[06-11-28-TF12]

Find dy/dx in each of the following problems (1 through

1. $y = \frac{x}{\sqrt{x^2 - 4}}$
2. $x^2 + xy + y^2 - 5x = 2$
3. $xy + y^2 = 1$
4. $x^3 + 4xy - 3y^3 = 2$
5. $x^2y + xy^2 = 6$
6. $y = (x + 1)^2(x^2 + 2x)^{-2}$
7. $y = \cos(1 - 2x)$
8. $y = \frac{\cos x}{\sin x}$
9. $y = \frac{x}{x + 1}$
10. $y = \sqrt{2x + 1}$
11. $y = x^2\sqrt{x^2 - a^2}$
12. $y = \frac{2x + 1}{2x - 1}$
13. $y = \frac{x^2}{1 - x^2}$
14. $y = (x^2 + x + 1)^3$
15. $y = \sec^2(5x)$
16. $y^3 = \sin^3 x + \cos^3 x$
17. $y = \frac{(2x^2 + 5x)^{3/2}}{3}$
18. $y = \frac{3}{(2x^2 + 5x)^{3/2}}$
19. $xy^2 + \sqrt{xy} = 2$
20. $x^2 - y^2 = xy$
21. $x^{2/3} + y^{2/3} = a^{2/3}$
22. $x^{1/2} + y^{1/2} = a^{1/2}$
23. $xy = 1$
24. $\sqrt{xy} = 1$
25. $(x + 2y)^2 + 2xy^2 = 6$
26. $y = \sqrt{\frac{1 - x}{1 + x^2}}$
27. $y^2 = \frac{x}{x + 1}$
28. $x^2y + xy^2 = 6(x^2 + y^2)$
29. $xy + 2x + 3y = 1$
30. $y = u^2 - 1, \quad x = u^2 + 1$
31. $y = \sqrt{2t + t^2}, \quad t = 2x + 3$
32. $x = \frac{t}{1 + t^2}, \quad y = 1 + t^2$
33. $t = \frac{x}{1 + x^2}, \quad y = x^2 + t^2$

34. Find the slope of $y = x/(x^2 + 1)$ at the origin. Write the equation of the tangent line at the origin.

35. Write the equation of the tangent at (2, 2) to the curve

$$x^2 - 2xy + y^2 + 2x + y - 6 = 0.$$

36. Determine the constant c such that the straight line joining the points (0, 3) and (5, -2) is tangent to the curve $y = c/(x + 1)$.

37. What is the slope of the curve $y = 2x^2 - 6x + 3$ at the point on the curve where $x = 2$? What is the equation of the tangent line to the curve at this point?

38. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the x-axis.

39. Find the derivatives of the following functions:

a) $y = (x^2 + 2x)^5$; b) $f(t) = \sqrt{3t^2 - 2t}$;

c) $f(r) = \sqrt{r^2 + 5} + \sqrt{r^2 - 5}$;

d) $f(x) = \frac{x^2 - 1}{x^2 + 1}$.

40. Find the equation of the tangent to the curve $y = 2/\sqrt{x - 1}$ at the point on the curve where $x = 10$.

41. Write the equation of the straight line passing through the point (2, 1) and normal to the curve $x^2 = 4y$.

42. Use the definition of the derivative to find dy/dx for $y = \sqrt{2x + 3}$ and then check the result by finding the same derivative by the power formula.

43. Find the value of

$$\lim_{\Delta x \to 0} \frac{[2 - 3(x + \Delta x)]^2 - [2 - 3x]^2}{\Delta x}$$

and specify the function $f(x)$ of which this is the derivative.

44. Find the slope of the curve $x^2y + xy^2 = 6$ at the point (1, 2).

45. A cylindrical can of height 6 (in.) and radius r (in.) has volume $V = 6\pi r^2$ (in.^{3}). What is the difference between ΔV and its principal part as r varies? What is the geometric significance of the principal part?

46. If a hemispherical bowl of radius 10 in. is filled with water to a depth of x in., the volume of water is given by $v = \pi[10 - (x/3)]x^2$. Find the rate of increase of the volume per inch increase of the depth.

47. A bus will hold 60 people. If the number x of persons per trip who use the bus is related to the fare charged (p nickels), by the law $p = [3 - (x/40)]^2$, write the function expressing the total revenue per trip received by the bus company. What is the number x_1 of people per trip that will make the marginal revenue equal to zero? What is the corresponding fare?

48. Prove Eq. (2), Article 2–3, by mathematical induction.

49. Given $y = x - x^2$, find the rate of change of y^2 with respect to x^2 (expressed in terms of x).

50. If $x = 3t + 1$ and $y = t^2 + t$, find dy/dt, dx/dt, and dy/dx. Eliminate t to obtain y as a function of x, and then determine dy/dx directly. Do the results check?

51. A particle projected vertically upward with a speed of a ft/sec reaches an elevation $s = at - 16t^2$ ft at the end of t sec. What must the initial velocity be in order for the particle to travel 49 ft upward before it starts coming back down?

52. Find the rate of change of $\sqrt{x^2 + 16}$ with respect to $x/(x - 1)$ at $x = 3$.

53. The circle $(x - h)^2 + (y - k)^2 = r^2$, center at (h, k), radius $= r$ (see Article 8–4), is tangent to the curve $y = x^2 + 1$ at the point $(1, 2)$. (a) Find the locus of the point (h, k). (b) If, also, the circle and the curve have the same second derivative at $(1, 2)$, find h, k, and r. Sketch the curve and the circle.

54. If $y = x^2 + 1$ and $u = \sqrt{x^2 + 1}$, find dy/du.

55. If $x = y^2 + y$ and $u = (x^2 + x)^{3/2}$, find dy/du.

56. If $f'(x) = \sqrt{3x^2 - 1}$ and $y = f(x^2)$, find dy/dx.

57. If $f'(x) = \sin(x^2)$ and $y = f((2x - 1)/(x + 1))$, find dy/dx.

58. Given $y = 3 \sin 2x$ and $x = u^2 + \pi$, find the value of dy/du when $u = 0$.

59. If $0 < x < \pi/2$, prove that $x > \sin x > 2x/\pi$.

60. If $y = x\sqrt{2x - 3}$, find d^2y/dx^2.

61. Find the value of d^2y/dx^2 in the equation $y^3 + y = x$ at the point $(2, 1)$.

62. If $x = t - t^2$, $y = t - t^3$, find the values of dy/dx and d^2y/dx^2 at $t = 1$.

63. Prove Leibniz's rule:

a) $$\frac{d^2(uv)}{dx^2} = \frac{d^2u}{dx^2} \cdot v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2},$$

b) $$\frac{d^3(uv)}{dx^3} = \frac{d^3u}{dx^3} \cdot v + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + u\frac{d^3v}{dx^3},$$

c) $$\frac{d^n(uv)}{dx^n} =$$

$$\frac{d^nu}{dx^n} \cdot v + n\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \cdots$$

$$+ \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{d^{n-k}u}{dx^{n-k}}\frac{d^kv}{dx^k} + \cdots + u\frac{d^nv}{dx^n}.$$

The terms on the right side of this equation may be obtained from the terms in the binomial expansion $(a + b)^n$ by replacing $a^{n-k}b^k$ by $(d^{n-k}u/dx^{n-k}) \cdot (d^kv/dx^k)$ for $k = 0, 1, 2, \ldots, n$, and interpreting d^0u/dx^0 as being u itself.

64. Find d^3y/dx^3 in each of the following cases:

a) $y = \sqrt{2x - 1}$ b) $y = \dfrac{1}{3x + 2}$

c) $y = ax^3 + bx^2 + cx + d$.

65. If $f(x) = (x - a)^n g(x)$, where $g(x)$ is a polynomial and $g(a) \neq 0$, show that $f(a) = 0 = f'(a) = \cdots = f^{(n-1)}(a)$; but $f^{(n)}(a) = n!\, g(a) \neq 0$.

66. If $y = 2x^2 - 3x + 5$, find Δy for $x = 3$ and $\Delta x = 0.1$. Approximate Δy by finding its principal part.

67. (a) Show that the perimeter P_n of an n-sided regular polygon inscribed in a circle of radius r is $P_n = 2nr \sin(\pi/n)$. (b) Find the limit of P_n as $n \to \infty$. Is the answer consistent with what you know about the circumference of a circle?

68. Find dy/dx and d^2y/dx^2 if $x = \cos 3t$ and $y = \sin^2 3t$.

69. To compute the height h of a lamppost, the length a of the shadow of a six-foot pole is measured. The pole is 20 ft from the lamppost. If $a = 15$ ft, with a possible error of less than one inch, find the height of the lamppost and estimate the possible error in height.

70. Find the differential dy in each of the following cases:

a) $y = x^2/(1 + x)$ b) $x^2 - y^2 = 1$ c) $xy + y^2 = 1$.

71. Suppose a function f satisfies the following two conditions for all x and y:

a) $f(x + y) = f(x) \cdot f(y)$;

b) $f(x) = 1 + xg(x)$, where $\lim_{x \to 0} g(x) = 1$.

Prove that (a) the derivative $f'(x)$ exists, (b) $f'(x) = f(x)$.

72. Let $f(x) = x^2 + 1$. Given $\epsilon > 0$, find $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ whenever $|x_1 - x_2| < \delta$ and x_1, x_2 both lie in the closed interval $-2 \le x \le 2$. State precisely what this means concerning the continuity of this function.

73. Given a function $f(x)$, defined for all real x, and a positive constant c such that $|f(x + h) - f(x)| \le ch^2$ for all real h. Prove that (a) $f(x)$ is uniformly continuous, (b) $f'(x) = 0$ for all x.

74. A function $f(x)$ is said to satisfy a Lipschitz condition of order m on the closed interval $a \le x \le b$ if there is a constant C such that

$$|f(x_2) - f(x_1)| \le C|x_2 - x_1|^m$$

for all values of x_1, x_2 on $[a, b]$. Prove that a function which satisfies a Lipschitz condition of order $m > 0$ on $[a, b]$ is uniformly continuous there.

75. Suppose $[a, b]$ is the interval $[-1, 1]$ and $f(x) = \sqrt{1 - x^2}$. Find appropriate values of C and of m to satisfy the conditions in Problem 74. [*Hint.* Show that if $y_2 > y_1 > 0$ and $y_2 - y_1 = h$, then $|\sqrt{y_2} - \sqrt{y_1}| \le \sqrt{h}$.]

Miscellaneous Problems Chapter 2, pp. 123–124

1. $-4(x^2-4)^{-(3/2)}$ **2.** $(5-2x-y)/(x+2y)$ **3.** $-y/(x+2y)$ **4.** $(3x^2+4y)/(9y^2-4x)$ **5.** $-y(2x+y)/(x(2y+x))$
6. $-2(x+1)(x^2+2x+2)/(x^3(x+2)^3)$ **7.** $2\sin(1-2x)$ **8.** $-1/\sin^2 x$ **9.** $(x+1)^{-2}$ **10.** $(2x+1)^{-(1/2)}$
11. $(3x^3-2a^2x)/\sqrt{x^2-a^2}$ **12.** $-4(2x-1)^{-2}$ **13.** $2x(1-x^2)^{-2}$ **14.** $3(x^2+x+1)^2(2x+1)$ **15.** $10\sin(5x)/\cos^3(5x)$
16. $(\sin^2 x\cos x-\cos^2 x\sin x)/(\sin^3 x+\cos^3 x)^{2/3}$ **17.** $[(4x+5)/2]\sqrt{2x^2+5x}$ **18.** $-(9/2)(4x+5)(2x^2+5x)^{-(5/2)}$
19. $-(y/x)[(2y\sqrt{xy}+1)/(4y\sqrt{xy}+1)]$ **20.** $(2x-y)/(x+2y)$ **21.** $-(y/x)^{1/3}$ **22.** $-\sqrt{y/x}$ **23.** $-(y/x)$
24. $-(y/x)$ **25.** $-\frac{1}{2}[(x+2y+y^2)/(x+2y+xy)]$ **26.** $(x^2-2x-1)/(2(1-x)^{1/2}(1+x^2)^{3/2})$
27. $1/(2y(x+1)^2)$ or $(x+1)^{-2}/(2y)$ **28.** $(12x-2xy-y^2)/(x^2+2xy-12y)$ **29.** $-(y+2)/(x+3)$ **30.** 1
31. $(4x+8)/\sqrt{4x^2+16x+15}$ **32.** $2t(1+t^2)^2/(1-t^2)$ **33.** $2x[1+((1-x^2)/(1+x^2)^3)]$ **34.** $m=1$, $y=x$ **35.** $2x+y=6$
36. $c=4$ **37.** $m=2$, $y-2x=-5$ **38.** $(2, 0)$ and $(-1, 27)$
39. **a)** $10(x^2+2x)^4(x+1)$ **b)** $(3t-1)/\sqrt{3t^2-2t}$ **c)** $r[(r^2+5)^{-(1/2)}+(r^2-5)^{-(1/2)}]$ **d)** $4x/(x^2+1)^2$ **40.** $x+27y=28$

41. $x+y=3$ **42.** $(2x+3)^{-(1/2)}$ **43.** $-6(2-3x)$, $f(x)=(2-3x)^2$ **44.** $-\frac{8}{5}$
45. $6\pi(\Delta r)^2$; a shell around the can with thickness Δr. **46.** $\pi x(20-x)$ **47.** $x_1=40$, $p=4$ (= 20¢) **49.** $(x-1)(2x-1)$
50. $(2x+1)/9$ **51.** 56 ft/sec **52.** $-\frac{12}{5}$ **53.** **a)** $h+2k=5$ **b)** $h=-4$, $k=\frac{9}{2}$, $r=5\sqrt{5/2}$ **54.** $2\sqrt{x^2+1}$
55. $2/[3(2y+1)(2x+1)\sqrt{x^2+x}]$ **56.** $2x\sqrt{3x^4-1}$ **57.** $(3/(x+1)^2)\sin[((2x-1)/(x+1))^2]$
58. $dy/du=(dy/dx)(dx/du)=12u\cos(2u^2+2\pi)$. At $u=0$, $dy/du=0$. **60.** $3(x-2)/(2x-3)^{3/2})$ **61.** $-\frac{3}{32}$ **62.** $y'=2$, $y''=-2$ **64.** **a)** $3(2x-1)^{-(5/2)}$ **b)** $-162(3x+2)^{-4}$ **c)** $6a$ **66.** $\Delta y=0.92$, principal part $=0.9$
67. **a)** Length of each side $=2r\sin(\pi/n)$ **b)** $2\pi r$; yes. **68.** $dy/dx=-2x$; $d^2y/dx^2=-2$ **69.** 14 ± 0.044 ft
70. **a)** $(x(x+2)/(x+1)^2)\,dx$ **b)** $(x/y)\,dx$ **c)** $(-y/(x+2y))\,dx$
72. $\delta=\epsilon/4$; the function is uniformly continuous for $-2\le x\le 2$. **75.** $m=\frac{1}{2}$, $c=\sqrt{2}$