Example 10

A man makes an annual deposit of \$200 in an account which pays 5% interest compounded annually. How much money should be in the account immediately after the 10^{th} deposit?

Solution

The total amount in the account immediately after the 10^{th} deposit is

$$[200 + 200(1 + 0{\cdot}05) + 200(1 + 0{\cdot}05)^2 + \dots + 200(1 + 0{\cdot}05)^9]$$
$$= 200[1 + 1{\cdot}05 + 1{\cdot}05^2 + \dots + (1{\cdot}05)^9]$$
$$= 200\left[\frac{(1{\cdot}05)^{10} - 1}{1{\cdot}05 - 1}\right]$$
$$= 2\,515{\cdot}58 \text{ (dollars)}$$
$$\simeq 2\,516 \text{ (dollars)}.$$

Exercise 2.2

1. Find the common ratio and write down the n^{th} term of each of the following geometric progressions:

   (a) $3, 12, 48, \dots$

   (b) $1, \frac{1}{2}, \frac{1}{4}, \dots$

   (c) $\frac{2}{3}, \frac{1}{3}, \frac{1}{6}, \dots$

   (d) $2, -2, 2, -2, \dots$

   (e) $a^3, -a, \frac{1}{a}, -\frac{1}{a^3}, \dots$

   (f) $a^2, ab, b^2, \dots$

2. Find the sum of each of the following geometric progressions:

   (a) $10, 5, 2\frac{1}{2}, \dots$ to 10 terms

   (b) $3, 12, 48, \dots$ to m terms

   (c) $144, -108, 81, \dots$ to 8 terms

(d) $6\frac{3}{4}, -4\frac{1}{2}, 3, \ldots$ to 20 terms

(e) $8, -4, 2, \ldots$ to n terms

(f) $24, 32, 4, \ldots$ to 15 terms

3. Find the two possible values of x if each of the following is in geometric progression:

   (i) $3, x, 48$

   (ii) $72, x, 18$

   (iii) $-5, x, -45$

   (iv) $\frac{7}{2}, x, \frac{1}{2}$

4. S_n denotes the sum of a geometric progression. Find the first term and the common ratio, if:

   (i) $S_n = 4\left[1 - \left(\frac{1}{4}\right)^n\right]$

   (ii) $S_n = \frac{3^n - 1}{3^{n-1}}$

   (iii) $S_n = \frac{a^n - b^n}{b^{n-1}}$

   (iv) $33S_5 = S_{10}$, $S_{10} = S_9 + 5$

5. The sum of the first n terms of a series is $1 - \left(\frac{3}{4}\right)^n$. Obtain an expression for the n^{th} term of the series. Prove that the series is geometric, and state the values of the first term and the common ratio.

6. How many terms of the geometric series, $4 + 6 + 9 \ldots$ must be taken to obtain a sum exceeding 3 000?

7. If the 1^{st}, 4^{th} and 8^{th} terms of an arithmetic progression are in geometric progression, find the common ratio of the geometric progression.

8. Find two numbers whose arithmetic mean is 10 and geometric mean is 8.

9. Find three numbers in geometric progression such that their sum is 21 and their product is 216.

10. If the first, fifth and tenth terms of an arithmetic progression are in geometric progression and the sum of the second and eighth terms is 20, find the first term and the (non-zero) common difference. (*C*)

11. The first three terms of a geometric series are 1, x, y, and the first three terms of an arithmetic series are 1, x, $-y$. Prove that $x^2 + 2x - 1 = 0$, and hence find y, given that x is positive. (*C*)

12. (a) The first term of a geometric progression is 7, its last term is 448 and its sum is 889. Find the common ratio.

(b) Give that $$\frac{1}{y - x}, \frac{1}{2y} \text{ and } \frac{1}{y - z}$$ are consecutive terms of an arithmetic progression, prove that x, y and z are consecutive terms of a geometric progression. (*C*)

13. Find the sum of n terms of the geometric series,

$$1 + \frac{5}{4} + \left(\frac{5}{4}\right)^2 + \ldots .$$

Find the least number of terms of this series which must be taken so that the sum of these terms exceeds 20.

14. The positive numbers a, b and c are consecutive terms in a geometric progression; express b in terms of a and c. Deduce that $\log a$, $\log b$, $\log c$ are consecutive terms in an arithmetic progression. (*C*)

15. The numbers 108, x, 48 are the first three terms of a geometric progression. Find the two possible values of x.
The numbers 108 and 48 are respectively the fourth and sixteenth terms of an arithmetic progression. Find the first term and the common difference of this A.P. Show that one of the values of x is a term of this A.P. while the other is not. (*C*)

16. In 1960 a man earned £2 000 and spent it all. During the next 10 years his salary increased by 5% per annum (compound interest), but inflation caused his expenditure to rise by 4% per annum (compound interest). Find how much he had saved by the end of 1970, giving your answer to two significant figures. (*C*)

17. The sum of the first n terms of a series is given by the expression

$$6 - \frac{2^{n+1}}{3^{n-1}}.$$

By finding an expression for the nth term of the series, or otherwise, show that this is a geometric series, and state the values of the first term and the common ratio. (*C*)

18. The seven numbers $a, x_1, x_2, x_3, x_4, x_5, b$ are in arithmetic progression. Express x_2 in terms of a and b, and show that $x_1 + x_3 + x_5 = \frac{3}{2}(a + b)$.
Given also that the numbers a, x_2, b are in geometric progression, and that $b \neq a$, express b in terms of a. (C)

19. A geometric series has first term 1 and the common ratio r is positive. The sum of the first 5 terms is twice the sum of the terms from the 6th to the 15th inclusive. Prove that $r^5 = \frac{1}{2}(\sqrt{3} - 1)$. (C)

20. The sum of the first 100 terms of an arithmetic progression is 10 000; the first, second and fifth terms of this progression are three consecutive terms of a geometric progression. Find the first term, a, and the non-zero common difference, d, of the arithmetic progression. (C)

Infinite Geometric Series

Examine the following three series:

Series 1	$1 + 3 + 5 + 7 + 9 + \ldots$
Series 2	$1 + 3 + 9 + 27 + 81 + \ldots$
Series 3	$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots$

Series 1 is an arithmetic series, and using the formula

$$S_n = \frac{n}{2}[2a + (n - 1)d],$$

we have

$$S_n = \frac{n}{2}[2 + (n - 1)2] = n^2.$$

For example, $S_1 = 1$, $S_2 = 4$, $S_3 = 9$, $S_4 = 16$, $S_5 = 25$, etc.

Notice that the magnitude of S_n increases rapidly without any limit.

Series 2 is a geometric series, and using

$$S_n = \frac{a(r^n - 1)}{r - 1},$$

we have

$$S_n = \frac{3^n - 1}{3 - 1} = \frac{1}{2}(3^n - 1).$$

Exercise 2.4

1. In each of the following, find the value of $\sum_{r=1}^{4} a_r$:

   (i) $a_r = 3r + 1$

   (ii) $a_r = r^2 + 3$

   (iii) $a_r = 2 + \sqrt{r}$

   (iv) $a_r = 2^{r+1}$

   (v) $a_r = \dfrac{1}{r}$

   (vi) $a_r = (-1)^r\, 2^{2r-1}$

2. Use the 'Σ' notation to express each of the following sums. (Such an expression is not unique in general).

   (i) $1 + 3 + 5 + \ldots + 99$

   (ii) $2 + 4 + 6 + \ldots + 1\,000$

   (iii) $5 + 9 + 13 + \ldots + 41$

   (iv) $\dfrac{1}{3} + \dfrac{2}{4} + \dfrac{3}{5} + \ldots + \dfrac{50}{52}$

   (v) $1.2 + 2.3 + 3.4 + \ldots + 99.100$

   (vi) $3.6.7 + 4.7.8 + 5.8.9 + \ldots + 20.23.24$

   (vii) $1 - 3 + 5 - 7 + \ldots + 57 - 59$

3. Find the value of each of the following sums:

   (i) $\sum_{r=1}^{100} (3r + 2)$

   (ii) $\sum_{i=1}^{20} (5^i - 2i)$

   (iii) $\sum_{k=1}^{10} (6k^2 + 1)$

   (iv) $\sum_{r=1}^{25} (r^2 + 3r + 4)$

   (v) $\sum_{r=5}^{100} r^2$

4. Find, in terms of n, the value of each of the following sums:

   (i) $\sum_{r=1}^{n} (2r - 1)^2$

   (ii) $\sum_{r=1}^{n} \left(2^r + r - \dfrac{1}{2}\right)$

   (iii) $\sum_{r=1}^{n} r(4r + 3)$

   (iv) $\sum_{r=1}^{n} (r + 1)(2r + 1)$

   (v) $\sum_{r=1}^{n} \dfrac{1}{r(r + 1)}$ $\left(\text{Hint: } \dfrac{1}{r(r + 1)} = \dfrac{1}{r} - \dfrac{1}{r + 1}\right)$

5. Find $\sum_{r=1}^{\infty} \dfrac{1}{10^{3r}}$, expressing your answer as a fraction in its lowest terms. Hence, or otherwise, express the infinite recurring decimal $0{\cdot}\dot{1}0\dot{8}$ ($= 0{\cdot}108\,108\,108\ldots$) as a fraction in its lowest terms. (*C*)

6. Show that $\sum_{r=1}^{\infty} \dfrac{1}{10^{2r+1}} = \dfrac{1}{990}$. Hence, or otherwise, express the infinite recurring decimal $2{\cdot}451\,515\,1\ldots$ as a fraction in its lowest terms.

Exercise 2.2

1. (a) $4, \frac{3}{4}(4)^n$ (d) $-1, 2(-1)^{n-1}$
   (b) $\frac{1}{2}, \left(\frac{1}{2}\right)^{n-1}$ (e) $-\frac{1}{a^2}, \frac{(-1)^{n-1}}{a^{2n-5}}$
   (c) $\frac{1}{2}, \frac{1}{3}\left(\frac{1}{2}\right)^{n-2}$ (f) $\frac{b}{a}, \frac{b^{n-1}}{a^{n-3}}$

2. (a) 19·98 (to 4 significant places) (d) 4·049 (to 4 significant places)
   (b) $4^m - 1$ (e) $\frac{16}{3}\left[1 - \left(-\frac{1}{2}\right)^n\right]$
   (c) 74·05 (to 4 significant places) (f) 5 315 (to 4 significant places)

3. (i) ± 12 (iii) ± 15
   (ii) ± 36 (iv) $\pm\frac{\sqrt{7}}{2}$

4. (i) $3, \frac{1}{4}$ (iii) $a - b, \frac{a}{b}$
   (ii) $2, \frac{1}{3}$ (iv) $\frac{5}{512}, 2$

5. $\frac{1}{3}\left(\frac{3}{4}\right)^n$, first term: $\frac{1}{4}$, common ratio: $\frac{3}{4}$

6. 15

7. $\frac{4}{3}$

8. 4, 16

9. 3, 6, 12

10. first term: 8, common difference: $\frac{1}{2}$

11. $3 - 2\sqrt{2}$

12. (a) 2

13. $4\left[\left(\frac{5}{4}\right)^n - 1\right]$, 9

14. $b = \sqrt{ac}$

15. $x = \pm 72$, first term: 123, common difference: -5,
    $x = -72$ is a term of the A.P.

16. £1 400

17. $G = 2, r = \frac{2}{3}$

18. $x_2 = \frac{2}{3}a + \frac{1}{3}b, b = 4a$

20. $a = 1, d = 2$

Exercise 2.4

1. (i) 34
   (ii) 42
   (iii) $11 + \sqrt{2} + \sqrt{3}$
   (iv) 60
   (v) $2\frac{1}{12}$
   (vi) 102

2. (i) $\sum_{r=0}^{49} (2r + 1)$
   (ii) $\sum_{r=1}^{500} 2r$
   (iii) $\sum_{r=1}^{10} (4r + 1)$
   (iv) $\sum_{r=1}^{50} \frac{r}{r + 2}$
   (v) $\sum_{r=1}^{99} r(r + 1)$
   (vi) $\sum_{r=3}^{20} r(r + 3)(r + 4)$
   (vii) $\sum_{r=1}^{30} (-1)^{r+1} (2r - 1)$

3. (i) 15 350
   (ii) $\frac{1}{4}(5^{21} - 5) - 420$
   (iii) 2 320
   (iv) 6 600
   (v) 338 320

4. (i) $\frac{1}{3}n(2n + 1)(2n - 1)$
   (ii) $2^{n+1} - 2 + \frac{n^2}{2}$
   (iii) $\frac{1}{6}n(n + 1)(8n + 13)$
   (iv) $\frac{1}{6}n(4n^2 + 15n + 17)$
   (v) $\frac{n}{n + 1}$

5. $\frac{1}{999}, \frac{4}{37}$

6. $\frac{809}{330}$