof, the analysis we made before starting helped us see where we might start. There are several other ways that this identity could be proved. One of them appears below.

We observe that the right side of the equation is $\cos x/(1 + \sin x)$. We *might multiply the left side of the equation* by 1, expressed in the form $(1 + \sin x)/(1 + \sin x)$, in order to get the expression $1 + \sin x$ in the denominator. If we are fortunate, we will see where to go from there.

(1) $\dfrac{1-\sin x}{\cos x} = \left(\dfrac{1-\sin x}{\cos x}\right)\left(\dfrac{1+\sin x}{1+\sin x}\right)$ (restrictions?)

(2) $\qquad = \dfrac{1-\sin^2 x}{\cos x\,(1+\sin x)}$

(3) $\qquad = \dfrac{\cos^2 x}{\cos x\,(1+\sin x)}$

(4) $\qquad = \dfrac{\cos x}{1+\sin x}$

$$\therefore \frac{\mathbf{1-\sin x}}{\mathbf{\cos x}} \equiv \frac{\mathbf{\cos x}}{\mathbf{1+\sin x}}, \quad \mathscr{D} = R - \left\{x \mid x = \frac{\pi}{2} + n\pi, n \in J\right\}.$$

As was suggested, we multiplied the left side of the equation by 1, expressed in an appropriate form. Once we had done this, the following steps were largely a matter of observation.

It should be noted that in proving trigonometric identities the most important tool is the power of observation. One needs to observe what he has and what he wants to get. Then he needs to select appropriate identities which have been proved for making the desired manipulations. There is no series of steps which will always work. The best way to develop the necessary power of observation is to practice. This practice can be obtained in only one way, through attempting to prove identities.

Another example appears below.

To prove: $\cos^6\phi + \sin^6\phi \equiv \cos^4\phi - \cos^2\phi\sin^2\phi + \sin^4\phi, \quad \mathscr{D} = R.$

In order to prove the identity above, it helps if we notice that $\cos^6\phi + \sin^6\phi$ is in the form $x^6 + y^6$, which can be factored as the sum of two cubes,

$$x^6 + y^6 = (x^2+y^2)(x^4 - x^2y^2 + y^4).$$

Thus,

$$\cos^6\phi + \sin^6\phi = (\cos^2\phi + \sin^2\phi)(\cos^4\phi - \cos^2\phi\sin^2\phi + \sin^4\phi)$$

$$= 1\cdot(\cos^4\phi - \cos^2\phi\sin^2\phi + \sin^4\phi)$$
$$= \cos^4\phi - \cos^2\phi\sin^2\phi + \sin^4\phi.$$

$$\therefore \boldsymbol{\cos^6\phi + \sin^6\phi \equiv \cos^4\phi - \cos^2\phi\sin^2\phi + \sin^4\phi}, \quad \mathscr{D} = R.$$

Exercises 2.2

Most of the following equations are identities. Some may be conditional equations. Determine in which category each belongs and prove your conclusion. Give the replacement set for each equation.

1. $\sec x \sin x \equiv \tan x.$
2. $\sec\phi\cot\phi \equiv \csc\phi.$
3. $\tan y\csc y \equiv \sec y.$
4. $\csc\theta\cos\theta \equiv \cot\theta.$
5. $\sin\alpha\cot\alpha \equiv \cos\alpha.$
6. $\cos\beta\tan\beta \equiv \sin\beta.$
7. $\sec^2 x - \tan^2 x \equiv 1.$
8. $\csc^2\beta - \cot^2\beta \equiv 1.$
9. $\csc^2 y - \sec^2 y \equiv \sin^2 y.$
10. $\cos(-\theta)\cos\theta - \sin\theta\sin(-\theta) \equiv 1.$
11. $\tan^2\theta\csc^2\theta - \sec^2\theta\sin^2\theta \equiv 1.$
12. $\cos^2 y\tan^2 y + \sin^2 y\cot^2 y \equiv 1.$
13. $\sec^2\phi\cot^2\phi - \csc^2\phi\cos^2\phi \equiv 1.$
14. $\cos^4\alpha - \sin^4\alpha \equiv \cos^2\alpha - \sin^2\alpha.$
15. $\sec^4 x - \tan^4 x \equiv \sec^2 x + \tan^2 x.$
16. $\sec^4\phi - 1 \equiv \tan^2\phi(\sec^2\phi + 1).$
17. $\csc^4 z - \cot^4 z \equiv \csc^2 z + \cot^2 z.$
18. $\tan^3\alpha + 1 \equiv (\tan\alpha + 1)(\sec^2\alpha - \tan\alpha).$
19. $\tan^3\alpha - 1 \equiv (\tan\alpha - 1)(\tan\alpha + \sec^2\alpha).$
20. $\cot^3\beta + 1 \equiv (\cot\beta + 1)(\csc^2\beta - \cot\beta).$
21. $\cot^3\beta - 1 \equiv (\cot\beta - 1)(\cot\beta + \csc^2\beta).$
22. $\cos^6\gamma - \sin^6\gamma \equiv \cos^2\gamma + \cos^4\gamma.$
23. $\cos^6\phi - \sin^6\phi \equiv \cos^4\phi + \sin^2\phi.$
24. $\cos^3 x + \sin^3 x \equiv \cos x(1 - \cos x\sin x) + \sin x(1 - \cos x\sin x).$
25. $\cos^3 y + \sin^3 y \equiv \cos y(1 - \cos y\sin y) - \sin y(\cos y\sin y - 1).$
26. $\cos^3\alpha - \sin^3\alpha \equiv \cos\alpha(1 + \sin\alpha\cos\alpha) - \sin\alpha(1 + \sin\alpha\cos\alpha).$
27. $(\tan^2\gamma + 1)^2 \equiv (-\sec\gamma)^4.$
28. $(\cos x - \sin x)^2 \equiv 1 - 2\sin x\cos x.$
29. $(\tan\alpha - 1)^2 \equiv \sec^2\alpha - 2\tan\alpha.$
30. $(1 + \cot z)^2 \equiv 2\cot z + \csc^2 z.$
31. $(\cos\beta + \sin\beta)^4 \equiv \cos^2\beta + \sin^2\beta + 4\cos^2\beta\sin^2\beta + 4\cos\beta\sin\beta.$

32. $(\cos\alpha - \sin\alpha)^4 \equiv \cos^2\alpha + \sin^2\alpha + 4\cos^2\alpha\sin^2\alpha - 4\cos\alpha\sin\alpha$.
33. $(\sin\theta + \cos\theta)^3 \equiv \sin\theta + 2\sin\theta\cos\theta(\sin\theta + \cos\theta) + \cos\theta$.
34. $\cos^2 x - \sin^2 x \equiv 1 - 2\sin^2 x$.
35. $\cos^2\alpha - \sin^2\alpha \equiv 2\cos^2\alpha - 1$.
36. $\cos^4\theta \equiv 1 - 2\sin^2\theta + \sin^4\theta$.
37. $\sec^2\phi + \csc^2\phi \equiv \sec^2\phi\csc^2\phi$.
38. $\sec^4 x - \tan^4 x \equiv \dfrac{\sin^2 x + 1}{\cos^2 x}$.
39. $\csc^4\beta - \cot^4\beta \equiv \dfrac{\cos^2\beta + 1}{\sin^2\beta}$.
40. $\csc^4\alpha - 1 \equiv \cot^2\alpha(\csc^2\alpha + 1)$.
41. $\tan^2\gamma - \cot^2\gamma \equiv \dfrac{\sin^2\gamma - \cos^2\gamma}{\cos^2\gamma\sin^2\gamma}$.
42. $\tan^2 z + \cot^2 z \equiv \dfrac{2\sin^4 z - 2\sin^2 z + 1}{\sin^2 z\cos^2 z}$.
43. $\tan^2\phi + \cot^2\phi \equiv 2\tan^2\phi - 2\sec^2\phi + \sec^2\phi\csc^2\phi$.
44. $\dfrac{\tan y + \cot y}{\sec y\csc y} \equiv 1$.
45. $\dfrac{\sin\phi + \cos^2\phi\csc\phi}{\csc\phi} \equiv 1$.

2.4 Sum and difference identities

Some of the many identities useful in trigonometry deal with expressions such as $\cos(\alpha \pm \beta)$ and $\sin(\alpha \pm \beta)$. Identities involving these expressions will be developed in this section.

First we shall develop the identity

$$\cos(\alpha + \beta) \equiv \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

if $\alpha, \beta \in R$. To do this we use the unit circle and the distance formula. Consider Figure 2.1, where the arcs $\widehat{PQ}$ and $\widehat{PR}$ are assumed to have lengths α and β, respectively. This means that the points Q and R must have the coordinates indicated. The points S and T depend upon the points P and Q. S is the point

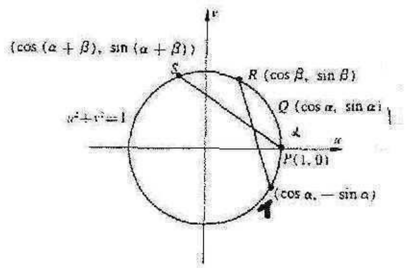

Figure 2.1

corresponding to $\alpha + \beta$, and T is the point corresponding to $-\alpha$. This determines the coordinates for the points S and T. Note that identities for $\cos(-\alpha)$ and $\sin(-\alpha)$ have been used in giving the coordinates of T.

Because of the way in which the points S and T were selected, the arcs $\widehat{PS}$ and $\widehat{TR}$ have the same length, $\alpha + \beta$. Since, within the same circle, equal arcs subtend equal chords, the lengths of the segments $\overline{PS}$ and $\overline{TR}$ are equal. This leads us to the following development.

$$\begin{aligned} d(PS) &= \sqrt{[\cos(\alpha+\beta) - 1]^2 + [\sin(\alpha+\beta) - 0]^2} \\ &= \sqrt{\cos^2(\alpha+\beta) - 2\cos(\alpha+\beta) + 1 + \sin^2(\alpha+\beta)} \\ &= \sqrt{2 - 2\cos(\alpha+\beta)}. \\ d(TR) &= \sqrt{(\cos\beta - \cos\alpha)^2 + [\sin\beta - (-\sin\alpha)]^2} \\ &= \sqrt{(\cos\beta - \cos\alpha)^2 + (\sin\beta + \sin\alpha)^2} \\ &= \sqrt{\cos^2\beta - 2\cos\beta\cos\alpha + \cos^2\alpha + \sin^2\beta + 2\sin\beta\sin\alpha + \sin^2\alpha} \\ &= \sqrt{2 - 2\cos\beta\cos\alpha + 2\sin\beta\sin\alpha}. \end{aligned}$$

Since $d(PS) = d(TR)$, we have

$$\begin{aligned} \sqrt{2 - 2\cos(\alpha+\beta)} &= \sqrt{2 - 2\cos\beta\cos\alpha + 2\sin\beta\sin\alpha}, \\ 2 - 2\cos(\alpha+\beta) &= 2 - 2\cos\beta\cos\alpha + 2\sin\beta\sin\alpha, \\ -2\cos(\alpha+\beta) &= -2\cos\beta\cos\alpha + 2\sin\beta\sin\alpha, \\ \cos(\alpha+\beta) &= \cos\beta\cos\alpha - \sin\beta\sin\alpha, \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta. \\ \therefore \cos(\alpha+\beta) &\equiv \cos\alpha\cos\beta - \sin\alpha\sin\beta, \quad \alpha \in R, \beta \in R. \end{aligned}$$

Any real numbers α and β could have been selected. α and β were chose[n] to be positive with a sum less than π for the convenience of the illustratio[n]. You may wish to experiment with other real numbers to demonstrate th[e] generality of the identity for yourself.

If we wish to use an expression for $\cos(\alpha - \beta)$, we note that $\alpha -$ [β] $= \alpha + (-\beta)$. This enables us to develop

$$\cos(\alpha - \beta) \equiv \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

as follows:

$$\begin{aligned} \cos(\alpha - \beta) &= \cos[\alpha + (-\beta)] \\ &= \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha(-\sin\beta) \\ &= \cos\alpha\cos\beta + \sin\alpha\sin\beta. \\ \therefore \cos(\alpha - \beta) &\equiv \cos\alpha\cos\beta + \sin\alpha\sin\beta, \quad \alpha \in R, \beta \in R. \end{aligned}$$

From the identity $\cos(\alpha - \beta) \equiv \cos\alpha\cos\beta + \sin\alpha\sin\beta$ we ca[n] develop the identity

$$\cos\left(\frac{\pi}{2} - \theta\right) \equiv \sin\theta.$$

This identity will be most helpful in developing expressions for $\sin(\alpha+\beta)$ and $\sin(\alpha-\beta)$. We note that

$$\cos\left(\frac{\pi}{2}-\theta\right) = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta$$
$$= (0)\cos\theta + (1)\sin\theta$$
$$= \sin\theta.$$
$$\therefore \cos\left(\frac{\pi}{2}-\theta\right) \equiv \sin\theta, \quad \mathscr{D} = R.$$

It is also true that $\sin(\pi/2-\theta) \equiv \cos\theta$ for any real number θ. We note that

$$\cos\left[\frac{\pi}{2}-\left(\frac{\pi}{2}-\theta\right)\right] = \sin\left(\frac{\pi}{2}-\theta\right).$$

But since

$$\frac{\pi}{2}-\left(\frac{\pi}{2}-\theta\right) = \frac{\pi}{2}-\frac{\pi}{2}+\theta = \theta,$$

we have

$$\cos\theta = \cos\left[\frac{\pi}{2}-\left(\frac{\pi}{2}-\theta\right)\right]$$
$$= \sin\left(\frac{\pi}{2}-\theta\right).$$
$$\therefore \sin\left(\frac{\pi}{2}-\theta\right) \equiv \cos\theta, \quad \mathscr{D} = R.$$

We now note that $\cos[\pi/2-(\alpha+\beta)] = \sin(\alpha+\beta)$. But since

$$\frac{\pi}{2}-(\alpha+\beta) = \left(\frac{\pi}{2}-\alpha\right)-\beta,$$

we have

$$\sin(\alpha+\beta) = \cos\left[\frac{\pi}{2}-(\alpha+\beta)\right]$$
$$= \cos\left[\left(\frac{\pi}{2}-\alpha\right)-\beta\right]$$
$$= \cos\left(\frac{\pi}{2}-\alpha\right)\cos\beta + \sin\left(\frac{\pi}{2}-\alpha\right)\sin\beta$$
$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$
$$\therefore \sin(\alpha+\beta) \equiv \sin\alpha\cos\beta + \cos\alpha\sin\beta, \quad \alpha\in R, \beta\in R.$$

To develop an expression for $\sin(\alpha-\beta)$, we note that $\alpha-\beta = \alpha+(-\beta)$ and use identities involving $\sin(-\beta)$ and $\cos(-\beta)$. The development of

$$\sin(\alpha-\beta) \equiv \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

appears below.

$$\sin(\alpha-\beta) = \sin[\alpha+(-\beta)]$$
$$= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$
$$= \sin\alpha\cos\beta + \cos\alpha(-\sin\beta)$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$
$$\therefore \sin(\alpha-\beta) \equiv \sin\alpha\cos\beta - \cos\alpha\sin\beta, \quad \alpha\in R, \beta\in R.$$

In this section we have derived the six identities which appear below. In each of them, α and β may be any real numbers.

1. $\cos(\alpha+\beta) \equiv \cos\alpha\cos\beta - \sin\alpha\sin\beta.$
2. $\cos(\alpha-\beta) \equiv \cos\alpha\cos\beta + \sin\alpha\sin\beta.$
3. $\cos\left(\frac{\pi}{2}-\alpha\right) \equiv \sin\alpha.$
4. $\sin\left(\frac{\pi}{2}-\alpha\right) \equiv \cos\alpha.$
5. $\sin(\alpha+\beta) \equiv \sin\alpha\cos\beta + \cos\alpha\sin\beta.$
6. $\sin(\alpha-\beta) \equiv \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

These six identities are all important in trigonometry, and they are used frequently. The student should learn them. Some applications of these identities are included in the exercises.

Exercises 2.3

Use the identities of this section to evaluate the following.

Example

$\cos\frac{\pi}{12}$

Solution

$\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$, so

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$$
$$= \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{6}+\sqrt{2}}{4}.$$

1. $\sin \frac{\pi}{12}$
2. $\sin \frac{5\pi}{12}$
3. $\cos \frac{5\pi}{12}$
4. $\sin \frac{11\pi}{12}$
5. $\cos \frac{11\pi}{12}$
6. $\cos \frac{7\pi}{12}$
7. $\sin \frac{7\pi}{12}$

Verify that each of the following equations is an identity.

8. $\cos\left(\frac{\pi}{6} + x\right) \equiv \frac{\sqrt{3}}{2} \cos x - \frac{1}{2} \sin x.$
9. $\sin\left(x + \frac{\pi}{4}\right) \equiv \frac{\sqrt{2}}{2}(\sin x + \cos x).$
10. $\sin\left(\theta + \frac{\pi}{2}\right) \equiv \cos \theta.$
11. $\cos\left(\theta + \frac{\pi}{2}\right) \equiv -\sin \theta.$
12. $\sin(\theta + \pi) \equiv -\sin \theta.$
13. $\cos(\theta + \pi) \equiv -\cos \theta.$
14. $\cos 4\theta \equiv \cos 3\theta \cos \theta - \sin 3\theta \sin \theta.$
15. $\sin 7x \equiv \sin 4x \cos 3x + \cos 4x \sin 3x.$
16. $\sin 3\phi \equiv \sin 5\phi \cos 2\phi - \cos 5\phi \sin 2\phi.$
17. $\cos 5y \equiv \cos 9y \cos 4y + \sin 9y \sin 4y.$
18. $\sin(x + \pi) - \sin(x - \pi) \equiv 0.$
19. $(\sin x + \cos \beta)^2 + (\cos x + \sin \beta)^2 \equiv 2[\sin(x + \beta) + 1].$
20. $1 - \tan \theta \tan \phi \equiv \frac{\cos(\theta + \phi)}{\cos \theta \cos \phi}.$
21. $\cos(\alpha + \beta)\cos(\alpha - \beta) \equiv (\cos \alpha \cos \beta)^2 - (\sin \alpha \sin \beta)^2.$
22. $(\cos \alpha \cos \beta)^2 - (\sin \alpha \sin \beta)^2 \equiv \cos^2 \alpha - \sin^2 \beta.$
23. $\tan(\alpha + \beta) \equiv \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$ [Use $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$.]
24. $\cot(\theta + \phi) \equiv \frac{\cot \theta \cot \phi - 1}{\cot \phi + \cot \theta}.$
25. $\sec(x + y) \equiv \frac{\sec x \sec y}{1 - \tan x \tan y}.$
26. $\csc(\alpha + \beta) \equiv \frac{\csc \alpha \csc \beta}{\cot \beta + \cot \alpha}.$
27. $\tan(\theta - \phi) \equiv \frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi}.$
28. $\cot(x - y) \equiv \frac{\cot x \cot y + 1}{\cot y - \cot x}.$
29. $\sec(\alpha - \beta) \equiv \frac{\sec \alpha \sec \beta}{1 + \tan \alpha \tan \beta}.$
30. $\csc(\theta - \phi) \equiv \frac{\csc \theta \csc \phi}{\cot \phi - \cot \theta}.$
31. $\tan\left(\theta + \frac{\pi}{2}\right) \equiv -\cot \theta.$
32. $\cot\left(\alpha + \frac{\pi}{2}\right) \equiv -\tan \alpha.$

33. $\sin 2\phi \equiv 2 \sin \phi \cos \phi.$
34. $\cos 2x \equiv \cos^2 x - \sin^2 x.$
35. $\cos 2y \equiv 2\cos^2 y - 1.$
36. $\cos 2z \equiv 1 - 2\sin^2 z.$
37. $\tan 2\alpha \equiv \frac{2 \tan \alpha}{1 - \tan^2 \alpha}.$
38. $\tan(\alpha + \beta) \equiv \frac{\cot \beta + \cot \alpha}{\cot \alpha \cot \beta - 1}.$
39. $(\sin x \cos y)^2 - (\cos x \sin y)^2 \equiv \sin^2 x - \sin^2 y.$
40. $\sin(x + y)\sin(x - y) \equiv \sin^2 x - \sin^2 y.$

2.5 The "double-angle" identities

In Section 2.4 we developed the identities

$$\cos(\alpha + \beta) \equiv \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

and

$$\sin(\alpha + \beta) \equiv \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

These identities are the basis for the development of the so-called **double-angle** formulas, which give expressions equal to $\cos 2\theta$ and $\sin 2\theta$. You may have derived these identities from the previous exercises.

First we shall derive the identity

$$\mathbf{\sin 2\theta \equiv 2 \sin \theta \cos \theta}, \quad \mathscr{D} = R.$$

If, in the identity $\sin(\alpha + \beta) \equiv \sin \alpha \cos \beta + \cos \alpha \sin \beta$, we let $\alpha = \theta$ and $\beta = \theta$, we have

$$\sin(\theta + \theta) = \sin \theta \cos \theta + \cos \theta \sin \theta,$$
$$\sin 2\theta = 2 \sin \theta \cos \theta,$$
$$\therefore \mathbf{\sin 2\theta \equiv 2 \sin \theta \cos \theta}, \quad \mathscr{D} = R.$$

In a similar fashion we can develop the identity

$$\mathbf{\cos 2\theta \equiv \cos^2 \theta - \sin^2 \theta}, \quad \mathscr{D} = R.$$

The derivation appears below.

$$\cos(\theta + \theta) = \cos \theta \cos \theta - \sin \theta \sin \theta,$$
$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta,$$
$$\therefore \mathbf{\cos 2\theta \equiv \cos^2 \theta - \sin^2 \theta}, \quad \mathscr{D} = R.$$

There are two other commonly used identities involving $\cos 2\theta$, which we can derive easily from the identity $\cos 2\theta \equiv \cos^2 \theta - \sin^2 \theta$. First,

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$
$$= \cos^2 \theta + \cos^2 \theta - \sin^2 \theta - \cos^2 \theta$$

$$= 2\cos^2\theta - (\sin^2\theta + \cos^2\theta)$$
$$= 2\cos^2\theta - 1.$$
$$\therefore\ \mathbf{\cos 2\theta \equiv 2\cos^2\theta - 1,}\quad \mathscr{D} = R.$$

Second,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \cos^2\theta + \sin^2\theta - \sin^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta.$$
$$\therefore\ \mathbf{\cos 2\theta \equiv 1 - 2\sin^2\theta,}\quad \mathscr{D} = R.$$

These last two identities involving $\cos 2\theta$ are quite useful in developing the so-called "half-angle" formulas, that is identities involving $\sin\theta/2$ and $\cos\theta/2$, which will be developed in the next section.

The identities developed in these two sections are also most helpful in deriving identities involving larger multiples of the variable. We would like to derive an identity expressing $\sin 3\theta$ in terms of $\sin\theta$.

$$\sin 3\theta = \sin(2\theta + \theta)$$
$$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$$
$$= (2\sin\theta\cos\theta)\cos\theta + (1 - 2\sin^2\theta)\sin\theta$$
$$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$$
$$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$$
$$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta.$$
$$\therefore\ \mathbf{\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta,}\quad \mathscr{D} = R.$$

The tangent of the sum of two numbers can be expressed, since we have developed identities for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$. You may already have derived this identity from the exercises.

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)},\quad \alpha+\beta \neq \frac{\pi}{2} + n\pi,\ n \in J,$$
$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$
$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}},\quad \alpha \neq \frac{\pi}{2} + n\pi,\ n \in J,\quad \beta \neq \frac{\pi}{2} + n\pi,\ n \in J.$$
$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$
$$\therefore\ \mathbf{\tan(\alpha+\beta) \equiv \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}},\quad \alpha, \beta, \alpha+\beta \neq \frac{\pi}{2} + n\pi,\ n \in J.$$

We should also note that if $\alpha = \beta = \theta$, we have

$$\mathbf{\tan 2\theta \equiv \frac{2\tan\theta}{1 - \tan^2\theta}},\quad \theta, 2\theta \neq \frac{\pi}{2} + n\pi,\ n \in J.$$

Exercises 2.4

Using $\cos\pi/6 = \sqrt{3}/2$, $\sin\pi/6 = 1/2$, and $\tan\pi/6 = 1/\sqrt{3}$ and the double-angle formulas, evaluate the following.

1. $\sin\frac{\pi}{3}$
2. $\cos\frac{\pi}{3}$
3. $\tan\frac{\pi}{3}$

Using $\cos\pi/4 = \sqrt{2}/2 = \sin\pi/4$ and the double-angle formulas, evaluate the following.

4. $\sin\frac{\pi}{2}$
5. $\cos\frac{\pi}{2}$

Using $\sin\pi/2 = 1$ and $\cos\pi/2 = 0$, evaluate the following.

6. $\sin\pi$
7. $\cos\pi$

Verify that each of the following equations is an identity.

8. $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta.$
9. $\sin 4\phi \equiv 4\cos\phi(\sin\phi - 2\sin^3\phi).$
10. $\cos 4\alpha \equiv 8\cos^4\alpha - 8\cos^2\alpha + 1.$
11. $\sin 5x \equiv 16\sin^5 x - 20\sin^3 x + 5\sin x.$
12. $\cos 5y \equiv 16\cos^5 y - 20\cos^3 y + 5\cos y.$
13. $\sin\frac{x}{2}\cos\frac{x}{2} \equiv \frac{1}{2}\sin x.$ [*Hint:* $x = 2\left(\frac{x}{2}\right)$]
14. $\cos^2\frac{z}{2} - \sin^2\frac{z}{2} \equiv \cos z.$
15. $\left(\cos\frac{\beta}{2} - \sin\frac{\beta}{2}\right)^2 \equiv 1 - \sin\beta.$
16. $\sin 2\alpha \equiv \frac{2\tan\alpha}{1 + \tan^2\alpha}.$
17. $\tan 3\phi \equiv \frac{3\tan\phi - \tan^3\phi}{1 - 3\tan^2\phi}.$
18. $\tan 2y \equiv \frac{2}{\cot y - \tan y}.$
19. $\cos 2x \equiv \cos^4 x - \sin^4 x.$
20. $1 - 2\sin^2\left(\frac{\pi}{4} - \theta\right) \equiv \sin 2\theta.$
21. $\cos 2z + 2\sin^2 z \equiv 1.$
22. $\sin^2\frac{\beta}{2} \equiv \frac{1 - \cos\beta}{2}.$
23. $\cos^2 2\gamma - 2\cos^2\gamma \equiv -1.$
24. $\sec 2\phi \equiv \frac{\sec^2\phi}{1 - \tan^2\phi}.$

25. $\sin\theta \sin 3\theta \equiv \sin^2 2\theta - \sin^2\theta$.
26. $\cot 2y \equiv \dfrac{\cot^2 y - 1}{2\cot y}$.
27. $\sec y \csc y \equiv 2\csc 2y$.
28. $\cos 4\alpha \equiv 1 - 8\sin^2\alpha\cos^2\alpha$.
29. $\cos 4z \equiv 8\sin^4 z - 8\sin^2 z + 1$.
30. $(\cos y - \sin y)^2 \equiv 1 - \sin 2y$.
31. $2\sin(\alpha + \beta)\cdot\cos(\alpha - \beta) \equiv \sin 2\alpha + \sin 2\beta$.
32. $\dfrac{\sin 3\phi}{\sin\phi} - \dfrac{\cos 3\phi}{\cos\phi} \equiv 2$.
33. $2\csc 2x \equiv \cot x + \tan x$.
34. $\cot\alpha - \cot 2\alpha \equiv \csc 2\alpha$.
35. $\csc^2 2y - \sec^2 2y \equiv 4\cot 4y \csc 4y$.
36. $\cos\theta\cos 3\theta \equiv \cos^2\theta - \sin^2 2\theta$.
37. $\tan\beta \equiv \dfrac{\sin 2\beta}{1 + \cos 2\beta}$.
38. $\cot^2\phi - \tan^2\phi \equiv \dfrac{4\cos 2\phi}{\sin^2 2\phi}$.
39. $4\sin^4\gamma \equiv 3 - 4\cos 2\gamma + \cos 4\gamma$.
40. $\dfrac{1 + \sin 2x + \cos 2x}{1 + \sin 2x - \cos 2x} \equiv \cot x$.

2.6 The "half-angle" identities

In Section 2.5 we developed the double-angle identities or formulas. These identities will be used in this section to develop **the half-angle** identities, involving $\sin\theta/2$, $\cos\theta/2$, and $\tan\theta/2$.

We derived three different expressions for $\cos 2\alpha$. One of these was

$$\cos 2\alpha \equiv 1 - 2\sin^2\alpha.$$

If we substitute $\theta/2$ for α, we have the following derivation.

$$\cos 2\alpha = 1 - 2\sin^2\alpha,$$
$$\cos 2\left(\frac{\theta}{2}\right) = 1 - 2\sin^2\frac{\theta}{2},$$
$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2},$$
$$2\sin^2\frac{\theta}{2} = 1 - \cos\theta,$$
$$\sin^2\frac{\theta}{2} = \frac{1 - \cos\theta}{2}.$$
$$\therefore \sin^2\frac{\theta}{2} \equiv \frac{1 - \cos\theta}{2}, \quad \mathscr{D} = R.$$

Using this identity we can find the value of $\sin^2\theta/2$ if we know $\cos\theta$. In order to evaluate $\sin\theta/2$, we need to know whether $\sin\theta/2$ is positive or negative. If $\theta/2$ is in the first or the second quadrant, $\sin\theta/2 > 0$, and we take the positive square root of $\sin^2\theta/2$. If $\theta/2$ is in the third or the fourth quadrant, $\sin\theta/2 < 0$, and we take the negative square root of $\sin^2\theta/2$. Thus, we cannot say in general which square root is appropriate. It is for this reason that we did not solve explicitly for $\sin\theta/2$.

In a similar fashion, we use the identity

$$\cos 2\alpha \equiv 2\cos^2\alpha - 1$$

to develop the identity

$$\cos^2\frac{\theta}{2} \equiv \frac{1 + \cos\theta}{2}, \quad \mathscr{D} = R.$$

Thus,

$$\cos 2\alpha = 2\cos^2\alpha - 1,$$
$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1,$$
$$1 + \cos\theta = 2\cos^2\frac{\theta}{2},$$
$$\frac{1 + \cos\theta}{2} = \cos^2\frac{\theta}{2}.$$
$$\therefore \cos^2\frac{\theta}{2} \equiv \frac{1 + \cos\theta}{2}, \quad \mathscr{D} = R.$$

Here we determine whether $\cos\theta/2$ is positive or negative by determining the quadrant in which $\cos\theta/2$ is located. If $\theta/2$ is in the first or the fourth quadrant, $\cos\theta/2 > 0$. If $\theta/2$ is in the second quadrant or the third quadrant, $\cos\theta/2 < 0$.

These two identities can be used to derive an identity for $\tan^2\theta/2$. We have the following derivation.

$$\tan^2\frac{\theta}{2} = \frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}$$
$$= \frac{\left(\frac{1 - \cos\theta}{2}\right)}{\left(\frac{1 + \cos\theta}{2}\right)}$$
$$= \frac{1 - \cos\theta}{1 + \cos\theta}.$$
$$\therefore \tan^2\frac{\theta}{2} \equiv \frac{1 - \cos\theta}{1 + \cos\theta}, \quad \mathscr{D} = R - \{\theta \mid \theta = \pi + 2n\pi, n \in J\}.$$

We see that if $\theta/2$ is in the first or the third quadrant, $\tan \theta/2$ is positive. If $\theta/2$ is in the second or the fourth quadrant, $\tan \theta/2$ is negative.

These identities can be used to evaluate some trigonometric expressions which we would *not* be able to evaluate otherwise. In some cases when we use these formulas, the results look different from the results obtained by other means. For example,

$$\cos^2 \frac{\pi}{12} = \frac{1 + \cos \frac{\pi}{6}}{2} = \frac{1 + \frac{\sqrt{3}}{2}}{2} = \frac{2 + \sqrt{3}}{4}.$$

$\cos \pi/12 = (\sqrt{2 + \sqrt{3}})/2$, since $\pi/12$ is in the first quadrant. But from the example in Exercises 3.3 we found that $\cos \pi/12 = (\sqrt{6} + \sqrt{2})/4$. Is $(\sqrt{6} + \sqrt{2})/4 = (\sqrt{2 + \sqrt{3}})/2$? We now show that it is.

$$\frac{\sqrt{6} + \sqrt{2}}{4} = \sqrt{\left(\frac{\sqrt{6} + \sqrt{2}}{4}\right)^2} = \sqrt{\frac{6 + 2\sqrt{12} + 2}{16}} = \sqrt{\frac{8 + 2\sqrt{12}}{16}} = \sqrt{\frac{8 + 2(2\sqrt{3})}{16}} = \sqrt{\frac{2 + \sqrt{3}}{4}} = \frac{\sqrt{2 + \sqrt{3}}}{2}.$$

$$\therefore \frac{\sqrt{6} + \sqrt{2}}{4} = \frac{\sqrt{2 + \sqrt{3}}}{2}.$$

This illustrates that sometimes expressions which look very different are actually equal and that either may be the solution to a given problem.

Exercises 2.5

Use the identities developed in this section to evaluate the following expressions.

1. $\sin \frac{\pi}{12}$ 2. $\sin \frac{\pi}{8}$ 3. $\cos \frac{\pi}{8}$ 4. $\sin \frac{\pi}{24}$

5. $\cos \frac{\pi}{24}$ 6. $\cos \frac{\pi}{16}$ 7. $\sin \frac{\pi}{16}$ 8. $\tan \frac{\pi}{8}$

9. $\tan \frac{\pi}{12}$ 10. $\tan \frac{\pi}{16}$ 11. $\tan \frac{5\pi}{12}$ 12. $\sin \frac{3\pi}{8}$

13. $\cos \frac{3\pi}{8}$ 14. $\cos \frac{7\pi}{8}$ 15. $\sin \frac{7\pi}{8}$ 16. $\sin \frac{11\pi}{12}$

17. $\cos \frac{11\pi}{12}$ 18. $\tan \frac{3\pi}{8}$ 19. $\tan \frac{7\pi}{8}$ 20. $\tan \frac{11\pi}{12}$

Verify that each of the following equations is an identity.

21. $\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \equiv \frac{1}{2} \sin \alpha.$

22. $\cos^2 \frac{\theta}{4} - \cos \frac{\theta}{2} \equiv \sin^2 \frac{\theta}{4}.$

23. $\tan \frac{\beta}{2} \equiv \frac{1 - \cos \beta}{\sin \beta}.$

24. $\tan \frac{\gamma}{2} \equiv \frac{\sin \gamma}{1 + \cos \gamma}.$

25. $\cot \frac{x}{2} \equiv \frac{1 + \cos x}{\sin x}.$

26. $\cot \frac{z}{2} \equiv \frac{\sin z}{1 - \cos z}.$

27. $\tan \frac{y}{2} \equiv \csc y - \cot y.$

28. $\sec^2 \frac{\phi}{2} \equiv \frac{2}{1 + \cos \phi}.$

29. $\sin \alpha \cot \frac{\alpha}{2} \equiv 2 \cos^2 \frac{\alpha}{2}.$

30. $\left(\sin \frac{y}{2} + \cos \frac{y}{2}\right)^2 \equiv 1 + \sin y.$

31. $\tan^2 \frac{\beta}{2} + 1 \equiv 2 \csc \beta \tan \frac{\beta}{2}.$

32. $\csc^2 \frac{x}{2} \equiv \frac{2}{1 - \cos x}.$

33. $\tan \frac{\phi}{2} \sin \phi \equiv 2 \sin^2 \frac{\phi}{2}.$

34. $\frac{1 + \tan \frac{z}{2}}{1 - \tan \frac{z}{2}} \equiv \sec z + \tan z.$

35. $2 \tan \frac{\alpha}{2} \csc \alpha \equiv \sec^2 \frac{\alpha}{2}.$

36. $\cot y + \csc y \equiv \cot \frac{y}{2}.$

37. $\csc (\alpha + \beta) - \cot (\alpha + \beta) \equiv \tan \left(\frac{\alpha + \beta}{2}\right).$

38. $2 \sin^2 \frac{y}{6} - \sin^2 \frac{y}{7} \equiv \cos^2 \frac{y}{7} - \cos$

39. $\cos^2 \frac{x}{18} \equiv \sin^2 \frac{x}{18} + \cos \frac{x}{9}.$

40. $\frac{1 - \tan \frac{\alpha}{2}}{1 + \tan \frac{\alpha}{2}} \equiv \frac{\cos \alpha}{1 + \sin \alpha}.$

2.7 Other important identities

There are occasions when we would like to work with expressions l $\sin \theta \cos \phi$, $\sin \theta \sin \phi$, and $\cos \theta \cos \phi$. There are identities involving each these expressions, but they are involved and probably difficult to rememl The basic knowledge that the student should gain from this section is hov develop the necessary identities if he needs them.

We remember that for any θ and ϕ which belong to R, $\sin (\theta + \phi$ $\sin \theta \cos \phi + \cos \phi \sin \theta$ and $\cos (\theta + \phi) \equiv \cos \theta \cos \phi - \sin \theta \sin \phi$. We develop the desired expressions by manipulating these identities.

First we will develop the identity of the form $\sin\theta\cos\phi \equiv$ ________ for $\theta, \phi \in R$. We observe that the identity

$$\sin(\theta+\phi) \equiv \sin\theta\cos\phi + \cos\theta\sin\phi$$

contains the desired expression, but it also contains the expression $\cos\theta\sin\phi$. However, the identity

$$\sin(\theta-\phi) \equiv \sin\theta\cos\phi - \cos\theta\sin\phi$$

contains the same two expressions with a difference in sign that is just what we need. The formal development appears below.

$$\begin{array}{rl} \sin(\theta+\phi) &\equiv \sin\theta\cos\phi + \cos\theta\sin\phi \\ (+)\ \sin(\theta-\phi) &\equiv \sin\theta\cos\phi - \cos\theta\sin\phi \\ \hline \sin(\theta+\phi)+\sin(\theta-\phi) &= 2\sin\theta\cos\phi, \end{array}$$

$$\sin\theta\cos\phi = \frac{1}{2}[\sin(\theta+\phi)+\sin(\theta-\phi)].$$

$$\therefore\ \boldsymbol{\sin\theta\cos\phi \equiv \frac{1}{2}[\sin(\theta+\phi)+\sin(\theta-\phi)],\quad \theta,\phi\in R.}$$

The development of the identity for $\cos\theta\cos\phi$ is quite similar. We observe that the identity

$$\cos(\theta+\phi) \equiv \cos\theta\cos\phi - \sin\theta\sin\phi$$

contains the desired expression and one other. This is also true of the identity

$$\cos(\theta-\phi) \equiv \cos\theta\cos\phi + \sin\theta\sin\phi.$$

Adding these two identities gives us the following derivation.

$$\begin{array}{rl} \cos(\theta+\phi) &\equiv \cos\theta\cos\phi - \sin\theta\sin\phi \\ (+)\ \cos(\theta-\phi) &\equiv \cos\theta\cos\phi + \sin\theta\sin\phi \\ \hline \cos(\theta+\phi)+\cos(\theta-\phi) &= 2\cos\theta\cos\phi, \end{array}$$

$$\cos\theta\cos\phi = \frac{1}{2}[\cos(\theta+\phi)+\cos(\theta-\phi)].$$

$$\therefore\ \boldsymbol{\cos\theta\cos\phi \equiv \frac{1}{2}[\cos(\theta+\phi)+\cos(\theta-\phi)],\quad \theta,\phi\in R.}$$

We can see that the identities

$$\cos(\theta+\phi) \equiv \cos\theta\cos\phi - \sin\theta\sin\phi$$

and

$$\cos(\theta-\phi) \equiv \cos\theta\cos\phi + \sin\theta\sin\phi$$

both contain the expression $\sin\theta\sin\phi$ as well as the expression $\cos\theta\cos\phi$. Adding these two identities gave us an expression equivalent to $\cos\theta\cos\phi$. If we subtract the first of these two identities from the second, we can derive an identity for $\sin\theta\sin\phi$.

$$\begin{array}{rl} \cos(\theta-\phi) &\equiv \cos\theta\cos\phi + \sin\theta\sin\phi \\ -\cos(\theta+\phi) &\equiv -\cos\theta\cos\phi + \sin\theta\sin\phi \\ \hline \cos(\theta-\phi)-\cos(\theta+\phi) &= 2\sin\theta\sin\phi, \end{array}$$

$$\sin\theta\sin\phi = \frac{1}{2}[\cos(\theta-\phi)-\cos(\theta+\phi)].$$

$$\therefore\ \boldsymbol{\sin\theta\sin\phi \equiv \frac{1}{2}[\cos(\theta-\phi)-\cos(\theta+\phi)],\quad \theta,\phi\in R.}$$

Exercises 2.6

Use the identities developed in this section to evaluate the following expressions.

1. $\sin\frac{5\pi}{12}\sin\frac{\pi}{12}$
2. $\sin\frac{\pi}{4}\cos\frac{\pi}{12}$
3. $\cos\frac{\pi}{8}\cos\frac{3\pi}{8}$
4. $\cos\frac{7\pi}{12}\cos\frac{\pi}{12}$
5. $\cos\frac{11\pi}{8}\sin\frac{3\pi}{8}$
6. $\sin\frac{\pi}{24}\sin\frac{5\pi}{24}$

Verify that each of the following equations is an identity.

7. $\cos x + \cos y \equiv 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$.
8. $\cos x - \cos y \equiv -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$.
9. $\sin x + \sin y \equiv 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$.
10. $\sin x - \sin y \equiv 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$.
11. $\sin 3x\sin 7x \equiv \frac{1}{2}[\cos 4x - \cos 10x]$.
12. $2\cos 2\alpha\sin 4\alpha \equiv \sin 6\alpha + \sin 2\alpha$.
13. $\cos 4x\cos 2x \equiv \frac{1}{2}[\cos 6x + \cos 2x]$.
14. $\sin 5x\sin(-x) \equiv \frac{1}{2}[\cos 6x - \cos 4x]$.
15. $\sin 3\phi\cos 7\phi \equiv \frac{1}{2}[\sin 10\phi - \sin 4\phi]$.
16. $\cos 3x\cos(-3x) \equiv \frac{1}{2}[1 + \cos 6x]$.
17. $\cos 2\theta\,[2\cos 4\theta - 1] \equiv \cos 6\theta$.
18. $\sin 2\beta\,[1 + 2\cos 4\beta] \equiv \sin 6\beta$.
19. $2\sin 2x\cos 2x \equiv 4\cos x\,(\sin x - 2\sin^3 x)$.
20. $\sin\frac{\phi}{4}\cos\frac{\phi}{4} \equiv \frac{1}{2}\sin\frac{\phi}{2}$.

21. $\cos 3x \cos\left(\frac{\pi}{2} - 3x\right) \sec\left(6x - \frac{\pi}{2}\right) \equiv \frac{1}{2}$.

22. $2 \cos 2\theta \equiv \frac{\sin 4\theta}{\sin 2\theta}$.

23. $\sin(u + h) - \sin u \equiv 2 \cos\left(u + \frac{h}{2}\right) \sin \frac{h}{2}$.

24. $4 \sin x \sin 2x \sin 3x \equiv \sin 4x + \sin 2x - \sin 6x$.

25. $\sin 5\phi \cos 5\phi \csc 10\phi \equiv \frac{1}{2}$.

26. $4 \cos x \cos 2x \cos 3x \equiv 1 + \cos 2x + \cos 4x + \cos 6x$.

27. $2 \cos 2\alpha \cos 2\alpha \equiv 2 + 8 \cos^4 \alpha - 8 \cos^2 \alpha$.

28. $\sin 5x \cos x \equiv \sin 3x \cos 3x + \sin 2x \cos 2x$.

29. $2 \cos 6\theta \cos 2\theta \equiv (2 \cos 4\theta - 1)(\cos 4\theta + 1)$.

Review Exercises

Verify the following identities.

1. $\sec^2 x (1 - \sin^2 x) \equiv 1$.

2. $\left(\sin \frac{\phi}{2} - \cos \frac{\phi}{2}\right)^2 \equiv 1 - \sin \phi$.

3. $\sin 5y \equiv \sin 7y \cos 2y - \cos 7y \sin 2y$.

4. $\cos 2\beta \equiv \cos^4 \beta - \sin^4 \beta$.

5. $\sin \alpha + \cos \alpha \cot \alpha \equiv \csc \alpha$.

6. $\sin 2z \equiv \frac{2 \tan z}{\sec^2 z}$.

7. $\frac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan \theta$.

8. $\sec^2 \frac{y}{2} \equiv 2 \csc y \tan \frac{y}{2}$.

9. $\tan \beta (\sin \beta + \cot \beta \cos \beta) \equiv \sec \beta$.

10. $\sin 3\phi \sin \phi \equiv \sin^2 2\phi - \sin^2 \phi$.

11. $\cos(x + \pi) - \cos(x - \pi) \equiv 0$.

12. $2 \csc 2\gamma \equiv \cot \gamma + \tan \gamma$.

13. $\tan^2 \phi + \cot^2 \phi \equiv \frac{1 - 2 \sin^2 \phi + 2 \sin^4 \phi}{\sin^2 \phi - \sin^4 \phi}$.

14. $\csc^2 \frac{\beta}{2} \equiv \frac{2}{1 - \cos \beta}$.

15. $(\cos x \cos z)^2 - (\sin x \sin z)^2 \equiv \cos^2 z - \sin^2 x$.

16. $\cos^2 2\gamma - 2 \cos^2 \gamma \equiv -1$.

17. $\frac{\cos y + \sin^2 y \sec y}{\sec y} \equiv 1$.

18. $\tan \frac{\alpha}{2} \equiv \csc \alpha - \cot \alpha$.

19. $\cot(\theta + \phi) \equiv \frac{\cot \theta \cot \phi - 1}{\cot \theta + \cot \phi}$.

20. $\tan 3z \equiv \frac{3 \tan z - \tan^3 z}{1 - 3 \tan^2 z}$.

21. $(\cos \alpha + \sin \alpha)^4 \equiv 1 + 4 \cos^2 \alpha \sin^2 \alpha + 4 \cos \alpha \sin \alpha$.

22. $\cos 4\alpha \equiv 1 - 8 \sin^2 \alpha \cos^2 \alpha$.

23. $(\sin \alpha \cos \theta)^2 - (\cos \alpha \sin \theta)^2 \equiv \sin^2 \alpha - \sin^2 \theta$.

24. $\frac{1 + \sin 2x + \cos 2x}{1 - \cos 2x + \sin 2x} \equiv \cot x$.

25. $\frac{1 - \tan \frac{\alpha}{2}}{1 + \tan \frac{\alpha}{2}} \equiv \sec \alpha - \tan \alpha$.

Evaluate the following expressions.

26. $\sin \frac{\pi}{12}$ 27. $\tan \frac{\pi}{8}$ 28. $\cos\left(-\frac{\pi}{8}\right)$ 29. $\sin \frac{3\pi}{8}$

30. $\tan \frac{\pi}{12}$ 31. $\cos\left(-\frac{5\pi}{12}\right)$ 32. $\sin\left(-\frac{\pi}{16}\right)$ 33. $\tan \frac{\pi}{16}$