[08-03-06-P2-10]

Locating a Rational Root of a Polynomial Equation

The result established for the possible rational roots of a cubic equation may be extended to all equations of the form $f(x) = 0$, where $f(x)$ is a polynomial. The following statement may be proved in a manner similar to that used for the cubic equation.

▶ **If $\frac{p}{q}$ is a root of $ax^n + bx^{n-1} + \cdots + k = 0$, p and q being integers having no common factor, and the coefficients $a, b, \cdots k$ also being integers, then p is a factor of k and q is a factor of a.**

The importance of the result is that it limits the number of possible rational roots of a particular equation. It thus enables us to decide, by trial of the possible values, whether or not the equation has a rational root. When there is a large number of possible rational roots, it is well to locate a real root of the equation by finding successive integral values of x for which $f(x)$ changes sign. The number of trial substitutions needed is thus reduced to those of the possible rational roots which lie between the successive integers.

Example. Find a rational root of $4x^3 + 4x^2 + 3x + 9 = 0$.

Solution: When x is positive, each of the four terms on the left-hand side of the equation is positive and their sum cannot be zero.

Hence, the equation does not have a positive root, and we consider only negative values of x.

Let $f(x) = 4x^3 + 4x^2 + 3x + 9$; then we find by substitution that

$$f(0) = 9, f(-1) = 6, f(-2) = -13.$$

Since $f(-1)$ is positive and $f(-2)$ is negative, there is a real root of the given equation between -1 and -2. Of the possible rational roots the only one between -1 and -2 is $-\frac{3}{2}$, so we make a trial substitution of this number.

$$\begin{aligned} f(-\tfrac{3}{2}) &= 4(-\tfrac{27}{8}) + 4(\tfrac{9}{4}) - \tfrac{9}{2} + 9 \\ &= -13.5 + 9 - 4.5 + 9 \\ &= 0 \end{aligned}$$

Hence, $-\frac{3}{2}$ is a rational root of $4x^3 + 4x^2 + 3x + 9 = 0$.

Exercises [A-1]

Factor:

1. $x^3 - 3x + 2$
2. $x^3 - 7x - 6$
3. $2x^3 + x^2 + x - 1$
4. $4x^3 - 7x - 3$
5. $2x^3 + 5x^2 + 6x + 2$
6. $x^4 - x^2 - 4x - 4$

Solve:

7. $x^3 - 5x - 4 = 0$
8. $x^3 - 6x^2 + 9x - 2 = 0$
9. $4x^3 = 1 - x$
10. $3x^3 - 13x = 2$
11. $3x^3 + 5x^2 + 4x = 2$
12. $2x^3 - x^2 + x - 6 = 0$
13. $x^4 = 5x + 6$
14. $2x^4 + 3x^3 + 2x^2 = 1$
15. Show that $3x^3 + 4x + 2 = 0$ does not have a rational root, but that it has a real root between $-\frac{1}{3}$ and $-\frac{2}{3}$.
16. Show that $6x^3 - x^2 - 3x - 20 = 0$ has a real root between 1 and 2, and find the rational root of the equation.

Exercises [A–2]

Factor:

1. $x^3 - 3x - 2$
2. $x^3 - 7x + 6$
3. $2x^3 + 3x^2 - 1$
4. $4x^3 - 5x - 6$
5. $3x^3 + 7x^2 + 7x + 4$
6. $x^4 - 2x^3 + 3x^2 - 4x + 2$

Solve:

7. $x^3 - 2x + 1 = 0$
8. $2x^3 - 5x^2 + 1 = 0$
9. $x^3 = x^2 + x + 2$
10. $2x^3 + 3x^2 + x + 6 = 0$
11. $2x^3 + 9x^2 = 2$
12. $3x^3 + 4x^2 + 2x - 4 = 0$
13. $x^4 + 20x = 21$
14. $4x^4 + x^2 - 3x + 1 = 0$
15. Show that $2x^3 + 4x + 3 = 0$ does not have a rational root, but that it has a real root between $-\frac{1}{2}$ and -1.
16. Show that $6x^3 - 13x^2 - x - 10 = 0$ has a real root between 2 and 3, and find the rational root of the equation.
17. (a) Factor $x^3 + x^2 - 2$. (b) Deduce the quotient and remainder when $x^3 + x^2 + 2$ is divided by $x - 1$. (c) Express $x^3 + x^2 + 2$ in the form $(x - 1)(ax^2 + bx + c) + d$.
18. (a) Factor $2x^3 + x^2 + 3x - 2$. (b) Deduce the quotient and remainder when $2x^3 + x^2 + 3x + 4$ is divided by $2x - 1$. (c) Express $2x^3 + x^2 + 3x + 4$ in the form $(2x - 1)(ax^2 + bx + c) + d$.

Synthetic Division

In the study of polynomials and the solution of equations it is frequently necessary to divide a polynomial by a binomial $x - k$, where k is a rational number. Efficiency in this operation is achieved by shortening the division process. Consider the division of $2x^3 - 3x^2 + 4x - 7$ by $x - 2$ as illustrated in forms (a), (b), (c) and (d) in the following discussion.

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1. $(x - 1)^2(x + 2)$
2. $(x + 1)(x + 2)(x - 3)$
3. $(2x - 1)(x^2 + x + 1)$
4. $(x + 1)(2x - 3)(2x + 1)$
5. $(2x + 1)(x^2 + 2x + 2)$
6. $(x^2 + x + 2)(x - 2)(x + 1)$
7. $\{-1, \frac{1}{2} + \frac{1}{2}\sqrt{17}, \frac{1}{2} - \frac{1}{2}\sqrt{17}\}$
8. $\{2, 2 + \sqrt{3}, 2 - \sqrt{3}\}$
9. $\{\frac{1}{2}, -\frac{1}{4} + \frac{1}{4}i\sqrt{7}, -\frac{1}{4} - \frac{1}{4}i\sqrt{7}\}$
10. $\{-2, 1 + \frac{2}{3}\sqrt{3}, 1 - \frac{2}{3}\sqrt{3}\}$
11. $\{\frac{1}{3}, -1 + i, -1 - i\}$
12. $\{\frac{3}{2}, -\frac{1}{2} + \frac{1}{2}i\sqrt{7}, -\frac{1}{2} - \frac{1}{2}i\sqrt{7}\}$
13. $\{-1, 2, -\frac{1}{2} + \frac{1}{2}i\sqrt{11}, -\frac{1}{2} - \frac{1}{2}i\sqrt{11}\}$
14. $\{\frac{1}{2}, -1, -\frac{1}{2} + \frac{1}{2}i\sqrt{3}, -\frac{1}{2} - \frac{1}{2}i\sqrt{3}\}$
15. $f(-\frac{1}{3}) = \frac{5}{9}$; $f(-\frac{2}{3}) = -1\frac{5}{9}$
16. $f(1) = -18$; $f(2) = +18$; $x = \frac{5}{3}$

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1. $(x + 1)^2(x - 2)$
2. $(x - 1)(x + 3)(x - 2)$
3. $(x + 1)^2(2x - 1)$
4. $(2x - 3)(2x^2 + 3x + 2)$
5. $(3x + 4)(x^2 + x + 1)$
6. $(x - 1)^2(x^2 + 2)$
7. $\{1, -\frac{1}{2} + \frac{1}{2}\sqrt{5}, -\frac{1}{2} - \frac{1}{2}\sqrt{5}\}$
8. $\{\frac{1}{2}, 1 + \sqrt{2}, 1 - \sqrt{2}\}$
9. $\{2, -\frac{1}{2} + \frac{1}{2}i\sqrt{3}, -\frac{1}{2} - \frac{1}{2}i\sqrt{3}\}$
10. $\{-2, \frac{1}{4} + \frac{1}{4}i\sqrt{23}, \frac{1}{4} - \frac{1}{4}i\sqrt{23}\}$
11. $\{-\frac{1}{2}, -2 + \sqrt{6}, -2 - \sqrt{6}\}$
12. $\{\frac{2}{3}, -1 + i, -1 - i\}$
13. $\{1, -3, 1 + i\sqrt{6}, 1 - i\sqrt{6}\}$
14. $\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} + \frac{1}{2}i\sqrt{3}, -\frac{1}{2} - \frac{1}{2}i\sqrt{3}\}$
15. $f(-\frac{1}{2}) = \frac{3}{4}$; $f(-1) = -3$
16. $f(2) = -16$; $f(3) = 32$; $x = \frac{5}{2}$
17. a. $(x - 1)(x^2 + 2x + 2)$
    b. $x^2 + 2x + 2$; $R = 4$
    c. $(x - 1)(x^2 + 2x + 2) + 4$
18. a. $(2x - 1)(x^2 + x + 2)$
    b. $x^2 + x + 2$; $R = 6$
    c. $(2x - 1)(x^2 + x + 2) + 6$