Modern Advanced Mathematics

By

ARTHUR W. WEEKS

JACKSON B. ADKINS

and

RANSOM V. LYNCH

The Phillips Exeter Academy

EXETER, NEW HAMPSHIRE

GINN AND COMPANY

A XEROX COMPANY

3

Functions

3.1 Definition of a Function

In Chapter 2 we used the name relation for any set of ordered pairs, without restrictions. In this chapter we shall focus our attention on relations which are subject to the restriction that each first element in the set of ordered pairs shall be paired with one and only one second element. This single restriction characterizes an important type of relation known as a *function*.

Suppose we have a set $A = \{J, L, V, S\}$, a set $B = \{8, 9, 10\}$, and a rule of correspondence which pairs J with 8, L with 8, V with 9, and S with 10. This rule of correspondence provides us with the set of ordered pairs $F = \{(J, 8), (L, 8), (V, 9), (S, 10)\}$ in which the first members are taken from set A and the second members from set B. Since each element of A is associated with one and only one element of B, the set F is a function. This example illustrates the most important property of any function, which may be described as follows: *Once an element of the first set (the set from which all first members are taken) is given, the corresponding element of the second set (the set from which all second members are taken) is then uniquely determined.* We formalize these introductory remarks with the following definition:

▶ **Definition.** A function is a set of ordered pairs such that each first element is paired with one and only one second element. That is, a relation F is a function if $(x_1, y_1) \in F$ and $(x_1, y_2) \in F$ implies $y_1 = y_2$.

Example. Is the relation $\{(x, y): y^2 = 4x\}$ a function?

Solution: When $x = 1$, we have $y^2 = 4$, which means that $y = 2$ or $y = -2$. Thus the distinct pairs $(1, 2)$, $(1, -2)$ both belong to the given relation, and the relation is therefore not a function.

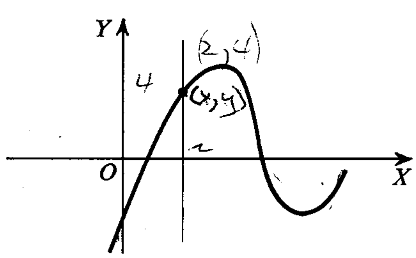
Terminology. The framework of any function consists of two sets and a rule by means of which each element of one set X is paired with just one element of the second set Y. This rule may be indicated by a verbal statement, by an algebraic formula, or by a table in which the pairing of the elements is exhibited.

A function is commonly represented by a single letter, in most cases f, F, g, or h. When a function is designated by the symbol f, then the set X of the first elements of the ordered pairs is called the *domain of f* and the set Y of the second elements is called the *range of f* (review definitions on page 69 in Chapter 2). When x is used to represent an arbitrary member of the domain X, then $f(x)$ is used to represent that element of the range Y which is paired with x by the rule of f. Since y is also frequently used to represent the second element of an ordered pair, either $f(x)$ or y may symbolize the *value of the function f for a particular x.* The symbol $f(x)$ is read "f of x" or "f at x" according to taste. The important point is that f is a name for a set of ordered pairs, while $f(x)$ is the second element of the pair belonging to the set f whose first element is x. In other words, a function f is a set of ordered pairs $(x, f(x))$ when x is an element of the domain of f.

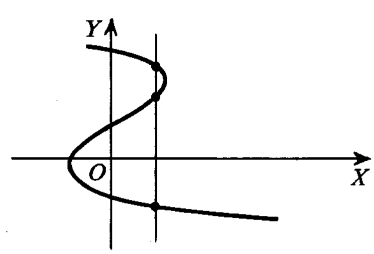
Functions f and g are equal if each ordered pair of f is in g and each ordered pair of g is in f. Thus f and g must have the same domain and for every x in the domain $f(x)$ must equal $g(x)$.

The general concept of a function merely requires a correspondence of the prescribed nature between the elements of a set X and the elements of a set Y, without specifying that these elements be numbers. Our concern, however, will be almost exclusively with cases in which X and Y are subsets of R. It is in connection with such cases that we speak of the *graph of a function.*

The graph of a function f is the set of all points with coordinates $(x, f(x))$ when x is an element of the domain of f. Since no two ordered pairs of a function can have the same first member but different second members, we know that *the graph of a function does not have more than one point on any vertical line in the xy-plane.*



Graph of a function



Graph of a relation which is not a function

In practice we shall be concerned with functions defined by algebraic rules such as $y = 3x + 2$, $f(x) = \sqrt{4 - x^2}$. Although the domain of a function may be arbitrarily limited, our main interest will be in functions defined over the largest possible subset of the real numbers. In fact, when the domain is not specified, we understand that it is the set of all real x for which $f(x)$ is a real number.

Example 1. (a) If a function is defined by $y = 3x + 2$, the domain is R, the set of real numbers, and the range is R.

(b) If a function is defined by $f(x) = \sqrt{4 - x^2}$, then the domain is $\{x: -2 \le x \le 2\}$ and the range is $\{y: 0 \le y \le 2\}$. (Note that $\sqrt{4 - x^2}$ is a non-negative real number if and only if $4 - x^2 \ge 0$. Review properties (1)–(3) in Chapter 2, pages 70 and 71).

Example 2. $f(x) = x^2 - 9$. (a) State the domain and range of f. (b) Evaluate $f(-4)$, $f(3 + h)$.

Solution:

(a) The domain of f is R. The range of f is $\{y: y \ge -9\}$.

(b) $f(-4) = (-4)^2 - 9 = 16 - 9 = 7$
$f(3 + h) = (3 + h)^2 - 9 = 9 + 6h + h^2 - 9 = 6h + h^2$

Exercises [A]

1. Write the domain and range of the functions defined below.
   (a) $f = \{(1, 4), (2, 9), (3, 19), (4, 51)\}$ (b) $f(x) = x^3$ (c) $f(x) = x^4 + 2$
   (d) $f(x) = \dfrac{6}{x - 2}$ (e) $f(x) = \sqrt{x^2 - 4}$

2. (a) Explain why $y^2 = 9 - x^2$ does not define a function.
   (b) Does $y = \sqrt{9 - x^2}$ define a function?

3. State the domain of the function defined by each equation.
   (a) $y = \dfrac{4}{x(x - 2)}$ (b) $f(x) = \dfrac{x}{x^2 - 1}$ (c) $y = \dfrac{2}{\sqrt{x - 1}}$

4. If $f(x) = 4\sqrt{x - 1} + x + 5$, find the least value of f.
   Hint: First find the domain of f.

5. $f(x) = 4x - 3$. Write the simplest form of the following expressions.
   (a) $f(3)$ (b) $f(0)$ (c) $f(-2)$ (d) $f(n)$ (e) $f(x^2)$
   (f) $f(n + 1) - f(n)$ (g) $\dfrac{f(a) - f(b)}{a - b}$ (h) $\dfrac{f(x_1 + h) - f(x_1)}{h}$

6. $f(x) = 9 - 3x$. Write the simplest form of the following expressions.
   (a) $f(0)$ (b) $f(4)$ (c) $f(-3)$ (d) $f(2k)$ (e) $f(x + 3)$
   (f) $f(n + 1) - f(n)$ (g) $\dfrac{f(a) - f(b)}{a - b}$ (h) $\dfrac{f(x_1 + h) - f(x_1)}{h}$

7. $f(x) = x^4 - 4x^2$. Show that $f(-x_1) = f(x_1)$.
   (b) What can be deduced about the symmetry of the graph of f?

8. (a) $f(x) = x^3 - 4x$. Show that $f(-x_1) = -f(x_1)$.
   (b) What can be deduced about the symmetry of the graph of f?

9. Which of the following statements are true for every function f?
   (a) The graph of f is symmetric about the y-axis.
   (b) The graph of f is symmetric about the x-axis.
   (c) If $x_1 = x_2$, then $f(x_1) = f(x_2)$.
   (d) If $f(x_1) = f(x_2)$, then $x_1 = x_2$.
   (e) A vertical line meets the graph of f in no more than one point.
   (f) A horizontal line meets the graph of f in no more than one point.

10. Show that if a linear function is defined by $f(x) = mx + k$, the expression $\dfrac{f(a) - f(b)}{a - b}$ is the slope of the graph of f when a, b are elements in the domain of f and $a \neq b$.

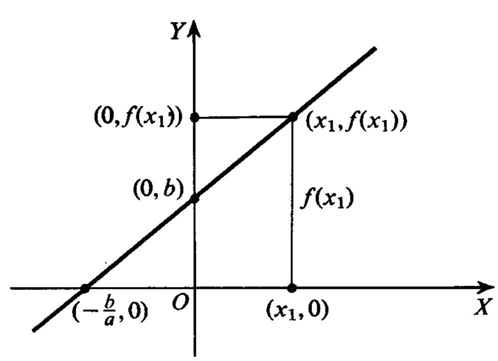
11. $f(x) = x^2$. Write the simplest form of the following expressions.
   (a) $f(3) - f(-3)$ (b) $f(x_1) - f(-x_1)$
   (c) $f(a + b) - f(a - b)$ (d) $\dfrac{f(x_1 + h) - f(x_1)}{h}$

12. $f(x) = x^2$. Evaluate $\dfrac{f(a) - f(b)}{a - b}$, where $a \neq b$, and give a geometric interpretation of the result.

3.2 Linear Functions

A function defined by $y = ax + b$ (or $f(x) = ax + b$), when a, b are real numbers and $a \neq 0$ is called a linear function. It is apparent from our work in Chapter 2 that the graph of the function defined by $y = ax + b$ is the straight line with slope a and y-intercept b.

For a linear function f, not only is there just one value of f corresponding to each real number x, but there is also just one real x corresponding to each real



number $f(x)$. Thus a linear function conforms to our intuitive idea of a one-to-one correspondence between the set of real numbers and the set of real numbers (see Chapter 1, pages 5 and 6), and it also satisfies the following formal definition:

▶ **Definition.** A function f is a one-to-one correspondence if, for all x_1, x_2 in the domain of f, $f(x_1) = f(x_2) \longrightarrow x_1 = x_2$.

NOTE: Since f is a function, $x_1 = x_2 \longrightarrow f(x_1) = f(x_2)$.

The particular x for which $f(x) = 0$ is called the zero of the function f. If $f(x) = ax + b$, the zero of f is $-\frac{b}{a}$.

Some typical questions concerning linear functions are dealt with in the following examples.

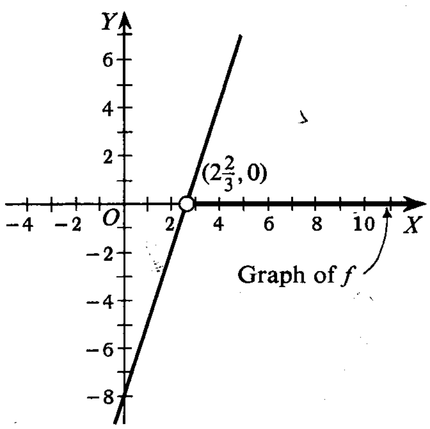
Example 1. If $f(x) = 3x - 8$, what is the solution set of $f(x) > 0$?

Solution:

$$f(x) > 0$$
$$\longleftrightarrow 3x - 8 > 0$$
$$\longleftrightarrow 3x > 8$$
$$\longleftrightarrow x > \tfrac{8}{3}$$

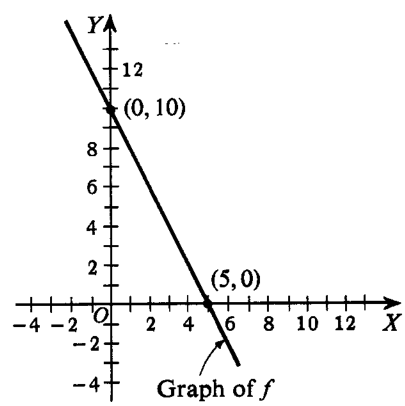
$\therefore$ the solution set is $\{x: x > 2\frac{2}{3}\}$.

We could also attack the problem as follows: The zero of f is $2\frac{2}{3}$ and the slope of the graph of f is a positive number. Hence the solution set of $f(x) > 0$ is $\{x: x > 2\frac{2}{3}\}$.



Example 2. If $f(x) = 10 - 2x$, what is the change in $f(x)$ when the value of x is increased by 4?

Solution: The slope of the graph of f is -2. This means that $\Delta y = -2 \cdot \Delta x$, or, in the function notation, $\Delta f(x) = -2 \cdot \Delta x$. Hence, when $\Delta x = 4$, $\Delta f(x) = -8$, or $f(x)$ is *decreased* by 8 when the value of x is *increased* by 4.



Example 3. If a linear function f contains the ordered pairs $(-1, 3)$, $(2, 9)$, find $f(t)$.

Solution: Since the graph of f contains the points with coordinates $(-1, 3)$, $(2, 9)$, the slope of the graph is $\frac{9-3}{2-(-1)} = \frac{6}{3} = 2$. Thus, $f(t) = 2t + b$, for which b is the y-intercept of the graph of f. Since $f(2) = 9$, we have $9 = 4 + b$, or $b = 5$. Hence, $f(t) = 2t + 5$.

Exercises [A]

1. $f(x) = 6 - 3x$. (a) State the zero of f. (b) Find the solution set of $f(x) > 0$. (c) Find the change in $f(x)$ when x is increased by 4.
2. $f(x) = 2x + 5$. (a) State the zero of f. (b) Find the solution set of $f(x) > 0$. (c) Find the change in $f(x)$ when x is increased by 7.
3. A function F is defined by the rule $y = 2x$ with the domain the set of positive integers. What is the range of F?
4. A function g is defined by the rule $y = \frac{1}{2}x$ with the range the set of positive integers. What is the domain of g?
5. $f(x) = 3x + 4$ and $g(x) = 10 - x$. (a) Find $f \cap g$. (b) Find the solution set of $f(x) \geq g(x)$.
6. $f(x) = 3x + 4$ and $g(x) = 3x + 8$. (a) Find $f \cap g$. (b) Find the solution set of $g(x) > f(x)$.
7. The number of units of length of a metal bar is given by the formula $l = 100.000 + 0.003(t - 55)$, when t is the temperature on the Fahrenheit scale. (a) Find l when $t = 55$. (b) Find l when $t = 0$. (c) Find the least integer t for which $l > 100.140$. (d) Find the increase in l corresponding to an increase in temperature of 20°.
8. (a) $f(x) = 12 - 3x$. Evaluate $f(4) - f(1)$.
   (b) $f(x) = 2.38x + 15.72$. Evaluate $f(17.5) - f(7.5)$.
   (c) $f(x) = mx + k$. Write the simplest expression for $f(n + 3) - f(n)$.
9. (a) If $f(x) = 2x - 5$, evaluate $f(a)$, $f(b)$, $f(a + b)$. (b) Is it true that $f(a + b) = f(a) + f(b)$?
10. (a) If $f(x) = 4.2x$, evaluate $f(a)$, $f(b)$, $f(a + b)$. (b) Is it true that $f(a + b) = f(a) + f(b)$?
11. If $f(x) = mx + k$, under what condition does $f(a + b) = f(a) + f(b)$ for all a and b?
12. If F is a linear function which contains $(2, -3)$ and $(0, 1)$, find $F(x)$.
13. If g is a linear function such that $g(0) = 4$ and $g(4) = 0$, find $g(t)$.
14. If f is a linear function such that $f(-2) = 4$ and $\Delta f(x) = 3 \cdot \Delta x$, find $f(x)$.
15. If $f(t) = 3t - 2$, find (a) $f(2x)$, (b) $f(x^2)$, (c) $f(2t + 1)$.
16. If $f(t) = 3t - 2$, find x in terms of t so that $f(x) = t$.

3.3 Composition of Functions

If a function F is defined by a rather intricate formula, the evaluation of $F(x)$ for a particular x may entail two or more separate processes. Consider, for example, F defined by $y = (3x - 1)^4$ with domain $\{0, 1, 2, 3, 4\}$. Values of $F(x)$ are found as follows:

$$\begin{array}{lll}
\text{When } x = 0,\ 3x - 1 = -1, \text{ and } (-1)^4 = & 1 = F(0). \\
\text{When } x = 1,\ 3x - 1 = 2, \text{ and } 2^4 = & 16 = F(1). \\
\text{When } x = 2,\ 3x - 1 = 5, \text{ and } 5^4 = & 625 = F(2). \\
\text{When } x = 3,\ 3x - 1 = 8, \text{ and } 8^4 = & 4096 = F(3). \\
\text{When } x = 4,\ 3x - 1 = 11, \text{ and } 11^4 = & 14641 = F(4).
\end{array}$$

In many situations it is helpful to think of F as being composed of the two functions which correspond to the two steps of computation. Since the second function must take up where the first function leaves off, so to speak, the domain of the second function must be the range of the first. If in this example we call the first function f and the second function g, then the rule for f is $y = 3x - 1$ with domain $\{0, 1, 2, 3, 4\}$ and the rule for g is $y = x^4$ with domain $\{-1, 2, 5, 8, 11\}$, which is the range of f. The value of $F(x)$ is obtained by first finding $f(x)$ and then $g(f(x))$. For instance, to find $F(2)$, we first find $f(2)$ and then find $g(f(2))$. Since $f(2) = 5$, then $F(2) = g(5) = 625$.

▶ **Definition.** **If for each x in the domain of F we have $F(x) = g(f(x))$ for two functions f and g, then F is called the composition of g with f and we write $F = g(f)$.**

For the time being we shall restrict our investigation of the operation of composition to the mathematical system consisting of the set of all *linear* functions under composition. Our first consideration will be closure.

Let f and g be any two linear functions, so that $f(x) = a_1x + b_1$ and $g(x) = a_2x + b_2$ for some a_1, a_2, b_1, b_2 in R, when a_1, $a_2 \neq 0$. Then, if $F = g(f)$, the following is true.

$$\begin{aligned}
F(x) &= g(f(x)) = g(a_1x + b_1) \\
&= a_2(a_1x + b_1) + b_2 \\
&= a_2a_1x + a_2b_1 + b_2 \qquad (a_2a_1 \neq 0)
\end{aligned}$$

Hence, F is a linear function defined by $y = a_1a_2x + (a_2b_1 + b_2)$.

Since f and g are arbitrarily chosen linear functions, we see that the composition of two linear functions is always a linear function. Thus the set of all linear functions, i.e., functions defined by $y = ax + b$, $a \neq 0$, is closed under the operation of composition.

Example. $f(x) = 3x - 2$, $g(x) = 2x + 5$, $F = g(f)$, and $G = f(g)$. Find (**a**) $F(3)$, (**b**) $G(3)$, (**c**) $F(x)$, (**d**) $G(x)$.

Solution:

(a) $f(3) = 3 \cdot 3 - 2 = 7$
$g(7) = 2 \cdot 7 + 5 = 19$
Hence, $F(3) = g(f(3)) = g(7) = 19$.

(b) $g(3) = 2 \cdot 3 + 5 = 11$
$f(11) = 3 \cdot 11 - 2 = 31$
Hence, $G(3) = f(g(3)) = f(11) = 31$.

(c) $F(x) = g(f(x))$
$= g(3x - 2)$
$= 2(3x - 2) + 5$
$= 6x + 1$

(d) $G(x) = f(g(x))$
$= f(2x + 5)$
$= 3(2x + 5) - 2$
$= 6x + 13$

Exercises [A]

1. $f = \{(1, 1), (2, 4), (3, 9)\}$, $g = \{(1, 3), (4, 15), (9, 35)\}$, and $F = g(f)$. List the ordered pairs of F. Also list the ordered pairs of $f(g)$.

2. f is defined by $y = 2x$, g is defined by $y = 4x - 3$, $F = g(f)$, and $G = f(g)$. (**a**) Write the rules which define F and G. (**b**) Evaluate $F(2)$ and $G(2)$ from the rules defining F and G and also from $g(f(2))$ and $f(g(2))$.

3. f is defined by $y = 3x - 4$, g is defined by $y = 7x + 1$, $F = g(f)$, and $G = f(g)$. (**a**) Write the rules which define F and G. (**b**) Evaluate $F(3)$ and $G(3)$.

For each of Exercises 4–8, write the rules which define $g(f)$ and $f(g)$.

4. f is defined by $y = x^2 + 1$ and g is defined by $y = 3x - 7$.

5. f is defined by $y = 4x$ and g is defined by $y = -2x$.

6. f is defined by $y = 4x$ and g is defined by $y = -2x + 5$.

7. f is defined by $y = x$ and g is defined by $y = 8x - 5$.

8. f is defined by $y = 15 - 2.4x$ and g is defined by $y = x$.

9. f is defined by $y = 3x + 1$ and g is defined by $y = \frac{1}{3}(x - 1)$. Show that both $g(f)$ and $f(g)$ are defined by $y = x$.

10. f is defined by $y = 6x - 3$ and g is defined by $y = \frac{1}{6}(x + 3)$. Show that both $g(f)$ and $f(g)$ are defined by $y = x$.

11. f is defined by $y = 2x + 4$ and g is defined by $y = mx + k$.
   (a) Find m and k so that $g(f)$ is defined by $y = x$.
   (b) Show that in this case $f(g)$ is also defined by $y = x$.

12. f is defined by $y = 5 - 3x$ and g is defined by $y = mx + k$.
   (a) Find m and k so that $g(f)$ is defined by $y = x$.
   (b) Show that in this case $f(g)$ is also defined by $y = x$.

13. I is the function defined by $y = x$. Complete these elements of I.
   (a) $(2, \quad)$ (b) $(\quad, -8)$ (c) $(x_1, \quad)$
   (d) $(\quad, x_1 + 2)$ (e) $(ax_1 + b, \quad)$

14. I is the function defined by $y = x$. Evaluate these elements of the range.
   (a) $I(4)$ (b) $I(-3)$ (c) $I(x)$
   (d) $I(2x)$ (e) $I(3x - 2)$ (f) $I(I(x))$

15. If I is the function defined by $y = x$ and f is the function defined by $y = mx + k$, find $I(f(x))$ and $f(I(x))$.

16. $f = \{(x, y) : y = 2x - 3\}$, $g = \{(x, y) : y = x + 2\}$, and $h = \{(x, y) : y = -3x + 1\}$.
   (a) Find the rules for $f(g)$ and $(f(g))(h)$.
   (b) Find the rules for $g(h)$ and $f(g(h))$.

Linear Functions under Composition. [B] We now briefly examine the set of linear functions, that is, functions defined by $y = ax + b$ (when a and b are real numbers, $a \neq 0$), as a mathematical system under the operation of composition.

1. **Closure.** We have already established that the set of linear functions is closed under the operation of composition (see page 102).

2. **Commutativity.** Some of the exercises in the preceding set illustrate the fact that composition is *not* a commutative operation. With f and g defined by $f(x) = a_1x + b_1$ and $g(x) = a_2x + b_2$, the student should find $f(g(x))$ and compare the result with $g(f(x))$ as found on page 102.

3. **Associativity.** If the function h is defined by $y = a_3x + b_3$, then $g(h)$ is a linear function defined by $y = a_2a_3x + (a_2b_3 + b_2)$.

Hence,
$$f(g(h(x))) = a_1(a_2a_3x + a_2b_3 + b_2) + b_1$$
$$= a_1a_2a_3x + (a_1a_2b_3 + a_1b_2 + b_1).$$

Similarly, since $f(g)$ is defined by $y = a_1a_2x + (a_1b_2 + b_1)$,

then
$$(f(g))(h(x)) = a_1a_2(a_3x + b_3) + (a_1b_2 + b_1)$$
$$= a_1a_2a_3x + (a_1a_2b_3 + a_1b_2 + b_1).$$

Thus, $f(g(h))$ and $(f(g))(h)$ are defined by the same rule over the set of real numbers, and hence they represent the same function. The operation of composition is therefore an associative operation in this system.

4. Identity element. If I is the function defined by $y = x$ and f is defined by $y = ax + b$, then $f(I) = I(f) = f$. We have $f(I(x)) = f(x) = ax + b$. Hence, $f(I) = f$. Also, $I(f(x)) = I(ax + b) = ax + b$. Hence, $I(f) = f$. Thus I as defined above is the identity element for composition.

5. Inverse elements. Once we have established that the function I defined by $I(x) = x$ is the identity element for the set of linear functions under the operation of composition, the rule defining the inverse element for a linear function under composition is obtained as follows:

Let f be defined by $f(x) = ax + b$ and let g be the linear function which is the inverse of f under composition if such a function exists.

Now $\quad f(g(x)) = a(g(x)) + b,$

so $\quad f(g(x)) = I(x) \longleftrightarrow a(g(x)) + b = x$

$$g(x) = \frac{1}{a}(x - b)$$

$$g(x) = \frac{1}{a}x - \frac{b}{a}.$$

Thus, if f has an inverse g, it is defined by $g(x) = \frac{1}{a}x - \frac{b}{a}$. Checking, we find that $g(f(x)) = \frac{1}{a}(f(x)) - \frac{b}{a} = \frac{1}{a}(ax + b) - \frac{b}{a} = x.$ Therefore, $f(g) = g(f) = I$ if g is the linear function defined by $g(x) = \frac{1}{a}x - \frac{b}{a}$, and we conclude that every linear function has an inverse under composition which is also in the set of linear functions.

The results established in the preceding paragraphs tell us that the system of linear functions under the operation of composition is an example of a *non-commutative group*.

The algebraic operations of addition and multiplication are defined for functions in much the same way as for real numbers. The operation of composition has special importance, however, in the algebra of functions. For example, the function defined by $y = x$ is called *the* identity function although it is the identity only under the operation of composition. Similarly, when we refer to the inverse of a function, we always mean the inverse under the operation of composition.

Example. If f is the function defined by $y = 3x - 5$, find the rule defining the function g which is the inverse of f.

Solution:

METHOD 1. We have established that if f is defined by $y = ax + b$, then the inverse g is defined by $y = \frac{1}{a}x - \frac{b}{a}$. In this case $a = 3$, $b = -5$. Hence g is defined by $y = \frac{1}{3}x + \frac{5}{3}$.

METHOD 2. If $f(x) = y$, then $g(f(x)) = x \longleftrightarrow g(y) = x$. This means that if f contains the ordered pair (x, y), then the inverse of f must contain the ordered pair (y, x) so that the composition of g with f will contain the ordered pair (x, x). Hence, the inverse of a function f reverses the order of the numbers in each ordered pair of f. Therefore, in order to find the rule for the inverse of f, we replace x by y and y by x in the rule defining f and solve for y. In our example we have $y = 3x - 5$ as the rule for f. Letting x be equal to $3y - 5$, we find that $y = \frac{1}{3}x + \frac{5}{3}$ is the rule for the inverse of f.

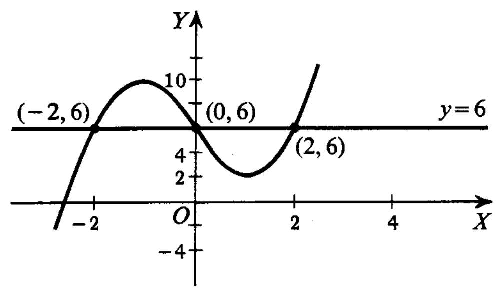
3.4 Inverse Functions in General

▶ **Definition.** Let f be a function which is a one-to-one correspondence between its domain and its range. Then $\{(y, x)\colon (x, y) \in f\}$ is also a function called the inverse of f.

The inverse of a function f is often denoted by f^{-1} (read "f inverse"). It is apparent from the definition that the domain of f is the range of f^{-1} and that the range of f is the domain of f^{-1}. It is essential that f be a one-to-one correspondence if f is to have an inverse since only under this condition is it possible for the relation formed by interchanging the first and second members of each ordered pair in f to be a function. The definition of one-to-one correspondence also makes it clear that whenever a function f satisfies this condition, then the relation formed from f in the manner described above is a function. Hence, the above definition of the inverse of a function is reasonable.

Graphs of Functions and their Inverses. A function has been defined as a set of ordered pairs such that each first element is paired with one and only one second element. Under this definition it is quite possible for several different first elements to have the same second element. We use a specific example to show why each ordered pair must have a different second element if a function is to have an inverse.

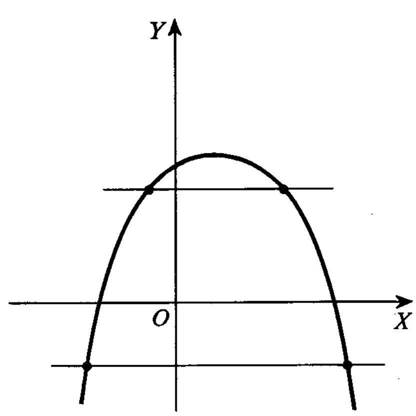
Consider the function f defined by $y = x^3 - 4x + 6$ with domain the set of real numbers. This function contains the ordered pairs $(-2, 6)$, $(0, 6)$, (2, 6). Now consider the set of ordered pairs formed by interchanging the first and second members of each ordered pair in f. This set contains



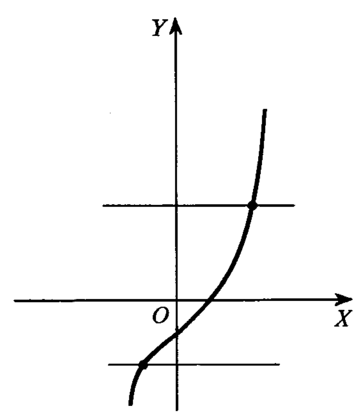
Graph of f

(6, −2), (6, 0), (6, 2) and therefore does not satisfy the definition of a function. A sketch of the graph of f shows that whereas each vertical line in the xy-plane meets the graph of f in just one point, infinitely many horizontal lines, including the graph of $y = 6$, meet the graph in three distinct points.

We have indicated why a necessary and sufficient condition for a function f to have an inverse function f^{-1} is that f be a one-to-one correspondence between its domain and its range. The graphical equivalent of this condition is that no vertical *or horizontal* line intersect the graph of f more than once. This condition is satisfied by every linear function, as we have seen, and by a number of other functions.

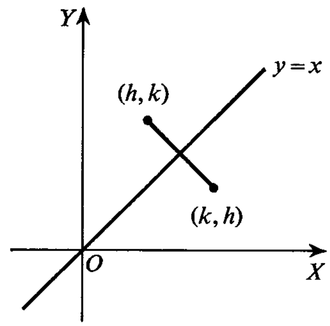


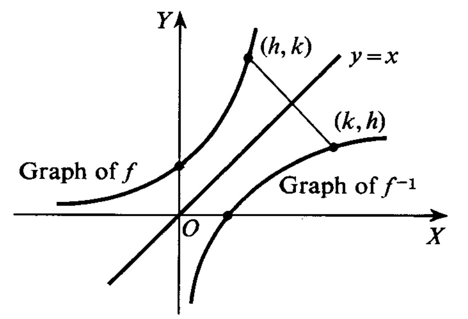
Graph of f
No function exists which is the inverse of f.



Graph of f
A function exists which is the inverse of f.

Now suppose that we have a function f which is a one-to-one correspondence, so that the inverse function f^{-1} exists. If (h, k) is any element of f, then f^{-1} contains the corresponding element (k, h). It was shown in Chapter 2 (Example 2, page 75) that an equation of the perpendicular bisector of the line segment with endpoints (h, k), (k, h) is $y = x$. Thus, the





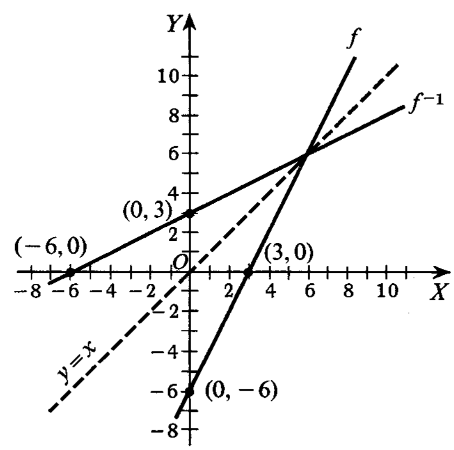
points (h, k), (k, h) are symmetric with respect to the line $y = x$. Since for every point with coordinates (h, k) on the graph of f there is a corresponding point with coordinates (k, h) on the graph of f^{-1}, the union of the graphs of f and f^{-1} is symmetric with respect to the line $y = x$. This symmetry is sometimes used to facilitate sketching the graph of f^{-1} once we have drawn the graph of f.

Example. (**a**) Draw the graph of the function f defined by $y = 2x - 6$. (**b**) Draw the graph of the function f^{-1}, which is the inverse of f.

Solution:

(**a**) The function f is defined by $y = 2x - 6$ and contains the ordered pairs $(0, -6)$, $(3, 0)$. The graph of f is the line which contains the corresponding points.

(**b**) The inverse function f^{-1} contains the ordered pairs $(-6, 0)$, $(0, 3)$. The graph of f^{-1} is the line which contains the corresponding points. The graphs of f and f^{-1} are shown in relation to the graph of $y = x$ in the diagram at the right.



Exercises [A]

1. $f = \{(-1, 3), (2, 4), (5, 6)\}$. (a) Write the function f^{-1} which is the inverse of f. (b) Evaluate $f^{-1}(f(5))$. (c) Draw the graphs of f and f^{-1}.
2. $f = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$. (a) Write the function f^{-1}. (b) Evaluate $f(f^{-1}(3))$. (c) Draw the graphs of f and f^{-1}.
3. $f = \{(1, 3), (2, 4), (3, 6), (4, 3)\}$. (a) Write the set of ordered pairs g obtained by interchanging the first and second elements of the pairs in f. (b) Why is g not a function? (c) Draw the graph of g.
4. f is defined by $y = 2x$. (a) Find the rule which defines f^{-1}. (b) Draw the graphs of f and f^{-1}.
5. f is defined by $y = x + 2$. (a) Find the rule which defines f^{-1}. (b) Draw the graphs of f and f^{-1}.
6. f is defined by $y = 3x - 5$. Find the rule which defines f^{-1}.
7. Does $y = |x|$ define a function g? Draw the graph of $y = |x|$. Is there a function g^{-1}?
8. f is defined by $y = 8 - 4x$. Find the rule which defines f^{-1}.
9. The rule $y = x^2$ defines a function F with domain the set of real numbers. (a) Sketch the graph of F. (b) Show that there is no function which is the inverse of F.
10. A function f is defined by the rule $y = x^2$ with domain the set of non-negative real numbers. (a) Sketch the graph of f. (b) Is f a one-to-one correspondence? (c) Write the rule which defines the function f^{-1}. (d) Sketch the graph of f^{-1}.
11. List the ordered pairs of a function f such that $f^{-1} = f$.
12. Write a rule for a function which is its own inverse (a) when the function is linear, (b) when the function is not linear.
13. If $f(x) = ax + b$, $ab \neq 0$, which of the following are true statements?

    (a) $f(x + h) - f(x) = ah$, all x.   (b) $f\left(\frac{1}{x}\right) = \frac{1}{f(x)}$ when $x \neq 0, f(x) \neq 0$

    (c) $f(x^2) = [f(x)]^2$ for all x.   (d) $f^{-1}(x) = \frac{1}{a}(x - b)$ for all x.

    (e) $\frac{f(x_1)}{f(x_2)} = \frac{x_1}{x_2}$ when $x_2 \neq 0$, $f(x_2) \neq 0$.   (f) $f(|x|) = |f(x)|$ for all x.

    (g) $f(x_1 + x_2) = f(x_1) + f(x_2)$ for all x_1 and x_2.

    (h) $f(x_1 x_2) = f(x_1) \cdot f(x_2)$ for all x_1 and x_2.
14. Repeat Exercise 13 with $f(x) = ax$, $a \neq 0$.

Pages 98–99

1. Domain (a) $\{1, 2, 3, 4\}$ (b) $\{x: x \in R\}$ (c) $\{x: x \in R\}$
(d) $\{x: x \neq 2, x \in R\}$ (e) $\{x: |x| \leq 2, x \in R\}$
Range (a) $\{4, 9, 19, 51\}$ (b) $\{f(x): f(x) \in R\}$
(c) $\{f(x): f(x) \geq 2, f(x) \in R\}$ (d) $\{f(x): f(x) \neq 0, f(x) \in R\}$
(e) $\{f(x): f(x) \geq 0, f(x) \in R\}$
3. (a) $\{x: x \neq 0, x \neq 2, x \in R\}$ (b) $\{x: x \neq 1, x \neq -1, x \in R\}$
(c) $\{x: x > 1, x \in R\}$
5. (a) 9 (b) -3 (c) -11 (d) $4n - 3$ (e) $4x^2 - 3$ (f) 4 (g) 4 (h) 4
9. (a) False (b) False (c) True (d) False (e) True (f) False
11. (a) 0 (b) 0 (c) $4ab$ (d) $2x_1 + h$

Page 101

1. (a) $f(2)$ (b) $x < 2$ (c) -12 **3.** even positive integers
5. (a) $\{\frac{3}{2}, \frac{17}{2}\}$ (b) $x \geq \frac{3}{2}$ **7.** (a) 100,000 (b) 99.835 (c) 102 (d) 0.06
9. (a) $2a - 5$; $2b - 5$; $2a + 2b - 5$ (b) No **13.** $-t + 4$
15. (a) $6x - 2$ (b) $3x^2 - 2$ (c) $6t + 1$

Pages 103–104

1. $\{(1, 3), (2, 15), (3, 35)\}$; $\{(1, 9)\}$ **3.** (a) $21x - 27$; $21x - 1$ (b) 36; 62
5. $-8x$; $-8x$ **7.** $8x - 5$; $8x - 5$ **11.** (a) $m = \frac{1}{2}, k = -2$
13. (a) 2 (b) -8 (c) x_1 (d) $x_1 + 2$ (e) $ax_1 + b$
15. $mx + k$; $mx + k$

Page 109

1. (a) $\{(3, -1), (4, 2), (6, 5)\}$ (b) 5 **5.** (a) $x - 2$ **7.** Yes; No
11. $\{(x, y): x = y, x, y \in R\}$
13. (a) T (b) F (c) F (d) T (e) F (f) F (g) F (h) F

Page 113

1. (a) $f(6), f(-2)$ (b) $\{y: y \geq -16\}$
3. (a) $f(2 + \sqrt{3}), f(2 - \sqrt{3})$ (b) $\{y: y \geq -3\}$
5. (a) $f(-2 + \sqrt{5}), f(-2 - \sqrt{5})$ (b) $\{y: y \leq 5\}$
7. $f(\frac{5}{6})$ **9.** (a) $\frac{16}{3}$ (b) $k < \frac{16}{3}$ **11.** -5 **13.** $-\frac{49}{8}$ **15.** (a) 1

Pages 118–119

1. (a) $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ **5.** (b) $x = 4, x = -2$ (c) $x > 4, x < -2$
7. (b) $x = -2 \pm \sqrt{2}$ (c) $-2 - 2\sqrt{2} < x < -2 + 2\sqrt{2}$ **9.** (b) $-4 \leq x \leq 1$
11. $x'y' = 2$ **15.** $-4t^2 + 10t + 6$