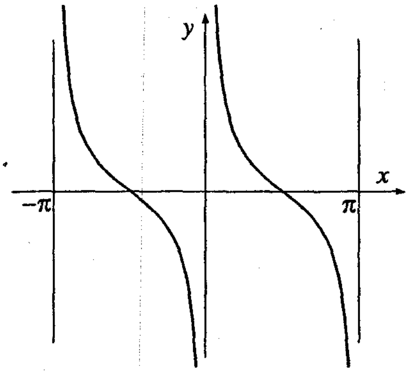
Therefore, the graph of $y = \cot x$ also takes on all real numbers as values. It is not defined for $x = n\pi$, where n is any integer, and has vertical asymptotes at $y = n\pi$:



Exercises

1. Draw the graphs of

   a) $y = \tan(x - \pi/6)$ b) $y = 3\tan x$ c) $y = \cot(x + \pi/4)$.

2. Suppose we graphed the equation $y = \tan x$. Is it possible to describe this graph with an equation of the form $y = \cot(x + \varphi)$, for some number φ? Why or why not?

9 An important question about sums of sinusoidal functions

We hope that from this material you have seen the importance, and the beauty, of the family of sinusoidal curves that we have been studying. Physicists call this family the curves of *harmonic oscillation.*

Let us now consider the following question. Suppose we have two sinusoidal curves (harmonic oscillations):

$$\begin{aligned} y_1 &= a_1 \sin k_1(x - \beta_1) \\ y_2 &= a_2 \sin k_2(x - \beta_2)\,. \end{aligned}$$

Will the sum of these two also be a sinusoidal curve (harmonic oscillation)? That is, will

$$y = a_1 \sin k_1(x - \beta_1) + a_2 \sin k_2(x - \beta_2)$$

be a sinusoidal curve? The answer is somewhat surprising. If $k_1 = k_2$, the answer is yes, but if $k_1 \neq k_2$, the answer is no.

That is, the sum of two harmonic oscillations is again a harmonic oscillation if and only if the original frequencies are the same. The results of the next few sections will allow us to explore this situation.

Exercises

Each of these exercises concerns the following three functions:

$$\begin{aligned} y_1 &= 2\sin x \\ y_2 &= \sin(x - \pi/4) \\ y_3 &= 3\sin 2x \end{aligned}$$

1. Use your calculator to draw the graph of (a) $y_1 + y_2$; (b) $y_1 + y_3$; (c) $y_2 + y_3$.
2. Which of the graphs in Exercise 1 appear to be sinusoidal functions?

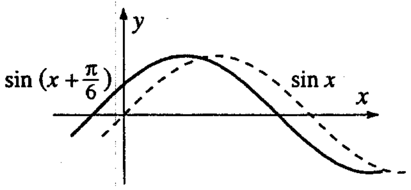
10 Linear combinations of sines and cosines

Definition: If we have two functions $f(x)$ and $g(x)$, and two constants a and b, then the expression $af(x) + bg(x)$ is called a *linear combination* of the functions $f(x)$ and $g(x)$.

Let us look at the graph of a linear combination of sinusoidal curves.

Example 65 Graph the function $y = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$.

Solution. Since $\frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$ and $\frac{1}{2} = \sin\frac{\pi}{6}$, we use the formula $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. Letting $\alpha = x$ and $\beta = \frac{\pi}{6}$, this formula tells us that the given function can be written as $y = \sin(x + \frac{\pi}{6})$. Now we can graph it as we did in Section 5:



This solution may seem artificial, but is in fact a general method. It works because there is an angle φ such that $\cos\varphi = 1/2$ and $\sin\varphi = \sqrt{3}/2$, and this happened because the values $A = 1/2$ and $B = \sqrt{3}/2$ satisfy the equation $A^2 + B^2 = 1$. (The reader is invited to do this computation.)

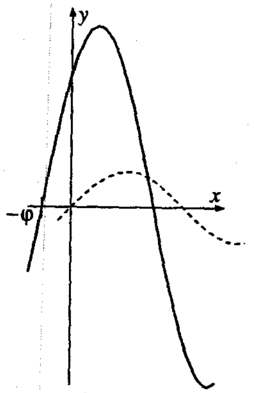
But what if $A^2 + B^2$ is not equal to 1?

Example 66 Draw the graph of the function $f(x) = 3\sin x + 4\cos x$.

Solution. Our "best friends" (of Chapter 1) are hiding in this expression: where we have 3 and 4, we try to look for the number 5. Indeed, $f(x)/5 = \frac{3}{5}\sin x + \frac{4}{5}\cos x$, and $\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1$, so we can use the method of Example 3. We know that there is an angle φ such that $\cos\varphi = 3/5$ and $\sin\varphi = 4/5$, and so

$$\frac{f(x)}{5} = \cos\varphi \sin x + \sin\varphi \cos x = \sin(x + \varphi)$$

or $f(x) = 5\sin(x + \varphi)$, for a certain angle φ. The graph is a sine curve, shifted to the left φ units, and with amplitude 5:



The same technique will work for linear combinations of $y = \sin kx$ and $y = \cos kx$, as long as the frequency of the two functions is the same. This is important enough to state as a theorem:

Theorem A linear combination of $y = \sin kx$ and $y = \cos kx$ can be expressed as $y = a\sin k(x + \varphi)$, for suitable constants a and φ.

Proof. A linear combination of $y = \sin kx$ and $y = \cos kx$ has the form $y = A\sin kx + B\cos kx$. We can rewrite it as

$$y = \sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2 + B^2}}\sin kx + \frac{B}{\sqrt{A^2 + B^2}}\cos kx\right).$$

Then

$$\left(\frac{A}{\sqrt{A^2 + B^2}}\right)^2 + \left(\frac{B}{\sqrt{A^2 + B^2}}\right)^2 = 1$$

so there exists an angle α such that

$$\cos\alpha = \frac{A}{\sqrt{A^2+B^2}} \text{ and } \sin\alpha = \frac{B}{\sqrt{A^2+B^2}}\,. \tag{1}$$

Now we can write

$$\begin{aligned} A\sin kx + B\cos kx &= \sqrt{A^2+B^2}(\cos\alpha\sin kx + \sin\alpha\cos kx) \\ &= \sqrt{A^2+B^2}\sin(kx+\alpha) \\ &= \sqrt{A^2+B^2}\sin k(x+\alpha/k)\,. \end{aligned}$$

Taking $a = \sqrt{A^2+B^2}$ and $\varphi = \alpha/k$, we have the required form. □

We have proved that $A\sin kx + B\cos kx$ can be written in the form $a\sin k(x+\gamma)$, where $a = \sqrt{A^2+B^2}$ and $\varphi = \alpha/k$ (for α defined by equations (1) above).

The converse statement is also correct:

Theorem The function $a\sin k(x+\varphi)$ can be written as a linear combination of the functions $\sin kx$ and $\cos kx$.

Proof. We have $a\sin k(x+\varphi) = a(\sin kx\cos k\varphi + \cos kx\sin k\varphi)$. Taking $A = a\cos k\varphi$ and $B = a\sin k\varphi$, we see that $a\sin k(x+\varphi) = A\sin kx + B\cos kx$. □

We can now write a sinusoidal curve in either of two standard forms: $y = a\sin k(x-\beta)$ or $y = A\sin kx + B\cos kx$.

Example 67 Write the function $y = 2\sin(x+\pi/3)$ as a linear combination of the function $y = \sin x$ and $y = \cos x$.

Solution. We have $2\sin(x+\pi/3) = 2(\sin x\cos\pi/3 + \cos x\sin\pi/3) = 2(1/2)\sin x + 2\sqrt{3}/2\cos x = \sin x + \sqrt{3}\cos x$.

Exercises

1. Write the function $y = 2\sin x + 3\cos x$ in the form $y = a\sin k(x-\beta)$. What is its amplitude?
2. What is the maximum value achieved by the function $y = 2\sin x + 3\cos x$?

3–6: Write each function in the form $y = a \sin k(x - \beta)$. What is the maximum value of each function?

3. $y = \sin x + \cos x$
4. $y = \sin x - \cos x$
5. $y = 4 \sin x + 3 \cos x$
6. $y = \sin 2x + 3 \cos 2x$

7, 8: Write each function in the form $A \sin x + B \cos x$:

7. $y = \sin(x - \pi/4)$
8. $y = 4 \sin(x + \pi/6)$

11 Linear combinations of sinusoidal curves with the same frequency

Now we are ready to address the important question of Section 9.

Theorem The sum of two sinusoidal curves with the same frequency is again a sinusoidal curve with this same frequency.

Proof. Let us take the two sinusoidal curves

$$a_1 \sin k(x - \beta_1) \text{ and}$$
$$a_2 \sin k(x - \beta_2) \,.$$

Using the addition formula, we can write:

$$\begin{aligned} a_1 \sin k(x - \beta_1) &= A_1 \sin kx + B_1 \cos kx \\ a_2 \sin k(x - \beta_2) &= A_2 \sin kx + B_2 \cos kx \end{aligned}$$

for suitable values of A_1, A_2, B_1, and B_2. Then our sum is equal to

$$(A_1 + A_2) \sin kx + (B_1 + B_2) \cos kx \,.$$

But we know, from the theorem of Section 9, that this sum is also a sinusoidal curve. Our theorem is proved. □

We invite the reader to fill in the details, by giving the expressions for A_1, A_2, B_1, and B_2.

Note that the two functions we are adding may have different amplitudes. The result depends only on their having the same period. This result is very important in working with electricity. Alternating electric current is described by a sinusoidal curve, and this theorem says that if we add two currents with the same periods, the resulting current will have this period as well. So if we are drawing electric power from different sources, we need not worry how to mix them (whether their phase shifts are aligned), as long as their periods are the same.

The next result is important in more advanced work:

Theorem If a linear combination of the functions $y = \sin kx$ and $y = \cos kx$ is shifted by an angle β, then the result can be expressed as a linear combination of the same two functions.

Proof. Let us take the linear combination

$$a \sin kx + b \cos kx$$

and shift it by an angle β. The result is

$$a \sin k(x - \beta) + b \cos k(x - \beta)\,.$$

We know that $\cos k(x-\beta)$ can be written as $\sin k(x-\gamma)$, for some angle γ. Thus we can write our shifted linear combination as

$$a \sin k(x - \beta) + b \sin k(x - \gamma)\,.$$

But this is a sum of sinusoidal curves with the same frequency k, so the previous theorem tells us that it can be written as a single sinusoidal curve with frequency k (even though the shifts are different!). And we know, from Section 9, that such a sum can be written as a linear combination of $\sin kx$ and $\cos kx$.

Example 68 Suppose we take the graph of a linear combination of $y = \sin x$ and $y = \cos x$:

$$y = 2 \sin x + 4 \cos x$$

and shift it $\pi/6$ units to the left. We get:

$$\begin{aligned}
y &= 2\sin(x + \pi/6) + 4\cos(x + \pi/6)\\
&= 2(\sin x \cos \pi/6 + \cos x \sin \pi/6) + 4(\cos x \cos \pi/6 - \sin x \sin \pi/6)\\
&= 2(\sqrt{3}/2 \sin x + 2(1/2)\cos x + 4(\sqrt{3}/2)\cos x - 4(1/2)\sin x\\
&= (\sqrt{3} - 2)\sin x + (2\sqrt{3} + 1)\cos x
\end{aligned}$$

which is again a linear combination of $y = \sin x$ and $y = \cos x$.

This technique works whenever we apply a shift to a linear combination of $y = \sin kx$ and $y = \cos kx$. The proof follows the reasoning of the above example.

A final comment: We have not considered linear combinations of sines and cosines with *different* frequencies. This is a more difficult situation, and leads to some very advanced mathematical topics, such as Fourier Series and almost periodic functions. We will return to this question a bit later.

Exercises

1. Express each function in the form $y = A \sin kx + B \cos kx$
   (a) $y = 2\sin(x + \pi/6) + \cos(x + \pi/6)$
   (b) $y = 2\sin 2(x + \pi/4) - \cos 2(x + \pi/4)$
2. Look at the exercises for Section 9 on page 189.
   (a) Write $y_1 + y_2$ as a linear combination of $\sin x$ and $\cos x$.
   (b) What goes wrong when you try to write $y_1 + y_3$ as a linear combination of $\sin x$ and $\cos x$?

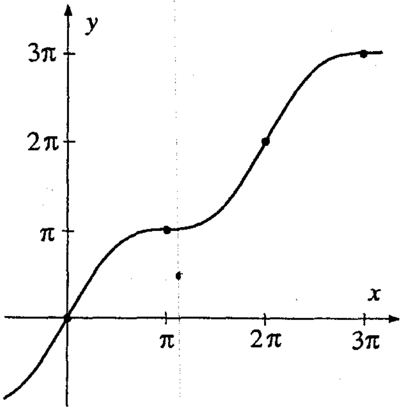
12 Linear combinations of functions with different frequencies

So far, we have some important results about linear combinations of sines and cosines with the same frequency. We would like to investigate the sum of two functions like $y = \sin k_1 x$ and $y = \sin k_2$, where $k_1 \neq k_2$. We start the discussion with some examples which may not at first appear related.

Example 69 Graph the function $y = x + \sin x$.

Solution. Each y-value on this graph is the sum of two other y-values: the value $y = \sin x$ and the value $y = x$. So we can take each point on the curve $y = \sin x$ and "lift it up" by adding the value $y = x$ to the value $y = \sin x$.

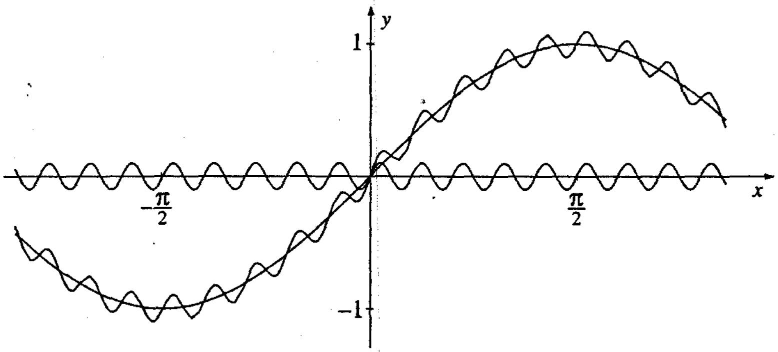
This is particularly easy to see for those points where $\sin x = 0$. For these points, the value of $x + \sin x$ is just x:



In between these points, the line $y = x$ is lifted up slightly, or brought down slightly, by positive or negative values of $\sin x$. We can think of the sine curve as "riding" on the line $y = x$.

Example 70 Graph the function $y = \sin x + 1/10 \sin 20x$.

Solution. This seems much more complicated, but in fact can be solved using the same method as the previous examples. We graph the two curves $y = \sin x$ and $y = 1/10 \sin 20x$ independently, then add their y-values at each point:



Again, we can think of one curve "riding" on the other. This time the curve $y = 1/10 \sin 20x$ "rides" on the curve $y = \sin x$, or *perturbs* it a bit at each point.

Note that our new curve is *not* a sinusoidal curve. We cannot express it either in the form $y = a \sin k(x - \beta)$ or in the form $y = A \sin kx + b \cos kx$.

Exercises

Construct graphs of the following functions.

1. $y = -x + \sin x$
2. $y = x^2 + \sin x$
3. $y = x^2 + \cos x$. Hint: Is the function odd? Is it even?
4. $y = x^3 + \sin x$
5. $y = x^2 + (1/10)\sin x$
6. $y = \cos x + (1/10)\sin 20x$
7. $y = 2\sin x + (1/10)\sin 20x$

13 Finding the period of a sum of sinusoidal curves with different periods

We know that the function $y = \sin 10x + \sin 15x$ is not a sinusoidal curve. Let us show that it is still periodic. Indeed, if we shift the curve by 2π, we have $y = \sin 10(x + 2\pi) + \sin 15(x + 2\pi) = \sin(10x + 20\pi) + \sin(15x + 30\pi) = \sin 10x + \sin 15x$.

But what is its smallest positive period? We can answer this by looking separately at all the periods of the two functions we are adding. Any period of $y = \sin 10x$ must have the form $m(2\pi/10)$, for some integer m. Any period of $y = \sin 15x$ must have the form $n(2\pi/15)$, for some integer n. To be a period of both functions, a number must be of both these forms. That is, we must have integers m and n such that $2m\pi/10 = 2n\pi/15$, or $3m = 2n$. If we take $m = 2$, $n = 3$, our problem is solved. The number $2\pi/5 = 2(2\pi/10) = 3(2\pi/15)$ is a period for both functions. And since we took the smallest positive values of m and n, this is the smallest positive period for the function $y = \sin 10x + \sin 15x$.

The argument above is drawn from number theory, where it is connected with the least common multiple of two numbers. This concept is used in elementary arithmetic, in finding the least common denominator for two fractions. The general statement, proved in number theory, is this:

> *The function* $y = \sin k_1 x + \sin k_2 x$ *is periodic if and only if the quotient* k_1/k_2 *is rational.*

But a function like $y = \sin x + \sin\sqrt{2}x$ has no period at all.

Exercises

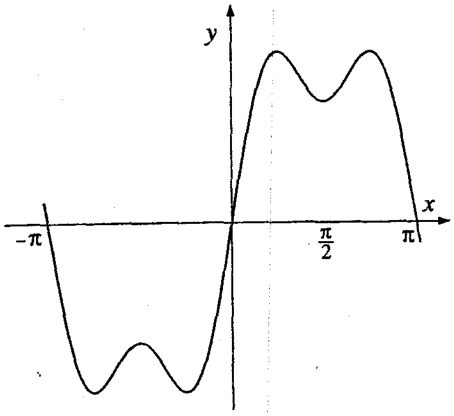
Find the (smallest positive) period for each of the following functions.

1. $y = \sin 2x + \sin 3x$
2. $y = \sin 3x + \sin 6x$
3. $y = \sin 4x + \sin 6x$
4. $y = \sin \sqrt{2}x + \sin 3\sqrt{2}x$

14 A discovery of Monsieur Fourier

Example 71 Graph the function $y = \sin x + (1/3)\sin 3x$.

Solution. This example is similar to Example 70. The values of $\sin x$ are "perturbed" by those of $\frac{1}{3}\sin 3x$:



Example 72 Graph the function $y = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x$. (Use a graphing calculator or software utility for this complicated function.)

Solution.