[08-03-06-HH10]

6 a Both $2x+1$ and $x-2$ are factors of $P(x)=2x^4+ax^3+bx^2-12x-8$. Find a and b and all zeros of $P(x)$.

b $x+3$ and $2x-1$ are factors of $2x^4+ax^3+bx^2+ax+3$. Find a and b and hence determine all zeros of the quartic.

7 a x^3+3x^2-9x+c has two identical linear factors. Prove that c is either 5 or -27 and factorise the cubic into linear factors in each case.

b $3x^3+4x^2-x+m$ has two identical linear factors. Find m and find the zeros of the polynomial in all possible cases.

THE REMAINDER THEOREM

Consider the following division: $\dfrac{x^3+5x^2-11x+3}{x-2}$. We can show by long division that

$$\frac{x^3+5x^2-11x+3}{x-2}=x^2+7x+3+\frac{9}{x-2} \leftarrow \text{remainder.}$$

i.e., on division by $x-2$ its remainder is 9.

Notice also that if $P(x)=x^3+5x^2-11x+3$

then
$$\begin{aligned}P(2)&=8+20-22+3\\&=9, \quad \text{which is the remainder.}\end{aligned}$$

After considering other examples like the one above we formulate the **Remainder theorem**.

THE REMAINDER THEOREM

When polynomial $P(x)$ is divided by $x-k$ until a constant remainder R is obtained then $R=P(k)$.

Proof: By the division algorithm, $P(x)=Q(x)(x-k)+R$

Now if $x=k$, $P(k)=Q(k)\times 0+R$

$\therefore$ $P(k)=R$

Example 29

Use the Remainder theorem to find the remainder when x^4-3x^3+x-4 is divided by $x+2$.

If $P(x)=x^4-3x^3+x-4$, then

$$\begin{aligned}P(-2)&=(-2)^4-3(-2)^3+(-2)-4\\&=16+24-2-4\\&=34\end{aligned}$$

$\therefore$ when $P(x)$ is divided by $x+2$, the remainder is 34. {Remainder theorem}

It is important to realise when doing Remainder theorem questions that $P(x)=(x+2)Q(x)+3$, $P(-2)=3$ and '$P(x)$ divided by $x+2$ leaves a remainder of 3' are all **equivalent statements**.

EXERCISE 8D.3

1. Write two equivalent statements for:
   a. If $P(2) = 7$, then ......
   b. If $P(x) = (x+3)Q(x) - 8$, then ......
   c. If $P(x)$ when divided by $x - 5$ has a remainder of 11 then ......

2. Without performing division, find the remainder when:
   a. $x^3 + 2x^2 - 7x + 5$ is divided by $x - 1$
   b. $x^4 - 2x^2 + 3x - 1$ is divided by $x + 2$.

3. Find a given that:
   a. when $x^3 - 2x + a$ is divided by $x - 2$, the remainder is 7
   b. when $2x^3 + x^2 + ax - 5$ is divided by $x + 1$, the remainder is -8.

4. Find a and b given that when $x^3 + 2x^2 + ax + b$ is divided by $x - 1$ the remainder is 4 and when divided by $x + 2$ the remainder is 16.

5. If $2x^n + ax^2 - 6$ leaves a remainder of -7 when divided by $x - 1$ and 129 when divided by $x + 3$, find a and n.

Example 30

When $P(x)$ is divided by $x^2 - 3x + 7$ the quotient is $x^2 + x - 1$ and the remainder is unknown. However, when $P(x)$ is divided by $x - 2$ the remainder is 29 and when divided by $x + 1$ the remainder is -16. If the remainder has the form $ax + b$, find a and b.

As the divisor is $x^2 - 3x + 7$ and the remainder has form $ax + b$,

$$\text{then}\quad P(x) = \underbrace{(x^2 + x - 1)}_{Q(x)}\underbrace{(x^2 - 3x + 7)}_{D(x)} + \underbrace{ax + b}_{R(x)}$$

But $P(2) = 29$ and $P(-1) = -16$ {Remainder theorem}

$$\therefore\quad (2^2 + 2 - 1)(2^2 - 6 + 7) + 2a + b = 29$$

$$\text{and}\quad ((-1)^2 + (-1) - 1)((-1)^2 - 3(-1) + 7) + (-a + b) = -16$$

$$\therefore \begin{cases} (5)(5) + 2a + b = 29 \\ (-1)(11) - a + b = -16 \end{cases}$$

$$\therefore \begin{cases} 2a + b = 4 \\ -a + b = -5 \end{cases}$$

Solving these gives $a = 3$ and $b = -2$.

6. When $P(z)$ is divided by $z^2 - 3z + 2$ the remainder is $4z - 7$.
   Find the remainder when $P(z)$ is divided by: a. $z - 1$ b. $z - 2$.

7. When $P(z)$ is divided by $z + 1$ the remainder is -8 and when divided by $z - 3$ the remainder is 4. Find the remainder when $P(z)$ is divided by $(z - 3)(z + 1)$.

8. If $P(x)$ is divided by $(x - a)(x - b)$, prove that the remainder is: $\left(\dfrac{P(b) - P(a)}{b - a}\right) \times (x - a) + P(a)$.

9 If $P(x)$ is divided by $(x-a)^2$, prove that the remainder is $P'(a)(x-a)+P(a)$, where $P'(x)$ is the derivative of $P(x)$.

An immediate consequence of the Remainder theorem is the **Factor theorem**.

THE FACTOR THEOREM

k is a zero of $P(x) \Leftrightarrow (x-k)$ is a factor of $P(x)$.

Proof: k is a zero of $P(x)$
$\Leftrightarrow P(k)=0$ {definition of a zero}
$\Leftrightarrow R=0$ {Remainder theorem}
$\Leftrightarrow P(x)=Q(x)(x-k)$ {division algorithm}
$\Leftrightarrow (x-k)$ is a factor of $P(x)$ {definition of factor}

The **Factor theorem** says that if 2 is a zero of $P(x)$ then $(x-2)$ is a factor of $P(x)$ and vice versa.

Example 31

Find k given that $x-2$ is a factor of x^3+kx^2-3x+6 and then fully factorise x^3+kx^2-3x+6.

Let $P(x)=x^3+kx^2-3x+6$

By the Factor theorem, as $x-2$ is a factor then $P(2)=0$

$\therefore\ 2^3+k2^2-3(2)+6=0$

$\therefore\ 8+4k=0$ and so $k=-2$

Now $x^3-2x^2-3x+6=(x-2)(x^2+ax-3)$

Equating coefficients of x^2 gives: $-2=-2+a$ i.e., $a=0$

Equating coefficients of x gives: $-3=-2a-3$ i.e., $a=0$

$$\therefore\ x^3-2x^2-3x+6=(x-2)(x^2-3)=(x-2)(x+\sqrt{3})(x-\sqrt{3})$$

or Using synthetic division

2	1	k	-3	6
	0	2	$2k+4$	$4k+2$
	1	$k+2$	$2k+1$	$4k+8$

$\therefore\ P(2)=4k+8$ and since $P(2)=0$, $k=-2$

$$\text{Now } P(x)=(x-2)(x^2+[k+2]x+[2k+1])=(x-2)(x^2-3)=(x-2)(x+\sqrt{3})(x-\sqrt{3})$$

EXERCISE 8D.4

1 Find k and hence factorise the polynomial if:

a $2x^3+x^2+kx-4$ has a factor of $x+2$

b $x^4-3x^3-kx^2+6x$ has a factor of $x-3$.

2 Find a and b given that $2x^3 + ax^2 + bx + 5$ has factors of $x - 1$ and $x + 5$.

3 **a** 3 is a zero of $P(z) = z^3 - z^2 + [k - 5]z + [k^2 - 7]$.
Find k and hence find all zeros of $P(z)$.

b Show that $z - 2$ is a factor of $P(z) = z^3 + mz^2 + (3m - 2)z - 10m - 4$ for all values of m. For what values of m is $(z - 2)^2$ a factor of $P(z)$?

4 **a** Consider $P(x) = x^3 - a^3$ where a is real.

i Find $P(a)$. What is the significance of this result?

ii Factorise $x^3 - a^3$ as a product of a real linear and quadratic factor.

b Now consider $P(x) = x^3 + a^3$, where a is real.

i Find $P(-a)$. What is the significance of this result?

ii Factorise $x^3 + a^3$ as a product of a real linear and quadratic factor.

5 **a** Prove that "$x + 1$ is a factor of $x^n + 1$ $\Leftrightarrow$ n is odd."

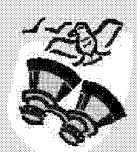
b Find real number a such that $x - 1 - a$ is a factor of $P(x) = x^3 - 3ax - 9$.

E GRAPHING POLYNOMIALS

In this section we are obviously only concerned with graphing **real** polynomials. Do you remember what is meant by a real polynomial?

Graphing using a **graphics calculator** or the **graphing package** provided would be invaluable.

INVESTIGATION 2 CUBIC GRAPHING



Possible types to consider are:

GRAPHING PACKAGE

Type 1: $P(x) = a(x - \alpha)(x - \beta)(x - \gamma), \ a \neq 0$

Type 2: $P(x) = a(x - \alpha)(x - \beta)^2, \ a \neq 0$

Type 3: $P(x) = (x - \alpha)(ax^2 + bx + c), \Delta = b^2 - 4ac < 0, \ a \neq 0$

Type 4: $P(x) = a(x - \alpha)^3, \ a \neq 0$

What to do: (Use transformations of **Chapter 6** wherever possible)

1 Experiment with *Type 1* graphs of cubics. Clearly state the effect of changing a (in size and sign). What is the geometrical significance of α, β and γ?

2 Experiment with *Type 2* graphs of cubics. What is the geometrical significance of the squared factor?

3 Experiment with *Type 3* graphs of cubics. What is the geometric significance of α and the quadratic factor which has imaginary zeros?

4 Experiment with *Type 4* graphs of cubics. What is the geometric significance of α? Do not forget to consider $a > 0$ and $a < 0$.

c $x^2+x+3+\dfrac{3x-4}{x^2-x+1}$ **d** $2x+4+\dfrac{5x+2}{(x-1)^2}$

e $x^2-2x+3-\dfrac{4x+3}{(x+1)^2}$

f $x^2-3x+5+\dfrac{15-10x}{(x-1)(x+2)}$

3 quotient is x^2+2x+3, remainder is 7

EXERCISE 8C.4

1 **a** $3x+1-\dfrac{2}{x-1}$ **b** $x^2+2x+\dfrac{5}{x+3}$

c $3z-4+\dfrac{6}{z+1}$ **d** x^2-3x+9

e $z^3+z^2+4z+12+\dfrac{32}{z-3}$ **f** $z^3-z^2+2z-3+\dfrac{3}{z+1}$

2 **a** $P(x)=3x^3-8x^2+5x+2$
b $P(x)=x^3-4x^2-13x+19$

EXERCISE 8D.1

1 **a** $4, -\frac{3}{2}$ **b** $-3\pm i$ **c** $3\pm\sqrt{3}$ **d** $0, \pm 2$

e $0, \pm i\sqrt{2}$ **f** $\pm 1, \pm i\sqrt{5}$

2 **a** $x=1, -\frac{2}{5}$ **b** $x=-\frac{1}{2}, \pm i\sqrt{3}$ **c** $z=0, 1\pm i$

d $x=0, \pm\sqrt{5}$ **e** $z=0, \pm i\sqrt{5}$ **f** $z=\pm i\sqrt{2}, \pm\sqrt{5}$

3 **a** $(2x+3)(x-5)$ **b** $(z-3+i\sqrt{7})(z-3-i\sqrt{7})$

c $x(x+1+\sqrt{5})(x+1-\sqrt{5})$ **d** $z(3z-2)(2z+1)$

e $(z+1)(z-1)(z+\sqrt{5})(z-\sqrt{5})$

f $(z+i)(z-i)(z+\sqrt{2})(z-\sqrt{2})$

5 **a** $P(z)=a(z^2-4)(z-3)$ $a\neq 0$
b $P(z)=a(z+2)(z^2+1)$ $a\neq 0$
c $P(z)=a(z-3)(z^2+2z+2)$ $a\neq 0$
d $P(z)=a(z+1)(z^2+4z+2)$ $a\neq 0$

6 **a** $P(z)=a(z^2-1)(z^2-2)$ $a\neq 0$
b $P(z)=a(z-2)(z+1)(z^2+3)$ $a\neq 0$
c $P(z)=a(z^2-3)(z^2-2z+2)$ $a\neq 0$
d $P(z)=a(z^2-4z-1)(z^2+4z+13)$ $a\neq 0$

EXERCISE 8D.2

1 **a** $a=2$, $b=5$, $c=5$ **b** $a=3$, $b=4$, $c=3$

2 **a** $a=2, b=-2$ or $a=-2, b=2$ **b** $a=3$, $b=-1$

4 $a=-2$, $b=2$, $x=1\pm i$ or $-1\pm\sqrt{3}$

5 **a** $a=-1$, zeros are $\frac{3}{2}$, $\dfrac{-1\pm i\sqrt{3}}{2}$

b $a=6$, zeros are $-\frac{2}{3}$, $\dfrac{1\pm i\sqrt{11}}{2}$

6 **a** $a=-3$, $b=6$ zeros are $-\frac{1}{2}$, 2, $\pm 2i$
b $a=1$, $b=-15$ zeros are $-3, \frac{1}{2}, 1\pm\sqrt{2}$

7 **a** $P(x)=(x+3)^2(x-3)$ or $P(x)=(x-1)^2(x+5)$
b If $m=-2$, zeros are -1 (repeated) and $\frac{2}{3}$.
If $m=\frac{14}{243}$, zeros are $\frac{1}{9}$ (repeated) and $-\frac{14}{9}$.

EXERCISE 8D.3

1 **a** $P(x)=(x-2)Q(x)+7$, $P(x)$ divided by $x-2$ leaves a remainder of 7.
b $P(-3)=-8$, $P(x)$ divided by $x+3$ leaves a remainder of -8.
c $P(5)=11$, $P(x)=(x-5)Q(x)+11$

2 **a** 1 **b** 1 **3** **a** $a=3$ **b** $a=2$ **4** $a=-5$, $b=6$

5 $a=-3$, $n=4$ **6** **a** -3 **b** 1 **7** $3z-5$

EXERCISE 8D.4

1 **a** $k=-8$, $P(x)=(x+2)(2x+1)(x-2)$
b $k=2$, $P(x)=x(x-3)(x+\sqrt{2})(x-\sqrt{2})$

2 $a=7$, $b=-14$

3 **a** If $k=1$, zeros are $3, -1\pm i$.
If $k=-4$, zeros are $\pm 3, 1$. **b** $m=-\frac{10}{7}$

4 **a** **i** $P(a)=0$, $x-a$ is a factor **ii** $(x-a)(x^2+ax+a^2)$
b **i** $P(-a)=0$, $x+a$ is a factor **ii** $(x+a)(x^2-ax+a^2)$

5 **b** $a=2$

EXERCISE 8E.1

1 **a** cuts the x-axis at α **b** touches the x-axis at α
c cuts the x-axis at α with a change in shape

2 **a** $P(x)=2(x+1)(x-2)(x-3)$
b $P(x)=-2(x+3)(2x+1)(2x-1)$
c $P(x)=\frac{1}{4}(x+4)^2(x-3)$
d $P(x)=\frac{1}{10}(x+5)(x+2)(x-5)$
e $P(x)=\frac{1}{4}(x+4)(x-3)^2$
f $P(x)=-2(x+3)(x+2)(2x+1)$

3 **a** $P(x)=(x-3)(x-1)(x+2)$
b $P(x)=x(x+2)(2x-1)$ **c** $P(x)=(x-1)^2(x+2)$
d $P(x)=(3x+2)^2(x-4)$

4 **a** F **b** C **c** A **d** E **e** D **f** B

5 **a** $P(x)=5(2x-1)(x+3)(x-2)$
b $P(x)=-2(x+2)^2(x-1)$
c $P(x)=(x-2)(2x^2-3x+2)$

EXERCISE 8E.2

1 **a** $P(x)=2(x+1)^2(x-1)^2$
b $P(x)=(x+3)(x+1)^2(3x-2)$
c $P(x)=-2(x+2)(x+1)(x-2)^2$
d $P(x)=-\frac{1}{3}(x+3)(x+1)(2x-3)(x-3)$
e $P(x)=\frac{1}{4}(x+1)(x-4)^3$ **f** $P(x)=x^2(x+2)(x-3)$

2 **a** C **b** F **c** A **d** E **e** B **f** D

3 **a** $P(x)=(x+4)(2x-1)(x-2)^2$
b $P(x)=\frac{1}{4}(3x-2)^2(x+3)^2$
c $P(x)=2(x-2)(2x-1)(x+2)(2x+1)$
d $P(x)=(x-1)^2\left(\frac{8}{3}x^2+\frac{8}{3}x-1\right)$

EXERCISE 8E.3

1 **a** $-1, 2\pm\sqrt{3}$ **b** $1, 1\pm i$ **c** $\frac{7}{2}, -1\pm 2i$
d $\frac{1}{2}, \pm i\sqrt{10}$ **e** $\pm\frac{1}{2}, 3, -2$ **f** $2, 1\pm 3i$

2 **a** $x=-2, \pm i\sqrt{3}$ **b** $x=-2, -\frac{1}{2}, 1$
c $x=2$ (treble root) **d** $x=-2, \frac{3}{2}, 3$
e $x=-3, 2, 1\pm\sqrt{2}$ **f** $x=-\frac{1}{2}, 3, 2\pm i$